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H.A.G. Wijshoff

J. van Leeuwen

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Rijksuniversiteit Utrecht

Vakgroep informatica

Princetonplein 5
Postbus 80.002
3508 TA Utrecht
Telefoon 030-53 1454
The Netherlands

vakgroep informatica R.U. Utrecht

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Department of Computer Science
University of Utrecht
P.O. Box 80.002, 3508 TA Utrecht
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Department of Computer Science, University of Utrecht
P.O. Box 80.002, 3508 TA Utrecht, the Netherlands

Abstract. In the study of data organisation for vector computations considerable attention has been given to mappings s (also called "skewing schemes") that map $\mathbb{Z}^2$ as the ultimate matrix domain to a finite set of memory banks in such a manner that cells of $\mathbb{Z}^2$ that are connected according to any one of a finite number of given data templates T_1 to T_k are mapped into different banks. We demonstrate that the (practical) class of periodic skewing schemes is best studied by methods from the classical theory of integral lattices. It is shown that every periodic skewing scheme is determined by a single fundamental polyomino that tiles the plane, and use this connection to derive a simple and effective storage mapping for every periodic scheme. (The fundamental polyomino of a skewing scheme can in fact be chosen as a rectangle.) We also prove that there is a polynomial time algorithm to determine for every set of (bounded) data templates T_1 to T_k the minimum number of memory banks required for a periodic skewing scheme that is valid for T_1 to T_k. (The algorithm also produces a corresponding "minimum" periodic skewing scheme in polynomial time.) Several interesting corollaries for periodic plane tessellations by polyominos are obtained as well.

1. Introduction.

SIMD-machines (Flynn [2]) are characterised by the availability of a multitude of arithmetic units and memory modules that can operate independently in parallel. Their effectiveness in vector processing applications derives from the fact that in one cycle a complete vector of M data items can be fetched through simultaneous access to each of the M memory banks provided in the architecture (see e.g. Thurber [8]). If the data items sought are not available in distinct memories and thus not retrievable as an M-vector in one cycle, then we say that a conflict occurs. Kuck [5] has shown that non-trivial storage schemes are needed if memory-conflicts are to be avoided in (numerical) algorithms. The problem arises in virtually all algorithms that operate on matrices of some size.

We follow a common paradigm (see e.g. Shapiro [7]) and study the problem of storing a matrix of arbitrary size in M memory banks such that all cells connected to one of a finite set of data templates $T_1, \dots, T_k$ necessarily belong to different memory banks. A template is just a (finite) configuration of locations in relative position. Any storage scheme s that satisfies some requirement of this type is called a "skewing scheme", and it is no restriction to assume that s is defined on the entire $\mathbb{Z}^2$ (identified with an infinite 2-dimensional matrix or "plane" of cells). For a justification of this assumption, see [7].

Definition. A skewing scheme s is valid for a template T if all cells connected according to T are mapped to different banks (no matter where T is positioned in the plane). A skewing scheme s is fitting for a template T if s is valid for T and $|T| = M$ (i.e., the size of T is exactly equal to the number of memory banks).

Note that a valid skewing scheme requires a number of banks at least equal to $|T|$. The following result is due to Shapiro [7] (see also [10])

Theorem. There exists a fitting skewing scheme for template T (thus using exactly $M = |T|$ memory banks) if and only if T tessellates the plane.

Clearly, if a template T does not tessellate the plane then a larger number of memory banks (i.e., larger than $|T|$) is needed for a valid skewing scheme to exist. In this paper we shall focus on the problem of determining the minimum number of memory banks required for the existence of a valid skewing scheme for an arbitrary set of templates T_i $(1 \le i \le k)$.

As expounded in [7] general skewing schemes are not of much practical interest if a high price must be paid for "storing" the scheme and computing the banknumbers where cells are mapped to. This lead Shapiro [7] to introduce "periodic" skewing schemes, which are completely defined by finite tabular information. We redefine these schemes in the following manner.

<u>Definition</u>. A skewing scheme s is called regular if and only if the following property is satisfied for all cells p and q ("points of the plane"): if $s(p) = s(q)$ then any pair of cells that are in the same relative position as p and q are mapped to equal banks.

(In section 2 we shall reconcile this definition with Shapiro's, after some non-trivial lemmas.) In section 2 we shall demonstrate that regular skewing schemes are closely connected to integral lattices as known in classical number theory (see Hardy and Wright [3]), with a useful relationship between the determinant of the corresponding lattice and the number of memory banks M used by the skewing scheme. This connection is instrumental for much of the technical results in this paper. There is an additional reason for restricting attention to regular (periodic) skewing schemes. Define a polyomino as a template of rook-wise connected cells and no "holes". Wijshoff and van Leeuwen [10] recently proved the following result:

<u>Theorem</u>. There exists a fitting skewing scheme for polyomino P if and

only if there exists a fitting regular (periodic) skewing scheme for P.

The remainder of the paper is organised as follows. In section 2 we develop the connection between regular (periodic) skewing schemes and integral lattices. Some familiarity with classical lattice theory will be assumed. A simple address mapping is derived that quickly gives the "bank" for any cell (i,j), given an arbitrary regular (periodic) skewing scheme. In section 3 the connection to lattices is explored further. It is shown that the lattice of a regular (periodic) skewing scheme always has a fundamental domain that is a polyomino. In fact, we show that the "fundamental polyomino" of a regular (periodic) s can be chosen to be a rectangle. Section 4 finally uses the results to show that the minimum number of memory banks required for valid periodic skewing of a (bounded) set of templates T_i $(1 \le i \le k)$ can be computed in time polynomial in the size of the templates. Several interesting corollaries for periodic plane tessellations by combinatorial objects are obtained, which extend the classical results in e.g. Hardy and Wright ([3], pp. 26-37) (see also Coxeter [1]).

