THE BOUNDED ASPECT RATIO PROBLEM FOR VLSI-LAYOUTS OF PERFECT BINARY TREES

J. van Leeuwen

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Rijksuniversiteit Utrecht

Vakgroep informatica

Princetonplein 5
Postbus 80.002
3508 TA Utrecht
Telefoon 030–53 1454
The Netherlands

vakgroep informatica R.U. Utrecht

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Department of Computer Science, University of Utrecht
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Abstract. Let T_k be the perfect binary tree of depth k and $n = 2^k$ leaves. It has been shown that there is a smallest constant γ such that all T_k have a VLSI-embedding of area $\gamma n + o(n)$ for $k \to \infty$. We prove that this, asymptotically best bound on area can be achieved using layouts with an aspect ratio converging to 1.

1. Introduction.

With current VLSI-technology it has become feasible to design large switching circuits that can be integrated on a single chip of silicon (Mead and Conway [2]). Taking many specific constraints of the technology into account, Thompson [4] has formulated a simple model of the surface of a chip that allows for a mathematical analysis of the basic questions concerning layout and performance of the integrated circuits. The model provides a surface that consists of a rectangular grid of unit size cells which can contain either a "node" (a transistor) or a wire. In one cell at most two wires are allowed to cross. For our purposes the area of an embedded circuit will be the size of the smallest enclosing rectangle.

Using Thompson's model the question of determining good or minimum area embeddings often leads to geometric and combinatorial considerations concerning the specific circuit structures. (See Leiserson [1] for a survey.) In this paper we shall study a detailed question concerning optimum area embeddings of perfect binary trees (see also van Leeuwen, Overmars and Wood [6]). Given an embedding for which the smallest enclosing rectangle has a shortest side of length s and a longest of length ℓ, we introduce the following notion.

Defenition. The aspect ratio of an embedding is $\sigma = \frac{s}{\ell}$.

For embeddings of large circuits there are practical reasons (cf. [1]) for requiring that the aspect ratio remains bounded away from 0. We shall consider the question of whether optimum area embeddings of perfect binary trees have a uniformly bounded aspect ratio.

Let T_k denote the perfect binary tree of depth k and $n = 2^k$ leaves. A well-known construction (the "H-pattern") due to Mead and Rem [3] proves that every T_k can be embedded in $4n + o(n)$ area and an aspect ratio bounded by $\frac{1}{2}$. (Leiserson [1] and Valiant [5] proved that all trees with n leaves can be embedded in $O(n)$ area and a uniformly bounded aspect ratio.) Let $A_{opt}(k)$ denote the minimum area required for an embedding of T_k. The following result was proved by van Leeuwen, Overmars and Wood [6].

Theorem 1.1. There is a constant γ such that $A_{opt}(k) = \gamma n + o(n)$ for $k \to \infty$.

They show that $2 < \gamma < 2.74306$.

In section 2 we shall prove that there are linear area embeddings for T_k with aspect ratio converging to 0 for $k \to \infty$. Thus there seems to be no a priori reason to believe that optimum embeddings must have a uniformly bounded aspect ratio. Some geometric considerations enable us to prove that in all embeddings with a "small" aspect ratio the nodes of a same level of T_k must be rather widely distributed over the occupied area (i.e., they cannot cluster).

In section 3 we shall prove a number of results, leading up to the conclusion that the T_k's can all be embedded in such a manner that area converges to γn (the optimum) while the aspect ratios remain uniformly bounded and, in fact, converge to 1 for $k \to \infty$. It follows that in determining asymptotically optimal layouts for perfect binary trees one can restrict attention to rather "square" embeddings. We shall use the following terminology.

Definition. A design consists of (i) an infinite sequence $0 \leq k_0 < k_1 < \ldots$ (the base of the design) and (ii) for each $j \geq 0$ an embedding of T_{k_j}.

Definition. A layout is a design with base 0, 1, 2, ...

Definition. A design (layout) is said to be asymptotically optimal if $A(k_j) = \gamma n + o(n)$ for $j \to \infty$ ($n = 2^{k_j}$), where $A(k_j)$ denotes the area of the embedding of a T_{k_j} provided by the design. (γ is the constant referred to in theorem 1.1.)

2. Linear area embeddings for T_k with a small aspect ratio.

Let the short side of a rectangle enclosing the embedding of some T_k have length $s(k)$, the long side length $\ell(k)$. The following result provides a lower bound on $s(k)$ irrespective of the area of the bounding rectangle. The result is stated in Leiserson [1] (p. 21), but not proved there in detail.

Theorem 2.1. $s(k) \geq \frac{1}{2}k - o(k)$.

Proof

Let the columns along the long side of the enclosing rectangle be numbered 1, 2, ... We shall inductively define a finite sequence of pairs d_j, e_j ($j = 0, \ldots$) such that (i) $d_0 \leq d_1 \leq \ldots$, (ii) $e_0 \geq e_1 \geq \ldots$, (iii) $d_j < e_j$, (iv) every vertical line positioned between columns d_j and e_j cuts through $\geq j$ independent paths in T_k and (v) there is a subtree T_{k-2j} which does not contain any of these paths and whose leaves are all located in columns d_j through e_j.

