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(revised version)

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Abstract. Uniform network emulations are a method to obtain efficient and structure preserving simulations of large networks on smaller networks. There are two slightly different types of graphs, both realizing Stone's concept of a shuffle-exchange network: the (classical) shuffle-exchange graph and the 4-pin shuffle. We analyze the uniform emulations both types of graphs allow, give a complete characterisation of the possible uniform emulations of the (classical) shuffle-exchange graph with 2^n nodes on itself and on the 4-pin shuffle with 2^{n-1} nodes, and show that the 4-pin shuffle allows uniform emulations in instances where the (classical) shuffle-exchange graph does not.

1. Introduction. Parallel algorithms are normally designed for execution on a suitable network with N processors, with N depending on the size of the problem to be solved. In practice the size of the processor network will be small and fixed whereas the size of the problem will be large and varying. In [3] Fishburn and Finkel introduced the concept of network emulation to obtain an efficient and structure preserving simulation of larger networks. An extensive analysis of this concept was made by Bodlaender and van Leeuwen [1,2]. In this paper we will study the notion for two slightly different types of networks, both realizing Stone's concept of a shuffle exchange network [4].

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Definition. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be networks of processors (graphs). We say that G can be emulated on H if there exists a function $f : V_G \to V_H$ such that for every edge $(g,g') \in E_G$: $f(g) = f(g')$ or $(f(g),f(g')) \in E_H$. The function f is called an emulation function or, in short, an emulation of G on H.

Clearly, emulation between networks is transitive. We shall only be interested in surjective emulations f.

Let f be an emulation of G on H. Any processor $h \in V_H$ must actively emulate the processors $\in f^{-1}(h)$ in G. When $g \in f^{-1}(h)$ communicates information to a neighbouring processor g', then h must communicate the corresponding information "internally", when it emulates g' itself, or to a neighbouring processor $h' = f(g')$ in H otherwise. If all processors act synchronously in G, then the emulaton will be slowed by a factor proportional to $\max_h |f^{-1}(h)|$.

Definition. Let G,H and f be as above. The emulation f is said to be (computationally) uniform if for all $h,h' \in V_H$: $|f^{-1}(h)| = |f^{-1}(h')|$.

Every uniform emulation f has associated with it a fixed constant c, called the computation factor, such that for all $h \in V_H$: $|f^{-1}(h)| = c$. It means that every processor of H emulates the same number of processors of G. Again, uniform emulation between networks is transitive. When G can be uniformly emulated on H, and H can be uniformly emulated on G, then G and H are necessarily isomorphic.

For graphs A,B, let A[B] denote the composition of A and B.

Lemma 1.1. [1] G can be uniformly emulated on H if and only if there exists a graph G' such that G is a spanning subgraph of H[G'].

Stone [4] proposed a network, called the shuffle exchange network, which has been successfully used as the interconnection network underlying a variety of parallel processing algorithms. However, there are two slightly different types of graphs, both realizing Stone's concept of a shuffle exchange network. We will use the terminology of [3] and call these graphs the shuffle-exchange graph* and the 4-pin shuffle, respectively. The nodes of the shuffle-exchange graph and the 4-pin shuffle are given n-bit addresses in the range $o..2^n-1$. In the shuffle-exchange graph there is an edge from node b to node c if and only if b can be "shuffled" (move the leading bit to tail position) or "exchanged" (flip the tail bit) into c. In the 4-pin shuffle there is an edge from node b to node c if and only if c can be reached from b by a shuffle or by a shuffle followed by an exchange. Computations proceed by iterating the networks some n or more times in a synchronized manner. We use the notation SE_n and S_n to denote the shuffle-exchange graph and the 4-pin shuffle, respectively, with 2^n nodes.

In [2] the problem to decide whether a connected graph G can be uniformly emulated on a connected graph H was shown to be NP-complete, even if various additional restrictions are imposed upon G, H and/or the computation factor. For instance the problem is NP-complete, if H is required to be a shuffle-exchange graph or a 4-pin shuffle, and the computation factor c is a fixed constant with $c \geq 15$ or $c \geq 7$, respectively.

An important question is whether (large) networks of some class C can be uniformly emulated by networks of a smaller size within the same class C. Fishburn and Finkel [3] showed that such emulations exist for the following classes of processor networks: the (4-pin) shuffle-exchange network, the grid-connected network, the n-dimensional cube, the plus-minus network, the binary lens, and the cube-connected cycles. (The definitions of these networks can be found in [3].) In [1] a detailed analysis was made of the possible uniform emulations of the (4-pin) shuffle exchange, the n-dimensional cube, the ring and the grid-connected network.

Fishburn and Finkel [3] showed that every shuffle-exchange graph and 4-pin shuffle can be uniformly emulated on a 4-pin shuffle with a

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* Warning: in [1] the term shuffle-exchange graph is used to denote the 4-pin shuffle.

smaller number of nodes. However they did not examine the question whether the shuffle-exchange graph can be uniformly emulated on the shuffle-exchange graph of smaller size. We conjecture that SE_n can be uniformly emulated on SE_k if and only if $k|n$ or $k\leq 2$.

The main results of this paper are the following: we give a complete characterisation of the uniform emulations of SE_n on S_{n-1} and of the graph isomorphisms of SE_n, and show there exist uniform emulations of SE_n on SE_k if $k|n$ or $k\leq 2$, and no uniform emulations of SE_n on SE_{n-1} for $n\geq 4$.