2. Periodic skewing schemes.

We shall prove some basic facts for regular skewing schemes and justify the claim that they have a succinct finite representation. First we summarise the notions from lattice theory that we need (see e.g. [6]).

A two-dimensional lattice L generated by integral vectors $\vec{x}$ and $\vec{y}$ (the basis of L) is the set of integer linear combinations $\lambda\vec{x} + \mu\vec{y}$. The set $\{\lambda\vec{x} + \mu\vec{y} \mid 0 \le \lambda < 1,\ 0 \le \mu < 1\}$ is the socalled fundamental parallellogram of L. Its "volume" is $\Delta(L)$, also called the determinant of L. Clearly $\Delta(L) = |\det(\vec{x}\ \vec{y})|$ and it can be shown that $\Delta(L)$ is independent of the basis chosen for L. Cells p and q are equivalent modulo L, notation. $p \equiv_L q$, if $\vec{p} - \vec{q} \in L$.

<u>Definition</u>. Let s be a regular skewing scheme. Any vector $\vec{v}$ that is the relative position of two cells p and q with $s(p) = s(q)$ is called a period of s.

<u>Proposition</u> 2.1. The periods of a regular skewing scheme form a discrete group in $\mathbb{Z}^2$, and hence form a lattice.

<u>Proof</u>.

Let $\vec{v}$ and $\vec{w}$ be periods and p an arbitrary cell. Then $s(p) = s(p+\vec{v}) = s(p+\vec{v}+\vec{w})$, hence $\vec{v}+\vec{w}$ is a period. The remaining group properties are easily verified. Discreteness follows because periods are integral vectors by definition. By a classical theorem (see e.g. Weyl [9], p. 172) discrete groups necessarily are lattices. □

Finding a lattice basis for the periods of an effective regular skewing scheme is not hard: any two vectors (periods) $\vec{v}$ and $\vec{w}$ such that the closed triangle with vertices 0, v and w does not contain any other lattice points will do ("theorem on lattice triangles", see Lekkerkerker [6] p. 20). Let the lattice of periods of s be L. Let p be an arbitrary cell and $s(p) = b$. Then all cells of $p + L = p + \{\lambda\vec{x} + \mu\vec{y} \mid \lambda, \mu \in \mathbb{Z}\}$ are mapped to bank b.

<u>Proposition</u> 2.2 Let $s(p) = b$. Then $p + L$ is the collection of all cells that are mapped to bank b, i.e., it characterizes the contents of this memory bank.

<u>Proof</u>.

We only need to show that $q \in p + L$ when $s(q) = b$. Clearly $s(q) = s(p)$ implies that $\vec{q} - \vec{p} \in L$ (it is a period), and hence $q = p + (\vec{q} - \vec{p}) \in p + L$.

□

These simple observations lead to the definition of a "periodic" skewing scheme as used in [10].

Definition. A skewing scheme $s : \mathbb{Z}^2 \to \{1..M\}$ is called periodic if there are M cells $a_1, \dots, a_M$ and a lattice L such that (i) the "co-sets" $a_i + L$ $(1 \le i \le M)$ are all disjoint but cover the entire $\mathbb{Z}^2$ and (ii) s maps all cells in $a_i + L$ to bank i. (We will often speak of a skewing scheme that is periodic "with lattice L", or say that L is the lattice of s.)

Proposition 2.3. A skewing scheme s is periodic if and only if it is regular.

Proof.

Let s be periodic with lattice L. If $s(p) = s(q)$, then p and q belong to the same $a_i + L$ and thus $\vec{p} - \vec{q} \in L$. It follows that all cells that are in the same relative position as p and q belong the same $a_j + L$ and are thus mapped to equal banks. Hence s is regular. The converse easily follows from proposition 2.2. Let L be the lattice of periods, and choose for a_i any point that is mapped to bank i. □

We shall from now on speak of periodic skewing schemes only.

Theorem 2.4. Let s be a periodic skewing scheme using M memory banks, L the underlying lattice. Then $\Delta(L) = M$, i.e., the determinant of L is precisely equal to the number of memory banks used.

Proof.

Identify each cell with its "midpoint". Consider the template T_L of cells that have their midpoint inside the fundamental parallellogram of L. Now observe that (i) no two cells of T_L are equivalent modulo L (and thus the cells of T_L are all mapped to different banks) and (ii) every cell of the plane is equivalent to a cell of T_L modulo L (and thus every bank receives some cell of T_L). It follows that $|T_L| = M$. On the other hand it can be seen that T_L covers exactly $\Delta(L)$ area. To this end, consider how T_L covers the fundamental parallellogram. Any cell of T_L fully contained in the parallellogram contributes a unit of 1 to $\Delta(L)$. Any cell p of T_L that has a part of area ε sticking out of the fundamental parallellogram (into one, two or three neighboring parallellograms) only covers an area of $1-\varepsilon$ of it, but this is compensated for by the cells not belonging to the instance of T_L that are situated like p in a neighboring parallellogram and that have a part sticking out into the fundamental parallellogram. (The total compensation of ε area thus comes from one, two or three bordering cells, respectively.) Hence all cells of T_L still account for precisely $\Delta(L)$ total area, and $\Delta(L) = M$. □

<u>Corollary</u> 2.5 Every periodic skewing scheme is fitting for some template.