Taking $d_0 = 1$ and $e_0 = \ell(k)$ gives a correct start. Assuming d_j and e_j are defined, the next pair is obtained as follows. Consider the T_{k-2j} that was identified with the j^{th} pair and split it in four T_{k-2j-2}'s. Let d_{j+1} be the first column $\geq d_j$ which contains a leaf of one of these subtrees and let e_{j+1} be the last column $\leq e_j$ which contains a leaf from a different one of these subtrees. Assuming d_{j+1} and e_{j+1} are well-defined and $d_{j+1} < e_{j+1}$, the path connecting the two leaves through the root of T_{k-2j} (see figure 1) will be a $(j+1)^{st}$ path independent of the previous j and any one of the two T_{k-2j-2}'s that do not contain one of these leaves will do to satisfy clause (v). Let J be the largest index for which a pair is obtained.

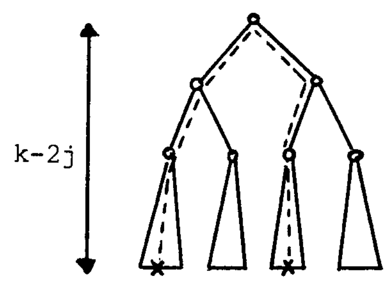

Figure 1.

A next pair can always be obtained as long as (i) $j \leq \frac{1}{2}k - 1$ and (ii) the leaves of the T_{k-2j} do not all lie in one single column. The latter is certainly the case as long as $s(k) < 2^{k-2j}$. It follows that $J \geq \frac{1}{2}k - \frac{1}{2}\log s(k)$. Observing that a vertical line can only cut $\geq J$

independent paths when there are $\gtrsim J$ squares in either column immediately bordering the line, we conclude that $s(k) \geq \frac{1}{2}k - \frac{1}{2}\log s(k)$ and (hence) that $s(k) \geq \frac{1}{2}k - o(k)$ for $k \to \infty$.

□

An embedding with $s(k) \sim \frac{1}{2}k$ is easily obtained by arranging the nodes of every level at equidistant positions in a separate row and collapsing the even-numbered rows (after the first) into the open "middle" positions of the odd-numbered rows (see figure 2). With the n leaves in row 1,

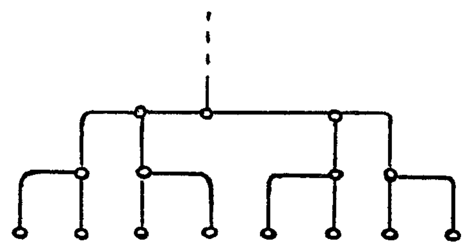

Figure 2.

the embedding uses area $\sim n.\frac{1}{2}k = O(n \log n)$. It is interesting that essentially for every $f(k) > \frac{1}{2}k$ there in an $O(n)$ area embedding for T_k with $f(k)$ as the length of the short side.

<u>Theorem</u> 2.2. Let $f(k) \geq ck$ for some (arbitrary) $c > \frac{1}{2}$ but $f(k) \leq \frac{4c}{c-1/2}\sqrt{n}$. Then T_k can be embedded in a rectangle of size $f(k)$ by $\frac{16c}{c-1/2}.n/f(k)$, for all $k > \frac{4}{c-1/2}$.

<u>Proof</u>

We shall make use of the following facts:

(a) every T_j can be embedded in a rectangle of size $\left\lceil\frac{j+1}{2}\right\rceil$ by 2^j, with all leaves appearing in the bottom row and the root appearing in the top row (as suggested above),

(b) every T_j can be embedded in a square of size $2.2^{\left\lceil\frac{j}{2}\right\rceil} - 1$ by $2.2^{\left\lceil\frac{j}{2}\right\rceil} - 1$, with access to the root (using the H-pattern).

Let $g = \frac{1}{4}(1 - \frac{1}{2c})\ f(k)$. It is easily verified that $1 < g \leq \sqrt{n}$. Determine $0 \leq j < k$ such that $2^j \leq n/g^2 = 2^k/g^2 < 2^{j+1}$. Imagine that T_k is cut at the j^{th} level. Lay out the complete tree by having a $\left\lceil\frac{j+1}{2}\right\rceil$ by 2^j

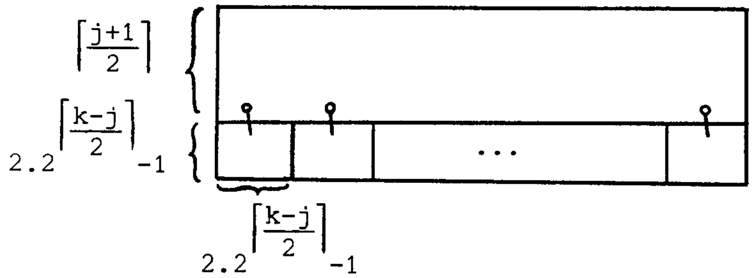

Figure 3.

embedding for the upper part, with the base line stretched by a factor $2.2^{\left\lceil \frac{k-j}{2} \right\rceil} - 1$, and a square embedding of a subtree of 2^{k-j} leaves appended to each of the nodes in the j^{th} level as they appear in the bottom row (see figure 3). Note that $1 \leqq 2^{k-j}/g^2 < 2$, hence $2^{\left\lceil \frac{k-j}{2} \right\rceil} \leqq \sqrt{2}.2^{\frac{k-j}{2}} < 2g$. The length of the short side can now be estimated as follows:

$$\left\lceil \frac{j+1}{2} \right\rceil + 2.2^{\left\lceil \frac{k-j}{2} \right\rceil} - 1 \leqq \frac{j+1}{2} + \frac{1}{2} + 2.2^{\left\lceil \frac{k-j}{2} \right\rceil} - 1 <$$

$$< \frac{j}{2} + 4g < \frac{k}{2} + 4g \leqq$$

$$\leqq \frac{1}{2c} f(k) + (1 - \frac{1}{2c}) f(k) = f(k)$$

The length of the long side can be bounded by

$$2^j.(2.2^{\left\lceil \frac{k-j}{2} \right\rceil} - 1) < 4.2^j.g \leqq$$

$$\leqq 4.\frac{n}{g^2}.g = 4.\frac{n}{g} = \frac{16c}{c-1/2}.n/f(k)$$

□

To obtain embeddings with s = s(k) small, it seems that the lower level nodes must be rather evenly spread over the "length" of the chip. To make this statement precise, we shall estimate the number of nodes of a same level that can occur in a single (narrow) slice.