The paper is organized as follows. In section 2 we give some preliminary definitions and results and recall some results from [1]. In section 3 we examine the uniform emulations of SE_n on S_{n-1}. In section 4 we examine the uniform emulations of SE_n on SE_k, for $k\leq n$. In section 5 we briefly discuss the results obtained.

2. Preliminaries.

First we introduce some notations:

$\frac{o}{1}$	:	a bit that can be o or 1
$\bar{\alpha}$	:	the complement of bit α ($\bar{o} = 1$, $\bar{1} = o$)
b	:	the n-bit address $b_1 \ldots b_n$
$\bar{b}$	:	the address one obtains by complementing every bit of b ($\overline{b_1 \ldots b_n} = \bar{b}_1 \ldots \bar{b}_n$)
$b\|_i$	:	$b_1 \ldots b_i$ (truncation after the i^{th} bit)
$_i\|b$	:	$b_i \ldots b_n$ (truncation "before" the i^{th} bit)
$(b)_i$	:	b_i (the i^{th} bit)

For functions f defined on n-bit numbers b we use:

$f_i(b) : (f(b))_i$ (projection on the i^{th} bit).

We use b,c,.. to denote full addresses and x,y,.. to denote segments of bits. Individual bits are denoted α, β.. .

Definition. The shuffle-exchange network is the graph $SE_n = (V_n, \tilde{E}_n)$ with $V_n = \{b_1 \ldots b_n \mid \forall 1\leq i\leq n\ b_i = \frac{o}{1}\}$ and $\tilde{E}_n = \{(b,c) \mid b,c \in V_n$ and $((\forall 2\leq i\leq n\ b_i = c_{i-1} \wedge b_1 = c_n)$ or $(\forall 1\leq i\leq n-1\ b_i = c_i \wedge b_n = \bar{c}_n))\}$. The 4-pin shuffle network is the graph $S_n = (V_n, E_n)$ with $E_n = \{ (b,c) \mid b,c \in V_n$ and $\forall 2 \le i \le n\ b_i = c_{i-1} \}$.

It follows that in SE_n a node $b_1 \ldots b_n$ is connected to $b_2 \ldots b_n b_1$ and $b_1 \ldots b_{n-1} \bar{b}_n$, and in S_n it is connected to $b_2 \ldots b_n o$ and $b_2 \ldots b_n 1$. The fact that S_n and SE_n can be (uniformly) emulated on S_{n-1} and, hence, on every S_k ($k<n$) derives from the following observation, using lemma 1.1. (Compare [3], theorem 1 and 2). Let K_2 denote the complete graph on two nodes.

<u>Lemma 2.1</u>. (a) [1] S_n is a spanning subgraph of $S_{n-1}[\bar{K}_2]$, for $n \ge 1$.
(b) SE_n is a spanning subgraph of $S_{n-1}[\bar{K}_2]$ for $n \ge 1$.

<u>Proof</u>.

Consider the mappings $h : S_n \to S_{n-1}[\bar{K}_2]$ and $h' : SE_n \to S_{n-1}[\bar{K}_2]$ defined by $h(b_1 .. b_n) = h'(b_1 .. b_n) = \langle b_1 .. b_{n-1}, b_n \rangle$. h, h' are clearly 1-1 and onto the set of nodes. One easily shows that h, h' are embeddings of S_n, SE_n respectively. □

From [1] we recall the following facts and definitions about emulations of the 4-pin shuffle.

<u>Lemma 2.2</u>. [1](a) f is an emulation of S_n on S_{n-k} if and only if for all $x \in (\frac{o}{1})^{n-1}$, $y \in (\frac{o}{1})^{n-k-1}$ and $\alpha, \beta \in (\frac{o}{1})$: if $f(\alpha x) = \beta y$ then $(f(xo) = \beta y$ or $f(xo) = y\frac{o}{1})$ and $(f(x1) = \beta y$ or $f(x1) = y\frac{o}{1})$.
(b) f is emulation of S_n on S_{n-k} if and only if for all $x \in (\frac{o}{1})^{n-1}$, $y \in (\frac{o}{1})^{n-k-1}$ and $\alpha, \beta \in (\frac{o}{1})$: if $f(x\alpha) = y\beta$ then $(f(ox) = y\beta$ or $f(ox) = \frac{o}{1}y)$ and $(f(1x) = y\beta$ or $f(1x) = \frac{o}{1}y)$.

<u>Definition</u>. [1] A mapping $g : S_n \to S_{n-k}$ is called <u>step-simulating</u> (or a step-simulation of S_n on S_{n-k}) if and only if for all $x \in (\frac{o}{1})^{n-1}$, $y \in (\frac{o}{1})^{n-k-1}$ and $\alpha, \beta \in (\frac{o}{1})$, if $f(\alpha x) = \beta y$ then $f(xo) = y\frac{o}{1}$ and $f(x1) = y\frac{o}{1}$.

<u>Lemma 2.3</u>. [1] Every step-simulation of S_n on S_{n-k} is an emulation.

Lemma 2.4. [1] A mapping $g : S_n \to S_{n-k}$ is step-simulating if and only if for all $x \in (\frac{o}{1})^{n-1}$, $y \in (\frac{o}{1})^{n-k-1}$ and $\beta \in (\frac{o}{1})$: if $f(x\alpha) = y\beta$ then $f(ox) = \frac{o}{1}y$ and $f(1x) = \frac{o}{1}y$.