<u>Proof</u>.

Consider the instances of T_L situated at every lattice point. The instances of T_L are disjoint but cover the complete plane (because every cell belongs to some parallellogram), and thus form a tessellation. By Shapiro's theorem ([7], see section 1) it follows that s is a fitting skewing scheme for T. □

(Given the observation that T_L tessellates the plane and has its instances arranged at every lattice point in the same manner, the conclusion of theorem 2.4 essentially follows also from Hardy and Wright [3], thm 41.)

In practice a (periodic) skewing scheme s will not be used on the entire $\mathbb{Z}^2$ but on the domain of an $N \times N$ matrix only, for some $N > 0$. If s is valid for a template T, then it is valid for the matrix as long as we only consider instances of T that are located entirely within the domain of the matrix. In some applications one also considers instances of T that lie in part across the border and uses the "wrap-around" convention for the cells that stick out, i.e., their coordinates are reduced modulo N to map them back into the matrix domain. We call this "skewing with wrap-around".

Theorem 2.6. Let s be a periodic skewing scheme using M memory banks, and assume that s is valid for a template T. Then s is valid for T on any $N \times N$ matrix with $M | N$, allowing "wrap-around" of the instances of T.

Proof.

Consider any cell p of T that is involved in a "wrap-around", $p \in a_i + L$ (thus p is mapped to bank i) and $p = \binom{a_{i1}}{a_{i2}} + \lambda \binom{x_1}{x_2} + \mu \binom{y_1}{y_2}$ for some integers λ and μ. Clearly cell p is wrapped around to the cell p' with

$$p' = \begin{pmatrix} (a_{i1} + \lambda x_1 + \mu y_1) \bmod N \\ (a_{i2} + \lambda x_2 + \mu y_2) \bmod N \end{pmatrix}$$

. We shall prove that $p' \in a_i + L$, and thus p' is mapped to the same bank as p and no conflict is introduced because of the wrap-around. Let $N = \gamma M$ for some integer $\gamma > 0$.

Clearly there exist integers α and β such that the following equalities hold for the coordinates of p':

$$\begin{aligned} (a_{i1} + \lambda x_1 + \mu y_1) \bmod N &= a_{i1} + \lambda x_1 + \mu y_1 + \alpha N \\ (a_{i2} + \lambda x_2 + \mu y_2) \bmod N &= a_{i2} + \lambda x_2 + \mu y_2 + \beta N \end{aligned} \qquad (1)$$

By theorem 2.4 $M = \Delta(L) = \left| \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \right| = | x_1 y_2 - x_2 y_1 |$ and hence $N = \gamma\, | x_1 y_2 - x_2 y_1 |$. By omitting the sign restriction on γ we can simply write $N = \gamma \cdot (x_1 y_2 - x_2 y_1)$. Substituting this into the right-hand sides of equations (1) we obtain after some rearrangements.

$$\begin{aligned} (a_{i1} + \lambda x_1 + \mu y_1) \bmod N &= a_{i1} + \lambda' x_1 + \mu' y_1 \\ (a_{i2} + \lambda x_2 + \mu y_2) \bmod N &= a_{i2} + \lambda' x_2 + \mu' y_2 \end{aligned} \qquad (2)$$

with $\lambda' = \lambda + \alpha\gamma y_2 - \beta\gamma y_1$ and $\mu' = \mu - \alpha\gamma x_2 + \beta\gamma x_1$ (thus $\lambda', \mu' \in \mathbb{Z}$). Thus $p' = a_i + \lambda' \vec{x} + \mu' \vec{y} \in a_i + L$, as was to be shown. $\square$

Crucial for the use of a periodic skewing scheme s is the question whether s-values can be efficiently computed. To compute $s(p)$ for a cell $p = (i,j)$ one needs to determine the (unique) k such that $p \in a_k + L$ $(1 \le k \le M)$, where L is the lattice corresponding to s.

<u>Proposition</u> 2.7. Every periodic skewing scheme s using M memory banks can be completely described by an $M \times M$ table a and a look-up procedure that is as simple as $s(i,j) = a[i \bmod M, j \bmod M]$.

<u>Proof</u>.

In the proof of theorem 2.6 was shown that the value of $s(p)$ does not change if we reduce the coordinates of p modulo M (take $N = M$ in the argument). It means that through this mechanism all values of s are suitably summarized in a table that lists the s-values for all cells (i,j) with $0 \le i,j < M$. □

(Note that proposition 2.7 reconciles our definition of periodic skewing schemes with one proposed by Shapiro [7].) Theorem 2.6 and proposition 2.7 are useful when a periodic skewing scheme must be designed using M banks that is valid for some template T: one only needs to try out on an $M \times M$ rectangle. From a practical point of view proposition 2.7 is not very useful, because the table is very large while only M essentially different values need to be recorded. A "minimum table" is obtained if we list the M different s-values corresponding to the (M) cells of the template T_L defined in the proof of theorem 2.4, and reduce cells modulo L to a cell of T_L. Define the "lattice matrix" A_L by

$$A_L = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

, where $\{\vec{x}, \vec{y}\}$ is the basis used for L. (Note that $\Delta(L) = |\det A_L|$.) Given a cell $p = (i,j)$ it follows that $A_L^{-1}(p)$ is the cell p "written in lattice coordinates" and (hence) that $\lfloor A_L^{-1}(p) \rfloor$ is the lattice-point that is the base-tip of the parallellogram containing p. Thus the reduction of p modulo L (to a cell of T_L) is simply computed as $A_L^{-1}(p) - \lfloor A_L^{-1}(p) \rfloor$ in lattice coordinates, or $p - A_L(\lfloor A_L^{-1}(p) \rfloor)$ in cartesion coordinates. Because T_L is not a very regular template, the look-up procedure requires one additional step depending on the representation of T_L. (We shall see in section 3 that T_L can be replaced by an equivalent template of a much more regular shape.) We shall proceed with a different argument and show that with every lattice L a simple expression can be associated that defines the representing periodic skewing scheme s_L using $M = \Delta(L)$ memory banks+.