Consider an arbitrary, i.e., not necessarily linear area embedding of T_k which fits in a rectangle with a shortest side of length s. Choose a vertical slice of width w somewhere in the rectangle (see figure 4) and pick a p with $1 < p < k$. Suppose the slice contains at least $\lceil \delta.2^p \rceil$ of the nodes at level p (some δ with $0 < \delta < 1$). It requires that

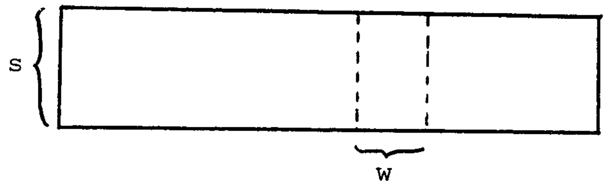

Figure 4.

$w.s \geq \lceil \delta.2^p \rceil$. It follows that the slice has only $w.s - \lceil \delta.2^p \rceil$ room left for nodes from other levels and thus, in particular, at least $2^k - w.s + \lceil \delta.2^p \rceil$ leaves of T_k definitely lie outside of the slice. Let all leaves outside of the slice be marked red. Suppose the slice contains r nodes of depth p whose subtree contains a red leaf. Note that outside of the slice there are at most $2^p - \lceil \delta.2^p \rceil \leq (1 - \delta)2^p$ nodes of depth p. Estimating the number of red leaves from above and from below, we obtain:

$$r.2^{k-p} + (1 - \delta)2^p.2^{k-p} \geq 2^k - w.s + \lceil \delta.2^p \rceil \Rightarrow$$

$$\Rightarrow r.2^{k-p} \geq \delta.2^k - w.s + \delta.2^p \Rightarrow$$

$$\Rightarrow r \geq \delta.2^p - (w.s - \delta.2^p)2^p/2^k.$$

Lemma 2.3. Suppose there is a vertical slice of width w which contains at least a fraction δ of the 2^p nodes at level p of T_k $(1 < p < k)$. Then $s \geq \delta.(2^k + 2^p)/(w + 2^{k-p+1})$.

Proof

We have identified r nodes among the 2^p at level p which must be connected to a red leaf outside of the slice. The connecting paths are independent and must cross the boundaries of the slice either to the left or to the right. Thus the 2.s squares along the boundaries of the slice must be sufficient to let $\geq r$ independent paths go through and we conclude that

$$s \geq \frac{1}{2}r \geq \delta.2^{p-1} - (w.s - \delta.2^p)2^{p-1}/2^k \Rightarrow$$

$$\Rightarrow 2^{k-p+1} s \geq \delta.2^k - w.s + \delta.2^p \Rightarrow$$

$$\Rightarrow s \geq \delta.(2^k + 2^p)/(w + 2^{k-p+1}).$$

□

Theorem 2.4. Consider any embedding of T_k with a shortest side of length $s = s(k) \leq 2^{\frac{k}{2}-h}$ (some $h \geq 0$). Then no slice of width $\leq \delta . 2^{\frac{k}{2}+h}$ can contain more than a fraction δ of the nodes at level p, for all $p > \frac{3}{4}k - \frac{1}{2}h + \frac{1}{2} + \frac{1}{2}\log\frac{1}{\delta}$ (and $p < k$).

Proof

Suppose there was a slice of width $w \leq \delta . 2^{\frac{k}{2}+h}$ which contained more than a fraction δ of the nodes at level p. By lemma 2.3 it follows that $s \geq \delta.(2^k + 2^p)/(\delta . 2^{\frac{k}{2}+h} + 2^{k-p+1})$. A contradiction can now be derived as follows.

We have $2^{\frac{3}{4}k-p-\frac{1}{2}h+1} < 2^{p-\log\frac{1}{\delta}-\frac{3}{4}k+\frac{1}{2}h}$ (using the assumption for p) and hence

$$\delta . 2^{\frac{1}{4}k+\frac{1}{2}h} + 2^{\frac{3}{4}k-p-\frac{1}{2}h+1} < \delta . 2^{\frac{1}{4}k+\frac{1}{2}h} + \delta . 2^{p-\frac{3}{4}k+\frac{1}{2}h} \Rightarrow$$

$$\Rightarrow \delta . 2^k + 2^{\frac{3}{2}k-p-h+1} < \delta . 2^k + \delta . 2^p \Rightarrow$$

$$\Rightarrow 2^{\frac{k}{2}-h}(\delta . 2^{\frac{k}{2}+h} + 2^{k-p+1}) < \delta.(2^k + 2^p) \Rightarrow$$

$$\Rightarrow 2^{\frac{k}{2}-h} < \delta.(2^k + 2^p)/(\delta . 2^{\frac{k}{2}+h} + 2^{k-p+1})$$

This contradicts the assumption on s.