Theorem 2.5. [1] Every uniform emulation of S_n on S_{n-1} is step-simulating.

Proposition 2.6. [1] For every n, S_n admits only the following two graph isomorphisms : $g : g(x) = x$,
and $\overline{g} : \overline{g}(x) = \overline{x}$.

The following fact follows directly from the definitions.

Lemma 2.7. (a) f is an emulation of SE_n on SE_{n-1} if and only if for all $b \in (\frac{o}{1})^n$, $y \in (\frac{o}{1})^{n-k}$: if $f(b) = y$ then $\{f(b_2 \ldots b_n b_1), f(b_1 \ldots b_{n-1}\overline{b}_n)\} \subseteq \{y, y_1 \ldots y_{n-1}\overline{y}_n, y_2 \ldots y_n y_1\}$.

(b) f is an emulation of SE_n on SE_{n-1} if and only if for all $b \in (\frac{o}{1})^n$, $y \in (\frac{o}{1})^{n-k}$: if $f(b) = y$ then $\{ f(b_n b_1 \ldots b_{n-1}), f(b_1 \ldots b_{n-1}\overline{b}_n)\} \subseteq \{y, y_1 \ldots y_{n-1}\overline{y}_n, y_n y_1 \ldots y_{n-1}\}$. □

For a mapping f, define its companion $\overline{f}$ by $\overline{f}_i(b) = \overline{f_i(b)}$.

Lemma 2.8. If f is an emulation of S_n on S_k (or of S_n on SE_k, SE_n on S_k, SE_n on SE_k), then so if $\overline{f}$.

In this paper we will use the notation (o1)* and (1o)* to denote a string consisting of a number of repetitions of o1 and 1o, respectively. With (o1)*[o] (resp. (1o)[1]) we denote a string consisting of alternating o's and 1's, starting with a o (resp. 1). The exact length of these strings will not be explicitly specified, but will always be clear from the context. For $b,c \in (\frac{o}{1})^n$ we denote by d(b,c) the shortest distance of b to c in the graph SE_n.

3. Uniform emulations of SE_n on S_{n-1}. In this section we examine the uniform emulations of SE_n on S_{n-1}. (Compare lemma 2.1b).

<u>Lemma 3.1</u>. Let f be a uniform emulation of SE_n on S_{n-1} with for all x $\in$ $(\frac{o}{1})^{n-1}$: f(xo) = f(x1). Then

(a) for all b $\in$ $(\frac{o}{1})^n$ $f(b_1 \ldots b_n) = b_1 \ldots b_{n-1}$, or

(b) for all b $\in$ $(\frac{o}{1})^n$ $f(b_1 \ldots b_n) = \overline{b}_1 \ldots \overline{b}_{n-1}$.

<u>Proof</u>.

Define g : $S_{n-1} \rightarrow S_{n-1}$ by g(x) = f(xo). g is uniform, (i.e. 1-1), because if there exist $x_1 \neq x_2$ with $g(x_1) = g(x_2)$, then $f(x_1o) = f(x_11) = f(x_2o) = f(x_21)$ and f is not uniform. Also g is an emulation of S_{n-1} on S_{n-1}. Suppose g(x) = y. Then f(xo) = y and f(x1) = y, hence f(ox) = y or f(ox) = $\frac{o}{1}y_1 \ldots y_{n-2}$, and f(1x) = y or f(1x) = $\frac{o}{1}y_1 \ldots y_{n-2}$, so $g(ox_1 \ldots x_{n-2}) = f(ox_1 \ldots x_{n-2}o) \in \{y, \frac{o}{1}y_1 \ldots y_{n-2}\}$ and $g(1x_1 \ldots x_{n-2}) = f(1x_1 \ldots x_{n-2}o) \in \{y, \frac{o}{1}y_1 \ldots y_{n-2}\}$. A uniform emulation of S_{n-1} on S_{n-1} necessarily is a graph isomorphism. We now use proposition 2.6 : either for all x $\in$ $(\frac{o}{1})^{n-1}$ g(x) = x or for all x $\in$ $(\frac{o}{1})^{n-1}$ g(x) = $\overline{x}$. If the former is the case, then for all b $\in$ $(\frac{o}{1})^n$ f(b) = $b_1 \ldots b_{n-1}$, and if the latter is the case, then for all b $\in$ $(\frac{o}{1})^n$ f(b) = $\overline{b}_1 \ldots \overline{b_{n-1}}$. □

<u>Lemma 3.2</u>. Let f be an emulation of SE_n on S_{n-1}. Then for all x $\in$ $(\frac{o}{1})^{n-1}$: f(xo) = f(x1) or { f(xo),f(x1) } = { (o1)*[o],(1o)*[1] }.

<u>Proof</u>.

If f(xo) $\neq$ f(x1) then f(xo) and f(x1) must be adjacent nodes, connected by edges in both directions. The only way to realize this in S_{n-1} is to map the nodes f(xo),f(x1) onto the nodes of the set { (o1)*[o],(1o)*[1] }. □

<u>Lemma 3.3</u>. Let f be a uniform emulation of SE_n on S_{n-1}, and let n be odd. Then for all x $\in$ $(\frac{o}{1})^{n-1}$: f(xo) = f(x1).

<u>Proof</u>.

Suppose the lemma does not hold. Then, by lemma 3.2, there is a x $\in$ $(\frac{o}{1})^{n-1}$ such that { f(xo),f(x1) } = { (o1)*,(1o)* }.