Lemma 2.8 Let $\vec{x} = \binom{x_1}{x_2}$ be such that $\gcd(x_1, x_2) = 1$ and $x_{1,2} \neq 0$. The mapping s with $s(i,j) = (-x_2 \cdot i + x_1 \cdot j) \bmod M$ is a periodic skewing scheme with the underlying lattice L generated by $\vec{x}$ and $\vec{y}$ ($\vec{y}$ arbitrary), with $M = \left| \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \right|$.

Proof.

By proposition 2.3 it suffices to show that s is regular, with L as its lattice of periods. Write $M = \gamma \cdot (x_1 y_2 - x_2 y_1)$ for $\gamma = \pm 1$.

Suppose $s(i_1, j_1) = s(i_2, j_2)$. It means that the following equality holds modulo M:

$$-x_2 \cdot i_1 + x_1 \cdot j_1 \equiv -x_2 \cdot i_2 + x_1 \cdot j_2 \pmod M$$

and hence that there exists an integer α such that

$$-x_2 \cdot i_1 + x_1 \cdot j_1 = -x_2 \cdot i_2 + x_1 \cdot j_2 + \alpha M \quad \Rightarrow$$

$$\Rightarrow \quad x_1 \cdot (j_1 - j_2) - x_2 \cdot (i_1 - i_2) = \alpha \gamma (x_1 y_2 - x_2 y_1) \quad \Rightarrow$$

+ The function s_L was obtained in joint work with H. L. Bodlaender.

$$\Rightarrow x_1 \cdot (j_1 - j_2 - \alpha\gamma y_2) = x_2 \cdot (i_1 - i_2 - \alpha\gamma y_1)$$

Because $\gcd(x_1, x_2) = 1$ and $x_{1,2} \neq 0$ it follows that there must be an integer β such that

$$i_1 - i_2 - \alpha\gamma y_1 = \beta x_1$$
$$j_1 - j_2 - \alpha\gamma y_2 = \beta x_2$$

Hence $(i_1, j_1) - (i_2, j_2) = \beta \cdot \vec{x} + \alpha\gamma \cdot \vec{y} \in L$.

Conversely, let $\alpha\vec{x} + \beta\vec{y} \in L$ be the relative position of two cells (i_1, j_1) and (i_2, j_2) : $(i_1, j_1) - (i_2, j_2) = \alpha \cdot (x_1, x_2) + \beta\,(y_1, y_2)$. It follows that

$$x_1 \cdot (j_1 - j_2) - x_2 \cdot (i_1 - i_2) = x_1 \cdot (\alpha x_2 + \beta y_2) - x_2 \cdot (\alpha x_1 + \beta y_1) \Rightarrow$$

$$\Rightarrow x_1 \cdot (j_1 - j_2) - x_2 \cdot (i_1 - i_2) = \beta\gamma \cdot \gamma\,(x_1 y_2 - x_2 y_1) = \beta\gamma M \Rightarrow$$

$$\Rightarrow -x_2 \cdot i_1 + x_1 \cdot j_1 \equiv -x_2 \cdot i_2 + x_1 \cdot j_2 \pmod{M}$$

and hence that $s(i_1, j_1) = s(i_2, j_2)$. □

Corollary 2.9. Let $\vec{x} = \binom{x_1}{x_2}$ be such that $\gcd(x_1, x_2) = g$ and $x_{1,2} \neq 0$. The mapping s defined by $s(i,j) = (-\frac{x_2}{g} \cdot i + \frac{x_1}{g} \cdot j) \bmod M/g$ is a periodic skewing scheme with the underlying lattice L generated by $\frac{1}{g} \cdot \vec{x}$ and $\vec{y}$ ($\vec{y}$ arbitrary), with $M := \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right|$.

(Note that M is indeed divisible by g.)

Lemma 2.10. Let $\vec{x} = \binom{x_1}{x_2}$ be such that $\gcd(x_1, x_2) = g$ and $x_{1,2} \neq 0$. The mapping s defined by

$$s(i,j) = (-\frac{x_2}{g} \cdot i + \frac{x_1}{g} \cdot j) \bmod M/g + (\lfloor \tfrac{g}{M}(y_2 \cdot i - y_1 \cdot j) \rfloor \bmod g) \cdot M/g$$

is a periodic skewing scheme with the underlying lattice L generated by $\vec{x}$ and $\vec{y}$ ($\vec{y}$ arbitrary), with $M = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right|$.

Proof.

Note that s is of the form $s_1(i,j) + s_2(i,j)\cdot M/g$ and that obviously: $s(i_1,j_1) = s(i_2,j_2) \Leftrightarrow (s_1(i_1,j_1) = s_1(i_2,j_2) \wedge s_2(i_1,j_1) = s_2(i_2,j_2))$. For the lemma it suffices to prove that s is regular, with L as its lattice of periods. Write $M = \gamma\cdot(x_1y_2 - x_2y_1)$.