□

Theorem 2.4. shows that the "lower level" nodes of T_k cannot cluster in slices of bounded width when s(k) is "small". Note that the theorem applies to slices of area up to about $\delta . 2^k$.

3. Designs and layouts with uniformly bounded aspect ratios.

It was argued that there is no reason to believe that optimal embeddings for T_k all have a uniformly bounded aspect ratio for $k \to \infty$. Leiserson [1] proved that every embedding with area A can be "folded" to fit into a rectangle with aspect ratio 1 (i.e., a square) and area $\leq 3A$. We shall give a number of ways to alter embeddings so as to bring their aspect ratio closer to 1, while inducing only a negligible amount of extra area. The first result (theorem 3.1.) is not restricted to embeddings for T_k, and applies to other "convergent" sequences as well. Recall the distinction made in Section 1 between designs and layouts.

Theorem 3.1. Given an asymptotically optimal design D with base B in which the embeddings do not have a uniformly bounded aspect ratio, and an arbitrary u with $0 < u < 1$, one can construct an asymptotically optimal design D' with base $B' \subseteq B$ in which the embeddings all have an aspect ratio $\geq u$.

Proof

Some purely arithmetic facts first. Let $t = \lceil -\frac{s}{2} + \frac{1}{2}\sqrt{s^2 + 4\ell s}\rceil$. It follows that $t^2 + st \geq \ell s$ and also that $t^2 + 2st \leq (-\frac{s}{2} + 1 + \frac{1}{2}\sqrt{s^2 + 4\ell s})^2 + 2s(-\frac{s}{2} + 1 + \frac{1}{2}\sqrt{s^2 + 4\ell s}) = \ell s + (-\frac{1}{2}s^2 + s + 1 + (\frac{s}{2} + 1)\sqrt{s^2 + 4\ell s}) \leq \leq \ell s + s.\sqrt{\ell s(\frac{s}{\ell} + 4)}$, provided $s \geq 2$. When $s \leq \alpha.\ell$, then $s \leq \sqrt{\alpha}.\sqrt{\ell s}$ and $t^2 + 2st \leq (1 + \sqrt{\alpha(\alpha + 4)})\ell s$.

Let D be as given. Use $s = s(k)$ and $\ell = \ell(k)$ to denote the length of the short and long sides, respectively, of the embeddings of T_k for $k \in B$. We may assume that $s(k)/\ell(k) \to 0$ for $k \to \infty$ and $k \in B$, perhaps after identifying B with a suitable infinite subsequence. Let $\alpha \leq 1$ be such that $1 + \sqrt{\alpha(\alpha + 4)} \leq \frac{1}{u}$ (as $\frac{1}{u} > 1$ such an α exists) and let N_0 be initialized such that for all $k \geq N_0$ and $k \in B$ $s(k)/\ell(k) \leq \alpha$. We shall construct a design D' with base B' consisting of suitably selected $k \in B$ with $k \geq N_0$ and suitable embeddings for the T_k with $k \in B'$. The i^{th} element of B' ($i = 0, 1, \ldots$) will be selected depending on a threshold N_i and tolerance $\varepsilon_i \leq \alpha$ as defined below. Let N_0 be as determined and ε_0 any real number with $0 < \varepsilon_0 \leq \alpha$. We use $A(k)$ (or $A'(k)$) to denote the area of the embedding of T_k according to D (D').

Starting with $i = 0$, the i^{th} element of B' will be the first $k \in B$ with $k \geq N_i$ such that $s(k)/\ell(k) \leq \varepsilon$ and $(\gamma - \varepsilon)n \leq A(k) \leq (\gamma + \varepsilon)n$, with $\varepsilon = \frac{1}{1369}\varepsilon_i^2$. (As $s(k)/\ell(k) \to 0$ and D is asymptotically optimal, such a k exists.) T_k will be embedded as follows. Cut its embedding according to D in at most $\lfloor\frac{\ell}{t}\rfloor + 1$ blocks of width t and fold it as suggested in figure 5, allowing an extra width of s on both sides of the embedding to route the existing wires between consecutive blocks

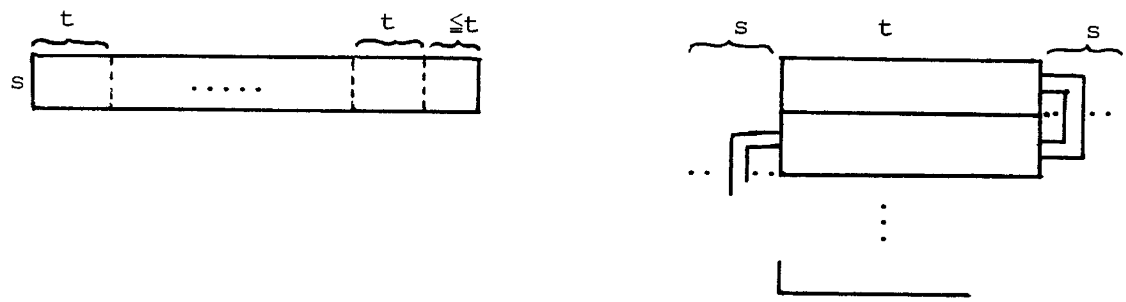

Figure 5.

around (t as above). The new layout has size $(\lfloor\frac{\ell}{t}\rfloor + 1)s$ by $t + 2s$. Observe that $t + 2s$ is the length of the longer side:

$$t^2 + 2st \geq \ell s + st \Rightarrow$$

$$\Rightarrow t + 2s \geq (\frac{\ell}{t} + 1)s \geq (\lfloor\frac{\ell}{t}\rfloor + 1)s.$$