Now suppose f((o1)*o) $\notin$ { (o1)*,(1o)* }. Then by lemma 3.2 f((o1)*1) = f((o1)*o). If f((1o)*o) = f((1o)*1) then f((o1)*o) can reach itself by performing two shuffle-exchange steps note that (f((o1)*o),f((1o)*o)) $\in$

S_{n-1} and $(f((1o)^*1), f((o1)^*1)) \in S_{n-1}$, due to the uniformity of f, and $f((1o)^*o) = f((1o)^*1)$ and $f((o1)^*1)=f((o1)^*o))$), which forces $f((o1)^*o) \in \{o^{n-1}, 1^{n-1}, (o1)^*, (1o)^*\}$. If $f((o1)^*o) = \alpha^{n-1} = f((o1)^*1)$, then $f(o(o1)^*) = f(1(o1)^*) = \bar{\alpha}\alpha^{n-2}$ (use uniformity of f). If $n \neq 3$, then, by lemma 3.2 $f(o(o1)^*) = f(1(o1)^*) = f(o(o1)^*oo) = f(1(o1)^*oo)$, which contradicts the uniformity of f. Is $n = 3$ then one obtains a contradiction by successively deriving that $f(oo1) = f(1o1) = \bar{\alpha}\alpha$, so $f(ooo) = f(1oo) = \alpha\bar{\alpha}$, and $f(11o) = \alpha\bar{\alpha}$ (use uniformity of f). So $f((o1)^*o) \in \{(o1)^*,(1o)^*\}$. Contradiction. If $f((1o)^*o) \neq f((1o)^*1)$, then, because of lemma 3.2, $f((1o)^*1) \in \{(1o)^*,(o1)^*\}$. Without loss of generality one may suppose $f((1o)^*1) = (1o)^*$, $f((1o)^*o) = (o1)^*$. Then $f((o1)^*o) = o(o1)^*$, $f((o1)^*1) = o(o1)^*$ and $f(1(o1)^*) = \frac{o}{1}o(o1)^*o$. Contradiction. So $f((o1)^*o) \in \{(o1)^*,(1o)^*\}$.

In the same way one can prove $f((1o)^*1) \in \{(o1)^*,(1o)^*\}$. We have now : $f^{-1}(\{(o1)^*,(1o)^*\}) = \{(o1)^*o,(o1)^*1,(1o)^*o,(1o)^*1\}$. From the assumption there is a $x \in (\frac{o}{1})^{n-1}$ with $\{f(xo),f(x1)\} = \{(o1)^*,(1o)^*\}$ now follows that one of the following cases must hold : I. $f((o1)^*o) = (o1)^* \wedge f((o1)^*1) = (1o)^*$; II. $f((o1)^*o) = (1o)^* \wedge f((o1)^*1) = (o1)^*$; III. $f((1o)^*1) = (o1)^* \wedge f((1o)^*o) = (1o)^*$; IV $f((1o)^*1) = (1o)^* \wedge f((1o)^*o) = (o1)^*$. We will only handle case I; the other cases are similar.

So suppose $f((o1)^*o) = (o1)^*$ and $f((o1)^*1) = (1o)^*$. With downward induction on k we prove : for all $x \in (\frac{o}{1})^{n-2k-2}$ there is a $y \in (\frac{o}{1})^{n-2k-3}$ such that $f((o1)^k oox) = (o1)^k ooy$. First we prove this fact for $n-2k-2=1$ i.e. $k=\frac{1}{2}(n-3)$. $f((o1)^*o) = (o1)^* \Rightarrow f(o(o1)^*) = o(o1)^*o \Rightarrow f(o(o1)^*oo) = o(o1)^*o \Rightarrow f((o1)^*oo\frac{o}{1}) = (o1)^*oo$ (use lemma 3.2 and the uniformity of f). Now let the proposition be true for a certain k. $f((o1)^k oox) = (o1)^k ooy$, so $f(1(o1)^{k-1}oox\frac{o}{1}) = (1o)^{k-1}1ooy\frac{o}{1}$. (Notice that $f((o1)^k oox) = f((o1)^k oox_1 \ldots x_{n-2k-3}\bar{x}_{n-2k-2})$, and due to the uniformity of f one gets $f(1(o1)^{k-1}oox\frac{o}{1}) \neq f((o1)^k oox)$. Using basically the same argument one proves $f((o1)^{k-1}oox\frac{oo}{11}) = (o1)^{k-1}ooy\frac{oo}{11}$, thus completing the inductional proof of the proposition.

In particular we now have $f(\{oox \mid x \in (\frac{o}{1})^{n-2}\}) = \{ooy \mid y \in (\frac{o}{1})^{n-3}\}$. Now $f((o1)^*1) = (1o)^* \Rightarrow f((1o)^*11o) = (o1)^*oo$. With downward induction on k we prove $f((1o)^k 11o(1o)^{(n-2k-3)/2}) = (o1)^k ooy$, for some y

$\in (\frac{o}{1})^{n-2k-3}$. We already proved this to be true for k=(n-2k-3)/2. Now let it be true for certain k. Notice $f((1o)^k 11o(1o)^*) = f((1o)^k 11o(1o)^* 11)$, so $f(o(1o)^{k-1} 11o(1o)^* 1) \neq f((1o)^k 11o(1o)^*)$, so $f(o(1o)^* 11o(1o)^* 1) = 1(o1)^{k-1} ooy'$ for some $y' \in (\frac{o}{1})^{n-2k-2}$. Using the same type of argument one proves $f((1o)^{k-1} 11o(1o)^*) = (o1)^{k-1} ooy''$ for some $y'' \in (\frac{o}{1})^{n-2k-1}$.