Suppose $s(i_1,j_1) = s(i_2,j_2)$. Then $s_1(i_1,j_1) = s_1(i_2,j_2)$ and by the argument of lemma 2.8 it follows that there must be integers α and β such that

$$\begin{aligned} i_1 - i_2 &= \beta/g\cdot x_1 + \alpha\gamma y_1 \\ j_1 - j_2 &= \beta/g\cdot x_2 + \alpha\gamma y_2 \end{aligned} \qquad (1)$$

(Note that $\beta/g\cdot x_i$ is integer because $g\,|\,x_i$, $1\le i\le 2$.) On the other hand we also have $s_2(i_1,j_1) = s_2(i_2,j_2)$. Substituting expressions for i_1 and j_1 from (1) this leads to

$$\left\lfloor \frac{g}{M}\left\{y_2\left(i_2 + \frac{\beta}{g}\cdot x_1 + \alpha\gamma y_1\right) - y_1\left(j_2 + \frac{\beta}{g}\cdot x_2 + \alpha\gamma y_2\right)\right\}\right\rfloor \equiv \left\lfloor\frac{g}{M}(y_2 i_2 - y_1 j_2)\right\rfloor \pmod g$$

$$\Rightarrow \left\lfloor \frac{g}{M}(y_2 i_2 - y_1 j_2) + \frac{g}{M}\cdot\frac{\beta}{g}\cdot(x_1y_2 - x_2y_1)\right\rfloor \equiv \left\lfloor\frac{g}{M}(y_2 i_2 - y_1 j_2)\right\rfloor \pmod g$$

$$\Rightarrow \left\lfloor \frac{g}{M}(y_2 i_2 - y_1 j_2)\right\rfloor + \gamma\beta \equiv \left\lfloor\frac{g}{M}(y_2 i_2 - y_1 j_2)\right\rfloor \pmod g$$

and hence $\beta \equiv 0 \pmod g$, and β/g must be integer. From (1) we conclude that $(i_1,j_1) - (i_2,j_2) = \beta/g\,\vec{x} + \alpha\gamma\vec{y} \in L$.

Conversely, let $\alpha\vec{x} + \beta\vec{y}$ be the relative position of two cells (i_1,j_1) and (i_2,j_2): $(i_1,j_1) - (i_2,j_2) = \alpha\cdot(x_1,x_2) + \beta\cdot(y_1,y_2)$. Rewriting this as $(i_1,j_1) - (i_2,j_2) = \alpha g\cdot(x_1/g, x_2/g) + \beta\cdot(y_1,y_2)$, the argument of lemma 2.8 shows that $s_1(i_1,j_1) = s_1(i_2,j_2)$. On the other hand we also have

$$y_2 \cdot i_1 - y_1 j_1 = y_2 \cdot (i_2 + \alpha x_1 + \beta y_1) - y_1 (j_2 + \alpha x_2 + \beta y_2) \Rightarrow$$

$$\Rightarrow y_2 \cdot i_1 - y_1 \cdot j_1 = y_2 \cdot i_2 - y_1 \cdot j_2 + \alpha (x_1 y_2 - x_2 y_1) \Rightarrow$$

$$\Rightarrow \frac{g}{M}(y_2 i_1 - y_1 j_1) = \frac{g}{M}(y_2 i_2 - y_1 j_2) + \alpha \gamma g \Rightarrow$$

$$\Rightarrow \lfloor \frac{g}{M}(y_2 i_1 - y_1 j_1) \rfloor \equiv \lfloor \frac{g}{M}(y_2 i_2 - y_1 j_2) \rfloor \pmod g$$

and thus $s_2(i_1, j_1) = s_2(i_2, j_2)$. We conclude that $s(i_1, j_1) = s(i_2, j_2)$ as was to be shown. □

Theorem 2.11. For every lattice L there is a periodic skewing scheme s_L corresponding to L whose values can be obtained by direct evaluation of a simple expression. All periodic skewing schemes s that have L as the underlying lattice can be obtained from s_L by a mere permutation of the "names" of the memory banks.

Proof.

Let L be generated by integral vectors $\vec{x}$ and $\vec{y}$. If $x_{1,2} \neq 0$ (or symmetrically, $y_{1,2} \neq 0$) then a periodic skewing scheme s_L as desired is given by the expression in lemma 2.10. It remains to define a scheme when x_1 and x_2 are not both $\neq 0$, and also y_1 and y_2 are not both $\neq 0$. Assume without loss of generality that $x_1 \neq 0$ and $x_2 = 0$. Then necessarily $y_1 = 0$ and $y_2 \neq 0$ (or else L would not be 2-dimensional), and $M = x_1 y_2$. Substituting in the expression in lemma 2.10 leads to a mapping s_L defined by $s_L(i,j) = j \bmod y_2 + (i \bmod x_1) \cdot y_2$, which is easily seen to be a periodic skewing scheme with underlying lattice L. This covers all cases, and the claim for an s_L is proved.

In the proof of theorem 2.4 was shown that every periodic skewing scheme s with underlying lattice L is fully determined by its values on a certain template T_L of size M, with $M = \Delta(L)$ the number of memory banks used. Thus every such skewing scheme s only differs from s_L by a permutation of the bank numbers. □

Theorem 2.11 substantiates the claim that periodic skewing schemes are "finitely represented" and, in fact, always easy to compute. In section 3 we shall find an even simpler sample scheme s_L.