For the aspect ratio σ of the embedding we obtain:

$$\sigma = (\lfloor\frac{\ell}{t}\rfloor + 1)s/(t + 2s) \geq \frac{\ell}{t}.s/(t + 2s) \Rightarrow$$

$$\Rightarrow \sigma \geq \ell s/(t^2 + 2st) \geq \ell.s/(1 + \sqrt{\varepsilon(\varepsilon + 4)})\ell.s \Rightarrow$$

$$\Rightarrow \sigma \geq 1/(1 + \sqrt{\alpha(\alpha + 4)}) \geq u,$$

as desired. It remains to estimate the area of the new embedding. As a lower bound we get:

$$A'(k) = (\lfloor\frac{\ell}{t}\rfloor + 1).s.(t + 2s) \geq \frac{\ell}{t}.s.(t + 2s) \Rightarrow$$

$$\Rightarrow A'(k) \geq \ell.s = A(k) \geq (\gamma - \varepsilon)n \Rightarrow$$

$$\Rightarrow A'(k) \geq (\gamma - \varepsilon_i)n$$

Before deriving an upper bound, we shall estimate $\frac{2s}{t}$ and $\frac{t}{\ell}$. Note that

$$\frac{2s}{t} \leq 2s/(-\frac{s}{2} + \frac{1}{2}\sqrt{s^2 + 4\ell s}) = 4/(-1 + \sqrt{1 + 4\frac{\ell}{s}}) \Rightarrow$$

$$\Rightarrow \frac{2s}{t} \leq 4/(-1 + \sqrt{1 + \frac{4}{\varepsilon}}) \leq 4\sqrt{\varepsilon}/(2 - \sqrt{\varepsilon}) \leq 4\sqrt{\varepsilon}$$

and that (use that $\varepsilon \leqq \sqrt{\varepsilon}$)

$$\frac{t}{\ell} \leqq \frac{1}{\ell} + \frac{1}{2}\sqrt{\frac{s^2}{\ell^2} + 4\frac{s}{\ell}} \leqq \varepsilon + \frac{1}{2}\sqrt{\varepsilon^2 + 4\varepsilon} \Rightarrow$$

$$\Rightarrow \frac{t}{\ell} \leqq 3\sqrt{\varepsilon}$$

The area of the new embedding can now be bounded as follows:

$$A'(k) \leqq (\frac{\ell}{t} + 1).s.(t + 2s) = \ell s(1 + 2\frac{s}{\ell} + \frac{2s}{t} + \frac{t}{\ell}) \Rightarrow$$

$$\Rightarrow A'(k) \leqq \ell s(1 + 2\varepsilon + 7\sqrt{\varepsilon}) = A(k)(1 + 2\varepsilon + 7\sqrt{\varepsilon}) \Rightarrow$$

$$\Rightarrow A'(k) \leqq (\gamma + \varepsilon)(1 + 2\varepsilon + 7\sqrt{\varepsilon})n \leqq (\gamma + 37\sqrt{\varepsilon})n \Rightarrow$$

$$\Rightarrow A'(k) \leqq (\gamma + \varepsilon_i)n$$

(where we have used that $\gamma \leqq 3$). To select the next element of B', set $N_{i+1} = k + 1$ (k the i^{th} index just determined) and $\varepsilon_{i+1} = \frac{1}{2}\varepsilon_i$.

The use of thresholds N_i guarantees that the elements of B' are chosen in numerically increasing order. As $\varepsilon_i \to 0$ for $i \to \infty$ the embeddings in D' have area converging to γn and (hence) D' is asymptotically optimal. □

Theorem 3.1. shows that we may restrict to asymptotically optimal designs with a uniformly bounded aspect ratio. (Note that designs do not give an embedding for every T_k yet.) A stronger result can be obtained that shows that even designs with a uniformly bounded aspect ratio can be improved, whenever the uniform bound is less than $\frac{1}{2}\sqrt{2}$.

Theorem 3.2. Given an asymptotically optimal design D and an arbitrary u with $\frac{1}{2} \leqq u < \frac{1}{2}\sqrt{2}$, one can construct an asymptotically optimal design D' in which the embeddings all have an aspect ratio $\geqq$u.

Proof

Let D have base B. Perhaps after omitting some indices from the beginning of B, we can assume that $A(k) \leqq 3n$ and $s(k) \geqq \frac{1}{3}k$ for all $k \in B$. Slice every embedding (of some T_k) in D under (or above) and right (or left) of the location of the root and insert an extra row and column, so room is obtained for a wire to access the root through the short side of the rectangle (see figure 6). If the original embedding

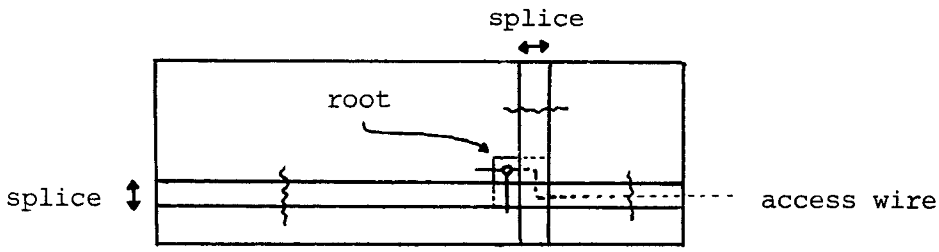

Figure 6.

had size s by ℓ, the new one has size s + 1 by ℓ + 1 and area is increased by $\ell + s + 1$. As $s \leqq \sqrt{3n}$ and $\ell \leqq 3n/s \leqq 9n/k = O(\frac{n}{\log n})$, the added amount is o(n) and the modified design D is still asymptotically optimal. We will construct a new design D' with base B' elementwise, with the i^{th} element depending on a threshold N_i and a tolerance ε_i. Let $\delta = 2u$. Let N_0 be such that $\ell(k) \geqq \frac{\delta}{2-\delta^2} \log 2\delta\ell(k)$ for $k \geqq N_0$ and let ε_0 be an arbitrary real number with $\varepsilon_0 > 0$. We use A(k) (and A'(k)) to denote the area of the embedding of T_k according to D(D').