This shows that $f^{-1}(\ \{ooy \mid y \in (\frac{o}{1})^{n-3}\}\ ) \supseteq \{oox \mid x \in (\frac{o}{1})^{n-2}\} \cup \{11o(1o)^*\}$, which contradicts the uniformity of f. □

<u>Lemma 3.4</u>. Let f be a uniform emulation of SE_n on S_{n-1}. Let n be even. Let $\tilde{f} : SE_n \to S_{n-1}$ be defined by $\tilde{f}((o1)^*) = f((1o)^*)$, $\tilde{f}((1o)^*) = f((o1)^*)$ and $\tilde{f}(b) = f(b)$, if $b \notin \{\ (o1)^*, (1o)^*\ \}$. Then either for all $x \in (\frac{o}{1})^{n-1}$: $f(xo) = f(x1)$ or for all $x \in (\frac{o}{1})^{n-1}$: $\tilde{f}(xo) = \tilde{f}(x1)$. In the latter case $\tilde{f}$ is a uniform emulation function of SE_n on S_{n-1}.

<u>Proof</u>.

Note that $f((o1)^*)$ and $f((1o)^*)$ must be adjacent with edges in both directions, or equal. If $f((o1)^*) = f((1o)^*)$, then $f((o1)^*oo) \neq f((o1)^*)$ by uniformity, hence $\{\ f((o1)^*oo), f((o1)^*)\ \} = \{\ (o1)^*o, (1o)^*1\ \}$, and $f((1o)^*11) \neq f((1o)^*)$, hence $\{\ f((1o)^*11), f((1o)^*)\ \} = \{(o1)^*o, (1o)^*1\}$. If $f((o1)^*) \neq f((1o)^*)$, then $f((o1)^*)$ and $f((1o)^*)$ must be mutually adjacent to each other, so $\{\ f((o1)^*), f((1o)^*)\ \} = \{\ (o1)^*o, (1o)^*1\ \}$. In both cases one has $f^{-1}\{(o1)^*o, (1o)^*1\} = \{(o1)^*, (1o)^*, (o1)^*oo, (1o)^*11\ \}$.

Now suppose there is a $x \in (\frac{o}{1})^{n-1}$ such that $f(xo) \neq f(x1)$. Then one of the following four cases must hold : I. $f((o1)^*) = (o1)^*$, $f((o1)^*oo) = (1o)^*1$, II. $f((o1)^*) = (1o)^*1$, $f((o1)^*oo) = (o1)^*o$, III. $f((1o)^*) = (o1)^*o$, $f((1o)^*11) = (1o)^*1$, IV. $f((1o)^*) = (1o)^*1$, $f((1o)11) = (o1)^*o$. We will only examine case I; the other cases are similar.

So suppose $f((o1)^*) = (o1)^*o$, and $f((o1)^*oo) = (1o)^*1$. Note that, due to the uniformity of f, $f(o(o1)^*o) \neq (o1)^*o$ and $f(o(o1)^*o) \neq (1o)^*o$, so $f(o(o1)^*o) = 1(1o)^*$. Now $f(o(o1)^*1) = 1(1o)^*$, $f((o1)^*1o) = (1o)^*o$ and $f((o1)^*11) = (1o)^*o$. So $f((1o)^*11) = (o1)^*o$, and $f((1o)^*) = (1o)^*1$. Notice that $(1o)^*$ is only adjacent to $(o1)^*$ and $(1o)^*11$, and $(o1)^*$ is only adjacent to $(1o)^*$ and $(o1)^*oo$ in SE_n; each of these nodes is mapped to the set $\{\ (o1)^*o, (1o)^*1\ \}$, so $\tilde{f}$ is also an emulation function. It is easy to verify (using lemma 3.2), that for all $x \in (\frac{o}{1})^{n-1}$ $\tilde{f}(xo) = \tilde{f}(x1)$, and that $\tilde{f}$ is uniform. □

Theorem 3.5. (Characterisation Theorem).

(a) If n is odd, then every uniform emulation of SE_n on S_{n-1} is one of the following list:

$f \;:\; f(b_1 \ldots b_n) = b_1 \ldots b_{n-1}$

$\bar{f} \;:\; \bar{f}(b_1 \ldots b_n) = \bar{b}_1 \ldots \bar{b}_{n-1}$

(b) If n is even, then every uniform emulation of SE_n on S_{n-1} is one of the following list :

$f_1 \;:\; f_1(b_1 \ldots b_n) = b_1 \ldots b_{n-1}$

$\bar{f}_1 \;:\; \bar{f}_1(b_1 \ldots b_n) = \bar{b}_1 \ldots \bar{b}_{n-1}$

$f_2 \;:\; f_2(b_1 \ldots b_n) = b_1 \ldots b_{n-1}$ if $b \in \{ (o1)^*, (1o)^* \}$

$\quad f_2((o1)^*) = (1o)^*1$

$\quad f_2((o1)^*) = (o1)^*o$

$\bar{f}_2 \;:\; \bar{f}_2(b_1 \ldots b_n) = \bar{b}_1 \ldots \bar{b}_{n-1}$ if $b \notin \{ (o1)^*, (o1)^* \}$

$\quad \bar{f}_2((o1)^*) = (o1)^*o$

$\quad \bar{f}_2((1o)^*) = (1o)^*1$.