Corollary 2.12. Every periodic skewing scheme s using M memory banks can be completely described by a table a of size M and a look-up procedure that is as simple as $s(i,j) = a[f(i,j)]$, where $f(i,j)$ is a simple expression.

Proof.

Immediate from theorem 2.11 using s_L for f and a table a to store the permutation by means of which s is obtained from s_L. □

3. Fundamental templates and their use.

We shall now delve deeper into the structure of two-dimensional lattices and (thus) of periodic skewing schemes. We shall always use L to denote a lattice, s its corresponding periodic skewing scheme.

Definition. A fundamental domain of a lattice L is any (viz. compact) domain $F \subseteq \mathbb{Z}^2$ such that (i) no two distinct points of F are equivalent modulo L and (ii) every point of the plane is equivalent to a point of F modulo L.

Clearly the fundamental parallellogram of L is a fundamental domain, but it normally will not be a template (i.e., be a composite of complete cells). In the proof of theorem 2.4 a fundamental domain T_L was obtained that is a correct template. Simple examples show that T_L need not be a polyomino and is, in fact, not necessarily even connected (take a lattice L in which the base vectors make a very small angle). Still it proves that fundamental templates exist.

Definition. Let L be a lattice, and let S and T be templates. S and T are said to be equivalent modulo L, notation: $S \equiv_L T$, if there is a 1-1 correspondence between the cells of S and T such that corresponding cells are equivalent modulo L.

(Thus $S \equiv_L T$ if and only if S and T have the same number of cells and S can be "moved" modulo L so as to completely and exactly cover T.) Clearly all fundamental templates of L are equivalent modulo L and consist of the same number of cells $M = \Delta(L)$. The use of fundamental templates derives from the following simple, but important observation.

Theorem 3.1 Let s be a periodic skewing scheme with underlying lattice L, and let F be a fundamental template for s. Then s is valid for a template T if and only if T is equivalent modulo L to a subtemplate of F.

Proof.

Let s be valid for T. Then every cell of T is mapped to a different memory bank, and (thus) cells correspond to unique cells of F modulo L. Conversely, it follows from Shapiro's theorem [7] (see also section 1) that s is valid for F and hence for every subtemplate of F or equivalent modulo L of it. □

We shall now demonstrate that every lattice L has a fundamental template that is a rectangle+. We need the following observations. If we draw a horizontal line through any lattice point, then other lattice points on this line at regular distances called the "horizontal yardstick" of L. If we draw horizontal lines through all lattice points

+ Whereas the underlying lattice theory is elementary, we have not found this observation in references we consulted like [1] and [4].

(identifying lines that coincide), then the horizontal lines appear at regular distances called the "vertical yardstick" of L.

<u>Lemma</u> 3.2 Let L be generated by integral vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. Then the horizontal yardstick of L has size $\Delta(L)/|\gcd(x_2, y_2)|$ and the vertical yardstick of L has size $|\gcd(x_2, y_2)|$.

<u>Proof</u>.

Lattice points have coordinates $(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2)$ with $\lambda, \mu \in \mathbb{Z}$. The y-coordinates $\lambda x_2 + \mu y_2$ precisely range over all multiples of the gcd of x_2 and y_2, as is well-known. Thus the vertical yardstick is as stated. To determine the horizontal yardstick, assume without loss of generality that $y_2 \neq 0$ and consider two lattice points $(\lambda_1 x_1 + \mu_1 y_1, \lambda_1 x_2 + \mu_1 y_2)$ and $(\lambda_2 x_1 + \mu_2 y_1, \lambda_2 x_2 + \mu_2 y_2)$ on the same horizontal line. Comparing y-coordinates we have

$$\lambda_1 x_2 + \mu_1 y_2 = \lambda_2 x_2 + \mu_2 y_2 \Rightarrow$$

$$\Rightarrow (\lambda_1 - \lambda_2) x_2 = -(\mu_1 - \mu_2) y_2$$

and thus either (i) there is an integer α such that $\lambda_1 - \lambda_2 = \alpha \cdot y_2/\gcd(x_2, y_2)$ and $\mu_1 - \mu_2 = -\alpha \cdot x_2/\gcd(x_2, y_2)$, or (ii) $\mu_1 = \mu_2$ and $x_2 = 0$ (which implies that $\Delta(L) = |x_1 y_2 - x_2 y_1| = |x_1 y_2|$ and $\Delta(L)/|\gcd(x_2, y_2)| = |x_1|$). Now consider the difference in x-coordinate

$$(\lambda_1 - \lambda_2) x_1 + (\mu_1 - \mu_2) y_1$$

In case (i) this evaluates to $\alpha \, (x_1 y_2 - x_2 y_1)/\gcd(x_2, y_2)$ and thus gives exactly all multiples of $\Delta(L)/|\gcd(x_2, y_2)|$ by varying α. In case (ii) we get $(\lambda_1 - \lambda_2) x_1$ and this also gives all multiples of $|x_1| = \Delta(L)/|\gcd(x_2, y_2)|$, by varying λ_1 and λ_2. Thus the horizontal yardstick is as stated in the lemma. □

Theorem 3.3 Let L be generated by integral vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. Then the (integral) rectangle of size $|\gcd(x_2, y_2)|$ (vertical) by $\Delta(L)/|\gcd(x_2, y_2)|$ (horizontal) is a fundamental template of L.

Proof.