Starting with i = 0, the i^{th} element of B' will be constructed from the first $k \in B$ with $k \geqq N_i$ such that $\log 2\delta\ell(k)/\ell(k) \leqq \varepsilon$ and $(\gamma - \varepsilon)n \leqq A(k) \leqq (\gamma + \varepsilon)n$, with $\varepsilon = \frac{1}{5}\varepsilon_i$. Let T_k have an s by ℓ embedding according to D, with access to the root by a wire through the short side. Let $j = j(k) \geqq 0$ be such that $2^j \leqq \delta.(\ell + \log 2\delta\ell)/s < 2^{j+1}$. Construct an

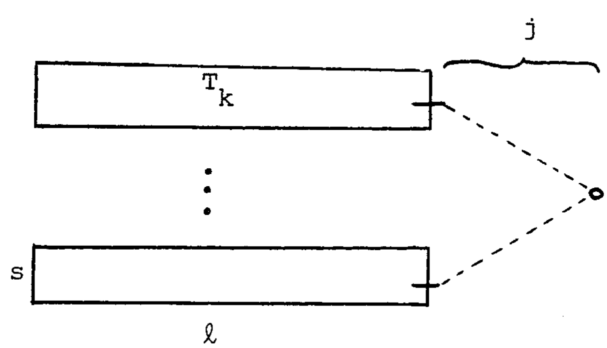

Figure 7.

embedding of T_{k+j} by stacking 2^j copies of the embedding of T_k with the long side horizontally, and connecting the T_k on one side of the stack by means of a simple tree-embedding of depth j (see figure 7). The resulting embedding has size $2^j s$ by $\ell + j$. When $2^j s \leq \ell + j$ we get the following bound for the aspect ratio:

$$\sigma = \frac{2^j s}{(\ell+j)} \geq \frac{1}{2}(\delta\ell + \delta \log 2\delta\ell)/(\ell + j) \geq \frac{\delta}{2} = u$$

(as $j \leq \log \frac{(\delta\ell+\delta \log 2\delta\ell)}{s} \leq \log \frac{(\delta\ell+\ell)}{s} \leq \log \frac{2\delta\ell}{s} \leq \log 2\delta\ell$). However, when we have $\ell + j \leq 2^j s$ the aspect ratio can be bounded by u as well:

$$\sigma = (\ell + j)/2^j s \geq (\ell + j)/\delta(\ell + \log 2\delta\ell) \geq \frac{\delta}{2} = u$$

(because $2\ell \geq \delta^2\ell + \delta \log 2\delta\ell$, hence $2\ell + 2j \geq \delta(\delta\ell + \log 2\delta\ell)$). It remains to estimate the area of the new embedding. As a lowerbound we get:

$$A'(k + j) = 2^j s.(\ell + j) \geq 2^j.\ell s = 2^j A(k) \Rightarrow$$
$$\Rightarrow A'(k + j) \geq 2^j.(\gamma - \varepsilon)2^k \geq (\gamma - \varepsilon_i)n$$

($n = 2^{k+j}$). As an upper bound we get:

$$A'(k + j) = 2^j s.(\ell + j) = 2^j.\ell s.(1 + \frac{j}{\ell}) = 2^j A(k)(1 + \frac{j}{\ell}) \Rightarrow$$
$$\Rightarrow A'(k + j) \leq (\gamma + \varepsilon)(1 + \frac{\log 2\delta\ell}{\ell})n \leq (\gamma + 5\varepsilon)n \Rightarrow$$
$$\Rightarrow A'(k + j) \leq (\gamma + \varepsilon_i)n.$$

We shall let k + j be the i^{th} element of B'. To prepare for selecting the next element of B', set $N_{i+1} = k + j + 1$ (k + j was the index just determined) and $\varepsilon_{i+1} = \frac{1}{2}\varepsilon_i$.

The use of thresholds N_i guarantees that the elements of B' are obtained in increasing order. As $\varepsilon_i \to 0$ for $i \to \infty$ the embeddings in D' have area converging to γn and (hence) D' is again asymptotically optimal.

□

We shall see momentarily that theorem 3.2. can be improved further, with the bound for u getting arbitrarily close to 1 (rather than just to $\frac{1}{2}\sqrt{2}$).

For the next result we assume that the root of T_k is accessible by a wire through the short side of the embedding. Let $N \geqq 3$ be such that $s(k) \geqq \frac{1}{3}k$ for all $k \geqq N$. (By theorem 2.1. such an N must exist.)

Lemma 3.3. Let T_k (some $k \geqq N$) have an embedding with aspect ratio $\geqq u$ ($u \geqq \frac{1}{2}$). Then T_{k+j} has an embedding with area $A(k + j)$ satisfying $2^j A(k) \leqq A(k + j) \leqq 2^j A(k)(1 + 11/2^{\frac{j}{2}})$ and aspect ratio $\geqq 1 - 8/2^{\frac{j}{2}}$, for all $60 \leqq j \leqq \frac{3}{7}k$.