Proof.

(a) Use lemma 3.1 and 3.3.

(b) Use lemma 3.4. If for all $x \in (\frac{o}{1})^{n-1}$ $f(xo) = f(x1)$, then f is of the form f_1 or $\bar{f}_1$ (use lemma 3.1). If for all $x \in (\frac{o}{1})^{n-1}$ $\tilde{f}(xo) = \tilde{f}(x1)$, then $\tilde{f}$ is of the form f_1 or $\bar{f}_1$, so f is of the form f_2 or $\bar{f}_2$. □

4. Uniform emulations of SE_n on SE_k $(k \leq n)$.

Proposition 4.1. For $n \geq 1$, SE_n admits precisely 2 graph isomorphisms.

Proof.

If n=1,2, verify directly.

Let $n \geq 3$. Clearly g, defined by $g(b) = b$ and $\bar{g}$, defined by $\bar{g}(b) = \bar{b}$ are isomorphisms. Suppose there is yet another isomorphism of SE_n, $\tilde{g}$. Define $h(b) = \tilde{g}(b)|_{n-1}$. h is a uniform emulation of SE_n on S_{n-1}. We use theorem 3.5 and consider 4 cases :

Case I : for all $b \in \binom{o}{1}^n$: $h(b) = b_1 \ldots b_n$.

Because $\tilde{g} \neq g$, there must be a b with $\tilde{g}(b) = b_1 \ldots b_{n-1}\bar{b}_n$. Now $\tilde{g}(b_n b_1 \ldots b_n) = b_n b_1 \ldots b_{n-2} \frac{o}{1}$ must be adjacent to $b_1 \ldots b_{n-1}\bar{b}_n$ in SE_n. So $b_1 \ldots b_n = \alpha^n$, for some $\alpha \in \binom{o}{1}$, and $g(\alpha^n) = \alpha^{n-1}\bar{\alpha}$. Now note that the outdegree of α^n is 1 and the outdegree of $\alpha^{n-1}\bar{\alpha}$ is 2 in SE_n, so $\tilde{g}$ is not a graph isomorphism. Contradiction.

Case II : for all $b \in \binom{o}{1}^n$: $h(b) = \bar{b}_1 \ldots \bar{b}_{n-1}$.

This case can be handled in the same way as case 1.

Case III : n is even, $h(b) = b_1 \ldots b_{n-1}$ if $b \notin \{(o1)^*, (o1)^*\}$, $h((o1)^*) = (1o)^*1$, $h((1o)^*) = (o1)^*o$.

If there is a b with $\tilde{g}(b) = b_1 \ldots b_{n-1}\bar{b}_n$, then we reach a contradiction in the same way as in case 1. So we may suppose $\tilde{g}(b) = b$ for all $b \notin \{ (o1)^*, (1o)^* \}$. So $\tilde{g}((o1)^*) = (1o)^*$, $\tilde{g}((1o)^*) = (o1)^*$, and now $\tilde{g}((o1)^*)$ is not adjacent to $\tilde{g}((o1)^*oo)$. Contradiction.

Case IV : n is even, $h(b) = \bar{b}_1 \ldots \bar{b}_{n-1}$, if $b \notin \{ (o1)^*, (1o)^* \}$, $h((o1)^*) = (o1)^*o$, $h((1o)^*) = (1o)^*1$.

This case can be handled in the same way as case 3. It follows that there are no other graph isomorphism of SE_n but g and $\bar{g}$. □

<u>Theorem 4.2</u>. Let $k,n \geq 1$, $k|n$. Then the function f, defined by $f_i(b_1 \ldots b_n) = \left(\sum_{j=o}^{n/k-1} b_{j.k+i} \right) \bmod 2$ $(i=1, \ldots, k)$ is a uniform emulation of SE_n on SE_k.

<u>Proof</u>.

By verifying that $f_i(b_1 \ldots b_n) = f_i(b_1 \ldots b_{n-1}\bar{b}_n)$ for $i \neq k$ and that $f_i(b_1 \ldots b_n) = f_{i+1}(b_2 \ldots b_n b_1)$ for $i \neq k$ and $f_k(b_1 \ldots b_n) = f_1(b_2 \ldots b_n b_1)$ one proves that f is an emulation of SE^n on SE^k. If x and $y \in SE_k$ differ only in the i'th bit position, then $f^{-1}(y) = (b_1 \ldots b_{i-1}\bar{b}_i b_{i+1} \ldots b_n \mid b \in f^{-1}(x) \}$, so $|f^{-1}(x)| = |f^{-1}(y)|$. With induction one now can prove that for all $x,y \in \binom{o}{1}^k$ $|f^{-1}(x)| = |f^{-1}(y)|$, so f is uniform. □

<u>Proposition 4.3</u>. Let $n \geq 2$. Then SE_n can be uniformly emulated on SE_2.

Proof.

If n is even, then use theorem 4.2.

Let n be odd. The graph SE_2 is shown in fig. 4.1.