Let the rectangle be R. Note that R has area equal to $\Delta(L)$, and thus consists of exactly $\Delta(L)$ cells. We claim that R contains no two points p and q ($p \neq q$) that are equivalent modulo L. For suppose there were. Then (by shifting p and q) there would be two lattice points whose y-coordinates differ by less than the vertical yardstick (which means they must lie on the same horizontal line) and whose x-coordinates differ by less than the horizontal yardstick (which is impossible if they lie on the same horizontal line), a contradiction. Because there are no more than $M = \Delta(L)$ distinct memory banks, this implies that R must be an exact fundamental domain (template) of L. (The same conclusion can be drawn from Hardy and Wright [3], thm 42.) □

Several useful conclusions can be drawn from this theorem.

Corollary 3.4. All fundamental templates of a lattice are equivalent to a rectangle.

Corollary 3.5. Every periodic skewing scheme is fitting for a rectangular template.

Theorem 3.3 is also of interest in conjunction with a theorem of Wijshoff and van Leeuwen [10] (see section 1) that asserts that every fitting skewing scheme for a polyomino can be transformed into a periodic scheme. We can now conclude that whenever a polyomino tessellates the plane, then it can tessellate the plane periodically and is equivalent to a rectangle modulo the underlying lattice!

A fundamental rectangle R naturally has the form of a table that we can use to obtain another "sample" periodic skewing scheme s_L with underlying lattice L. Let L be generated by integral vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ and take R as in theorem 3.3. Write M for $\Delta(L)$ as the number of memory banks in use.

<u>Theorem</u> 3.6 The mapping s defined by $s(i,j) = (i - c_L \cdot f(j)) \bmod M/|\gcd(x_2,y_2)| + (j \bmod |\gcd(x_2,y_2)|) \cdot M/|\gcd(x_2 y_2)|$ (where f is a simple function specified below) is a periodic skewing scheme corresponding to L.

<u>Proof</u>.

Think of the lattice as being divided into horizontal strips of width $|\gcd(x_2,y_2)|$ and copies of R at every lattice point. (Thus each horizontal strip is a layer of R-"bricks".) All we need to do is locate a point (i,j) in the proper brick and assign it to the memory bank of the corresponding cell. Clearly (i,j) is in the strip whose "bottom line" has y-coordinate fixed at $j - (j \bmod |\gcd(x_2,y_2)|) = f'(x_2, y_2, j)$, a multiple of $|\gcd(x_2,y_2)|$. To find the lattice points $(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2)$ on this line, note that the equation $\lambda x_2 + \mu y_2 = f'(x_2, y_2, j)$ fixes (λ,μ) to a collection $(f'', f''') + \alpha\,(y_2/|\gcd(x_2,y_2)|, -x_2/|\gcd(x_2,y_2)|)$, where f'' and f''' are a standard solution. (We assume that $y_2 \neq 0$ and omit the special case that $x_2 = 0$, compare the proof of lemma 3.2.) Then the x-coordinates of lattice points on the line are obtained by applying horizontal yardsticks beginning at $f(L,j) = f'' \cdot x_1 + f''' \cdot y_1$. Thus (i,j) lies at position $((i - f(L,j)) \bmod M/|\gcd(x_2,y_2)|,\ j \bmod |\gcd(x_2,y_2)|)$ in its brick, and its assignment to a memory bank easily follows. To complete the argument we show $f(L,j)$ factors. Let $a = x_2/|\gcd(x_2,y_2)|$, $b = y_2/|\gcd(x_2,y_2)|$ and $f(j) = (j - j \bmod |\gcd(x_2,y_2)|)/|\gcd(x_2,y_2)|$, then a and b are relatively prime, $c = b^{-1} \pmod a$ exists, f'' and f''' can be chosen as $f''' = c \cdot f(j)$ and $f'' = (1-bc)/a \cdot f(j)$ and $f(L,j) = \{\frac{1-bc}{a} \cdot x_1 + c \cdot y_1\} \cdot f(j) = c_L \cdot f(j)$. (Note that $c_L = x_1/a + (acy_1 - bcx_1)/a = x_1/a - \Delta(L) \cdot c/x_2$.) A very similar argument holds in case $x_2 = 0$. □

As c_L is a fixed lattice constant, the function s from theorem 3.6 is no more complicated than the s_L determined in section 2. By theorem 2.11 all other periodic skewing schemes induced by L can be obtained from the sample scheme by simple permutation of the memory numbers.

4. Periodic skewing with a minimum number of memory banks.

We shall now deal with the problem of determining a periodic skewing scheme that is valid for a set of templates T_1 to T_k and uses a smallest number of memory banks M. First consider the case of a single template. We shall make use of Shapiro's theorem [7] (see also section 1) that relates (periodic) skewing schemes to (periodic) plane tessellations if the number of memory banks is to be $M = |T|$. In general (viz. if T does not tessellate the plane) the minimum number of memory banks required will be larger. Testing the validity of a periodic skewing scheme is rather easy.

Proposition 4.1 Let s be a periodic skewing scheme, T an arbitrary template. Then s is valid for T if and only if s is conflict-free on a single (freely chosen) instance of T.

Proof.

We only need to show the if-part. Let s be conflict-free on an instance T' of T (located anywhere). Consider any other instance T'' of T and suppose there were two cells $p, q \in T''$ with $s(p) = s(q)$. Then the two corresponding cells of T' must be mapped to equal banks too, by the regularity of s (cf. proposition 2.3). Hence s would not be conflict-free on T', a contradiction. □

(Another simple test can be derived from theorem 3.1.) If s is valid for a template T, then s is valid for all templates equivalent to T modulo L.

To obtain a periodic skewing scheme s valid for T one could enclose T by an N×N rectangle (N sufficiently large) and use a valid periodic scheme for the rectangle. Most likely this will not give a smallest number of memory banks.