Proof

Let the given embedding of T_k have size s by ℓ (with s the length of the shortest side) and aspect ratio of $\frac{s}{\ell} \geqq u$. Determine the smallest $a \geqq 1$ such that $\lfloor\frac{\ell}{s}a\rfloor . a \geqq 2^j$. Clearly $a \leqq \sqrt{\frac{s}{\ell}}.2^{\frac{j}{2}} + 1$. Construct an embedding for T_{k+j} by arranging 2^j copies of T_k in a stacks of at most $\lfloor\frac{\ell}{s}a\rfloor$ copies high, with $\lceil\log\lfloor\frac{\ell}{s}a\rfloor\rceil \leqq \lceil\log\sqrt{\frac{\ell}{s}}.2^{\frac{j}{2}} + 2\rceil \leqq \lceil\frac{1}{2u} + 1 + \frac{j}{2}\rceil \leqq j$ extra columns right of every stack to connect blocks into perfect subtrees and a layer of $\lceil\log a.(\frac{j}{2} + 3)\rceil \leqq \lceil\frac{j}{2} + \log(\frac{j}{2} + 3)\rceil < j$ extra rows over the top of the arrangement to connect the "loose" $\lceil 2 + \frac{j}{2}\rceil$ subtrees along every stack together (see figure 8). The embedding has size $\lfloor\frac{\ell}{s}a\rfloor s + j$ by $a.(\ell + j)$, where $\lfloor\frac{\ell}{s}a\rfloor s + j$ clearly is the length of the shortest side. For the aspect ratio we get the following bound:

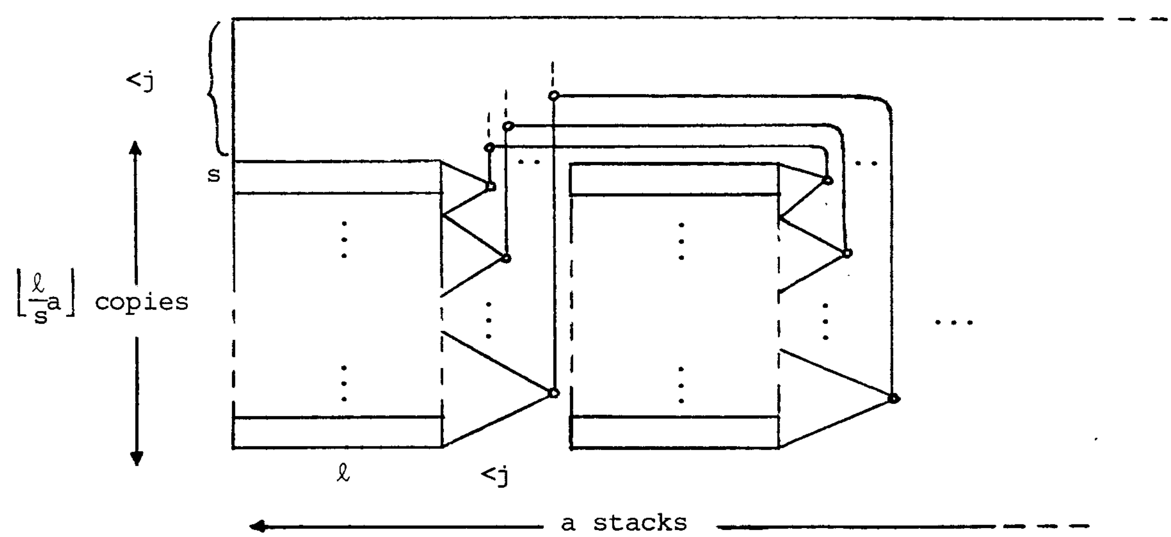

Figure 8.

$$\sigma = (\lfloor \tfrac{\ell}{s}a \rfloor s + j)/(a\ell + aj) \geqq (a\ell - s + j)/(a\ell + aj) \geqq (a\ell - s)/(a\ell + aj) \Rightarrow$$
$$\Rightarrow \sigma \geqq 1 - (aj + s)/(a\ell + aj) \geqq 1 - \frac{2}{a} \Rightarrow$$
$$\Rightarrow \sigma \geqq 1 - 1/2^{\frac{j}{2}-3}$$

where we have used that $a \geqq \frac{s}{\ell}2^{\frac{j}{2}} - 1 \geqq u2^{\frac{j}{2}} - 1 \geqq 2^{\frac{j}{2}-2}$ and $aj \leqq (2^{\frac{j}{2}} + 1)j \leqq 2^{\frac{k}{2}} < \ell$ for $j \leqq k - 3 \log k$ and k sufficiently large. The embedding uses 2^j copies of T_k and thus $A(k + j) \geqq 2^jA(k)$. It remains to bound the area of the new embedding from above. By the choice of a we have

$$\lfloor \tfrac{\ell}{s}a \rfloor . a \leqq 2^j + \{\lfloor \tfrac{\ell}{s}a \rfloor . a - \lfloor \tfrac{\ell}{s}(a - 1) \rfloor . (a - 1)\} \Rightarrow$$
$$\Rightarrow \lfloor \tfrac{\ell}{s}a \rfloor . a \leqq 2^j + \{\tfrac{\ell}{s}.a^2 - (\tfrac{\ell}{s}(a - 1) - 1).(a - 1)\} \Rightarrow$$
$$\Rightarrow \lfloor \tfrac{\ell}{s}a \rfloor . a \leqq 2^j + (2\tfrac{\ell}{s} + 1)a - \tfrac{\ell}{s} - 1 \leqq 2^j + 5a$$