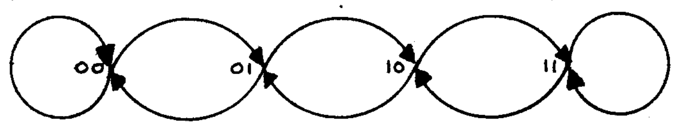

fig. 4.1. SE_2

Let f be a mapping of SE_n on SE_2. We will show that f can be chosen such that f is a uniform emulation. We first want f to fulfill the following conditions :

$$f_1(b_1 \ldots b_n) = \begin{cases} o \text{ if } \sum_{i=1}^{n} b_i < \frac{1}{2}n \\ \\ 1 \text{ if } \sum_{i=1}^{n} b_i > \frac{1}{2}n \end{cases}$$

f maps one half of the nodes of SE_n on {oo,o1} and the other half on {10,11}. We can choose f in such a way that every string with $\sum_{i=1}^{n} b_i = \lfloor \frac{1}{2}n \rfloor$ and $b_n = o$ is mapped to o1 and every string with $\sum_{i=1}^{n} b_i = \lceil \frac{1}{2}n \rceil$ and $b_n = 1$ is mapped to 1o and f is uniform. We show that f is an emulation of SE_n on SE_2: If b,b' are adjacent in SE_n, then there are four cases : I. $\sum_{i=1}^{n} b_i \leq \lfloor \frac{1}{2}n \rfloor$ and $\sum_{i=1}^{n} b_i' \leq \lfloor \frac{1}{2}n \rfloor$, then b, b' are mapped to nodes in the set {oo,o1}; II. $\sum_{i=1}^{n} b_i \leq \lfloor \frac{1}{2}n \rfloor$ and $\sum_{i=1}^{n} b_i' \geq \lceil \frac{1}{2}n \rceil$, then necessarily b=xo and b'=x1 for some $x \in (\frac{o}{1})^{n-1}$ and $\sum_{i=1}^{n-1} x_i = \lfloor \frac{1}{2}n \rfloor$, so b is mapped upon o1 and b' is mapped upon 1o; III. $\sum_{i=1}^{n} b_i \geq \lceil \frac{1}{2}n \rceil$, and $\sum_{i=1}^{n} b_i' \leq \lfloor \frac{1}{2}n \rfloor$; this case is similar to case II; IV. $\sum_{i=1}^{n} b_i \geq \lceil \frac{1}{2}n \rceil$ and $\sum_{i=1}^{n} b_i' \geq \lceil \frac{1}{2}n \rceil$, then $\{f(b_i), f(b_i')\} \subseteq \{10,11\}$. So f is a uniform emulation of SE_n on SE_2. □

For choices of n and k other than $k|n$ and $k\leq 2$ there are presently no uniform emulation functions of SE_n on SE_k known. We conjecture that for $n,k>2$ with $n>k$, $k\nmid n$, no such function exists. The following results show that the conjecture is at least plausible. We show that from a uniform emulation of SE_n on SE_k with $n>k>2$ and $k\nmid n$, an emulation function of S_{n-1} on S_{k-1} can be derived, that is uniform, but not step-simulating. Presently no functions of this sort are known. We also show that for $n\geq 4$, there indeed are no uniform emulations of SE_n on SE_{n-1}.

<u>Lemma 4.4</u>. Let f be an emulation function $SE_n \to SE_k$ $(n,k\geq 1)$.
(a) If k is odd, then for all $x \in (\frac{o}{1})^{n-1}: f(xo)|_{k-1} = f(x1)|_{k-1}$.
(b) If k is even, then for all $x \in (\frac{o}{1})^{n-1}$: $f(xo)|_{k-1} = f(x1)|_{k-1}$ or $\{f(xo),f(x1)\} = \{ (o1)^*,(1o)^* \}$.

<u>Proof</u>.

f(xo) and f(x1) must be adjacent or equal. □

<u>Lemma 4.5</u>. Let f be an emulation of SE_n on SE_k $(n,k\geq 2)$. Let ψ be a function $(\frac{o}{1})^{n-1} \to (\frac{o}{1})$. Let $g : S_{n-1} \to S_{k-1}$ be defined by $g(x)= f(x\,\psi(x))|_{k-1}$ (for all $x \in (\frac{o}{1})^{n-1}$). Then g emulates S_{n-1} on S_{k-1}.

<u>Proof</u>.

We first show that $g(\psi(x)(x|_{n-2}))$ is adjacent to $g(x)$. Let $g(x) = y$. Then $f(x\psi(x)) = y\frac{o}{1}$, so $f(\psi(x)x) = y\frac{o}{1}$ or $f(\psi(x)x) = \frac{o}{1}y$. If $x_{n-1} = \psi(\psi(x)(x|_{n-2}))$, then $g(\psi(x)(x|_{n-2})) = y$ or $g(\psi(x)(x|_{n-2})) = \frac{o}{1}(y|_{k-1})$, and adjacency is proved. If $x_{n-1} = \overline{\psi(\psi(x)(x|_{n-2}))}$ then either $f(\psi(x)(x|_{n-2})o)|_{n-1} = f(\psi(x)(x|_{n-2})1)|_{n-1}$ or $\{ f(\psi(x)(x|_{n-2})o), f(\psi(x)(x|_{n-2})1)\} = \{(o1)^*,(1o)^*\}$. (In the latter case k must be even.) In the former case $g(\psi(x)(x|_{n-2})) \in \{y, \frac{o}{1}(y|_{k-1})\}$, and adjacency follows. In the latter case $g(\psi(x)(x|_{n-2})) \in \{(o1)^*o, (1o)^*1\}$, and $f(\psi(x)x) \in \{(o1)^*,(1o)^* \}$, hence $f(x\psi(x)) \in \{(o1)^*, (1o)^*, (o1)^*oo, (1o)^*11\}$ and $g(x) \in \{(o1)^*o, (1o)^*1\}$, and adjacency follows again.