Proposition 4.2. Let s be a periodic skewing scheme using M memory banks that is valid for T. Then there is a template S of size M that encloses T such that s is fitting for S (or equivalently, S tessellates the plane according to the underlying lattice).

Proof.

Define S in the following manner. Consider an arbitrary instance of T laid down in the plane (and skewed conflict-free by assumption) and extend it to an instance of S by "appending" M - |T| cells to it, one cell corresponding to every memory bank that did not receive an element from the instance of T. By construction this instance of S is skewed conflict-free and (hence) s is valid for S by proposition 4.1. The skewing scheme is fitting because M = |S|, and by Shapiro's theorem [7] (see section 1) this is equivalent to asserting that S tessellates the plane according to the same underlying lattice. □

Definition. A minimal hull of a template T is a template S of smallest possible size that encloses T and periodically tessellates the plane.

(Minimal hulls are not unique but have the same size.)

Theorem 4.3 The minimum number of memory banks required for a periodic skewing scheme valid for a template T is equal to the size of a minimal hull of T.

We shall see momentarily that theorem 4.3 is effective and that minimal hulls can be constructed in polynomial time. We briefly digress and consider the case of polyominos in more detail.

Theorem 4.4 Every polyomino has a minimal hull that is again a polyomino.

Proof.

Let T be a polyomino, and consider a periodic skewing scheme s that is valid for T and uses the smallest possible number of banks M. The underlying lattice L can be divided into horizontal layers of bricks as in the proof of theorem 3.6, each brick being a copy of the fundamental rectangle R and located at a lattice point. (Thus reductions modulo L can be computed as reductions "modulo R".) Lay down an instance of T and observe the finitely many parts $T^{(1)}$ to $T^{(l)}$, some l, as they appear in different bricks. Each part is a polyomino within its brick, and when reduced modulo L to a single copy of R the parts appear as disjoint "islands" within R. (Disjointness follows because s was conflict-free on T.) Now extend the islands by adding bordering cells such that they remain polyominos but cover the entire R. Unfolding this and extending the polyominal parts $T^{(i)}$ thusly within the bricks where they are located effectively extends T to a larger polyomino of size M that must be a minimal hull by the same argument as in proposition 4.2. □

Theorem 4.4 has an obvious interpretation for periodic plane tessellations by polyominos.

Now consider the problem of effectively computing the smallest number of memory banks required for a periodic skewing scheme valid for a set of templates T_1 to T_k. Suppose all templates can be fitted in an $N \times N$ rectangle. The rectangle is merely used to delimit the size of the templates. Clearly N^2 is an upperbound on the number of memory banks minimally required. The number of periodic skewing schemes to test that use N^2 memory banks or less is unfeasibly large, but fortunately many are equivalent (in the sense of theorem 2.11) and use the same underlying lattice.

Theorem 4.5 The minimum number of memory banks required for a periodic skewing scheme that is valid for T_1 to T_k can be computed in time polynomial in N and k.

Proof.

By theorem 2.7 we must test all lattices L that have $\Delta(L) \leq N^2$. For a given value k of the determinant there are at most $O(k^2)$ possible choices of a single base vector, hence $O(k^4)$ different lattices in all. Thus the number of lattices to inspect is polynomially bounded in N (and the lattice bases can be enumerated within this bound). With every lattice L a simple mapping s_L is associated that can act as a representative of all periodic skewing schemes that correspond to L (see theorems 2.11 and 3.6). By proposition 4.1 the validity of s_L for each of the templates T_1 to T_k can be tested in linear time per template. The smallest value of $\Delta(L)$ that leads to a successful scheme is the minimum number of memory banks we were after. The method requires only polynomial time in N and k. □

Corollary 4.6 For every set of templates T_1 to T_k that fit in an $N \times N$ box one can determine a valid periodic skewing scheme that uses the smallest possible number of memory banks in time polynomial in N and k.

For k = 1 the method of theorem 4.5 is easily extended to an effective algorithm to compute a minimal hull of T (proposition 4.2). We only formulate this for the interesting case of a single polyomino.

Theorem 4.7 For every polyomino P one can determine a smallest enclosing polyomino that periodically tessellates the plane in time polynomial in the size of P.

Proof.

Theorem 4.4 implies that any smallest polyomino that encloses P and tessellates the plane must be a minimal hull. Thus apply the method of theorem 4.5 to find a lattice of minimum determinant that is valid for skewing P, and carry out the construction of an enclosing polyomino as in the proof of theorem 4.4. Since any polyomino P obviously fits in a $|P| \times |P|$ box (thus $N = |P|$ in theorem 4.5), the complete algorithm is easily seen to be polynomially bounded in $|P|$. □

Theorem 4.5 and the underlying method can be extended in several ways For example, consider periodic skewing schemes in which it is permitted that up to r cells of a template are mapped to every memory bank. (Thus instances of T could be retrieved in up to r parallel memory accesses.) Call this "r-fold periodic skewing". Clearly a scheme is valid as an r-fold periodic skewing scheme for T if and only if T is "r-fold equivalent" to a subtemplate of a fundamental template of the underlying lattice, i.e., if it can be reduced to it modulo L such that at most r cells of T map to a same cell of the fundamental template at a time (compare theorem 3.1). With this notion of validity the minimum number of memory banks required for a valid r-fold periodic skewing scheme for templates T_1 to T_k can again be determined in polynomial time (in N, k and r), by the same method of theorem 4.5.

5. References

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