For $A(k + j)$ we obtain:

$$A(k + j) = (\lfloor \tfrac{\ell}{s}a \rfloor s + j)(a\ell + aj) = \lfloor \tfrac{\ell}{s}a \rfloor a.\ell s + \lfloor \tfrac{\ell}{s}a \rfloor a.sj + aj\ell + aj^2 \Rightarrow$$
$$\Rightarrow A(k + j) \leqq 2^j\ell s(1 + \frac{5a}{2^j} + \frac{j}{\ell} + \frac{5aj}{\ell} + \frac{aj}{2^js} + \frac{aj^2}{2^j\ell s}) \Rightarrow$$
$$\Rightarrow A(k + j) \leqq 2^jA(k)(1 + (9.2^{\frac{j}{2}} + 3)/2^j + (5.2^{\frac{j}{2}} + 6)/2^{\frac{k}{2}} + (2^{\frac{j}{2}} + 1)j^2/2^j.2.2^k) \Rightarrow$$
$$\Rightarrow A(k + j) \leqq 2^jA(k)(1 + 10/2^{\frac{j}{2}} + 2^{\frac{2}{3}j}/2^{\frac{k}{2}}) \leqq 2^jA(k)(1 + 11/2^{\frac{j}{2}})$$

, provided j is large enough (e.g., $j \geqq 60$) and $j \leqq \frac{3}{7}k$.

□

<u>Lemma</u> 3.4. Let T_k (some $k \geqq N$) have an embedding with aspect ratio $\geqq u$. Then T_{k+j} has an embedding with area $A(k + j)$ satisfying $2^jA(k) \leqq A(k + j) \leqq 2^jA(k)(1 + \frac{7}{k})$ and aspect ratio $\geqq u$, for all $j \geqq 0$ and j even.

<u>Proof</u>

Let the given embedding of T_k have size s by ℓ, with $\frac{s}{\ell} \geqq u$. Construct an embedding of T_{k+j} (j even) by means of the H-pattern of Mead and Rem [3] (with access to the root through the short side all through the recursion), starting with the embedding of T_k. It is easily verified that for the length of the short and long sides, respectively, one has

$$s(k + j) = 2^{\frac{j}{2}}(s + 1) - 1$$

$$\ell(k + j) = 2^{\frac{j}{2}}(\ell + 1) - 1$$

Note that indeed $s(k + j) \leqq \ell(k + j)$ and that the aspect ratio can only have improved:

$$\sigma = (2^{\frac{j}{2}}(s + 1) - 1)/(2^{\frac{j}{2}}(\ell + 1) - 1) \geqq \frac{s}{\ell} \geqq u$$

As the H-pattern construction for T_{k+j} uses exactly 2^j copies of the embedding for T_k one has $A(k + j) \geqq 2^j A(k)$. On the other hand:

$$A(k + j) = 2^j \ell s + 2^{\frac{j}{2}}(2^{\frac{j}{2}} - 1)(\ell + s) + (2^{\frac{j}{2}} - 1)^2 \Rightarrow$$

$$\Rightarrow A(k + j) \leqq 2^j A(k)(1 + \frac{(\ell + s + 1)}{\ell s}) \leqq 2^j A(k)(1 + (2 + \frac{1}{\ell})/s) \Rightarrow$$

$$\Rightarrow A(k + j) \leqq 2^j A(k)(1 + \frac{7}{k})$$

□

Note that the H-pattern (in lemma 3.4) cannot be used for j odd, as it only guarantees an aspect ratio $\geqq \frac{1}{2}$ in this case.

We can now prove the main result relating optimal area embeddings and aspect ratios. Note that a complete layout (not just a design) is obtained.

Theorem 3.5. Given an asymptotically optimal design D and an arbitrary $\varepsilon > 0$, one can construct an asymptotically optimal layout L in which all embeddings have aspect ratio $\geqq 1 - \varepsilon$.

Proof

Given D, one can construct an asymptotically optimal design D' in which all embeddings have an aspect ratio $\geqq \frac{1}{2}$ (theorem 3.2.). Perhaps after omitting an initial segment of D' (so we can choose j approximately equal to $\frac{k}{4}$ while $j \geqq \max\{60, 2 \log \frac{8}{\varepsilon}\}$), lemma 3.3. can be used to obtain two asymptotically optimal designs D" and D"' with bases B" and B"' respectively, such that B" (B"') contains only even (odd) indices and the embeddings in both designs have aspect ratios $\geq 1 - \varepsilon$. Now construct a full layout L of the T_k ($k \geq 0$) as follows. Use an arbitrary embedding of aspect ratio 1 for all even (odd) k up to the first element of B" (B"'). From there, take the embeddings as given by D" and D"', using lemma 3.4. to generate embeddings for all T_k with even (odd) index k in between every two consecutive elements of B" (B"'). L is again asymptotically optimal and all aspect ratios are $\geq 1 - \varepsilon$.

□

Using theorem 1.1. (which asserts the existence of an asymptotically optimal layout, namely the sequence of optimal embeddings) it follows that for every $\varepsilon > 0$ there is an asymptotically optimal layout with all aspect ratios $\geq 1 - \varepsilon$. By a straightforward diagonal construction one obtains the following result:

Theorem 3.6. There is an asymptotically optimal layout for the T_k with aspect ratio converging to 1 for $k \to \infty$.

4. References.

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