Next we verify that $g(\overline{\psi(x)}(x|_{n-2}))$ is adjacent (or equal) to $g(x)$. Let $g(x) = y$. Then $f(x\psi(x)) = y\frac{o}{1}$. If $f(x\psi(x))|_{k-1} = f(x\overline{\psi(x)})|_{k-1}$, then $f(\overline{\psi(x)}x) = y\frac{o}{1}$ or $f(\overline{\psi(x)}x) = \frac{o}{1}y$ and the argument can proceed as in the first part of the proof. So suppose $\{f(x\psi(x)), f(x\overline{\psi(x)})\} = \{(o1)^*,(1o)^*\}$. Now $g(x) \in \{(o1)^*o,(1o)^*1\}$ and $f(\overline{\psi(x)}x) \in \{(o1)^*,(1o)^*, (o1)^*oo, (1o)^*11\}$. Hence $f(\overline{\psi(x)}(x|_{n-2}) \psi(\overline{\psi(x)}(x|_{n-2}))) \in \{(o1)^*,(1o)^*, (o1)^*oo, (1o)^*11\}$ and $g(\overline{\psi(x)}(x|_{n-2})) \in \{(o1)^*o, (1o)^*1\}$. Adjacency now follows again. □

<u>Lemma 4.6</u>. Let f be a uniform emulation of SE_n on SE_k $(n>k)$. There exists a $\psi : (\frac{o}{1})^{n-1} \to (\frac{o}{1})$, such that the function g, defined by $g(x) = f(x\psi(x))|_{k-1}$ is a uniform emulation of S_{n-1} on S_{k-1}.

<u>Proof</u>.

Lemma 4.5 shows that g is an emulation for every choice of ψ. So we have to show that ψ can be chosen such that g is uniform. If k is odd, then any $\psi : (\frac{o}{1})^{n-1} \to (\frac{o}{1})$ will do. Suppose g is not uniform. Then there is an $x \in (\frac{o}{1})^{k-1}$ with $|g^{-1}(x)| \neq 2^{n-k}$. One has $g(b) = x \Leftrightarrow f(b\psi(b)) \in \{xo,x1\} \Leftrightarrow f(b\overline{\psi(b)}) \in \{xo,x1\}$ (use lemma 4.4.), so $|f^{-1}(\{xo,x1\})| = 2.|g^{-1}(x)| \neq 2^{n-k+1}$. So f is not uniform. Contradiction.

Now suppose k is even. For every $x \in (\frac{o}{1})^{n-1}$ with $f(xo)|_{k-1} = f(x1)|_{k-1}$ we can choose $\psi(x)$ arbitrarily. Let $X = \{x \in (\frac{o}{1})^{n-1} | f(xo)|_{k-1} \neq f(x1)|_{k-1}\} = \{x \in (\frac{o}{1})^{n-1} | \{f(xo),f(x1)\} = \{(o1)^*,(1o)^*\}\}$. There must be an even number of x with $\{f(xo),f(x1)\} = \{(o1)^*,(o1)^*oo\}$, else there would be an odd number of nodes mapped upon $(o1)^*oo$. Likewise there must be an even number of x with $\{f(xo), f(x1)\} = \{(1o)^*,(1o)^*11\}$. Hence there must be an even number of x with $\{f(xo),f(x1)\} = \{(o1)^*,(1o)^*\}$. $|X|$ is even. Choose $X_1,X_2 \subseteq X$, such that $|X_1|=|X_2|$, $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$.) For $x \in X_1$ we choose $\psi(x)$, such that $f(x\psi(x)) = (o1)^*$. (This is possible, because $\{f(xo),f(x1)\} = \{(o1)^*,(1o)^*\}$. For $x \in X_2$ we choose $\psi(x)$, such that $f(x\psi(x)) = (1o)^*$. Now g is uniform. For $y \notin \{(o1)^*o, (1o)^*1\}$, $g(b) = y \Leftrightarrow \{f(bo),f(b1)\} \subseteq \{yo,y1\}$, so $2^{n-k+1} = |f^{-1}(\{yo,y1\})| = 2|g^{-1}(y)|$, and $|g^{-1}(y)| = 2^{n-k}$. If $g(b) = (o1)^*o$, then either $b \in \{x| \{f(xo),f(x1)\} \subseteq \{(o1)^*, (o1)^*oo\}\} = Z_1$ or $b \in X_1$. If $g(b) = (1o)^*1$, then either $b \in \{x|\{f(xo), f(x1)\} \subseteq \{(1o)^*, (1o)^*11\}\} = Z_2$ or $b \in X_2$. Finally notice that $|Z_1| = |Z_2|$ and $|X_1|=|X_2|$. Hence $|g^{-1}(\{(o1)^*o\})| = |g^{-1}(\{(10)^*1\})|$, which shows that $|g^{-1}(y)| = 2^{n-k}$, for all $y \in (\frac{o}{1})^{k-1}$. □

<u>Lemma 4.7</u>. Let f be an emulation of SE_n on SE_k and let f be surjective*. Let $n \geq k \geq 3$ and $k \nmid n$. Let ψ be a function $(\frac{o}{1})^{n-1} \to (\frac{o}{1})$, and let g be the emulation of S_{n-1} on S_{k-1} defined by $g(x) = f(x\psi(x))|_{k-1}$. Then g is not step-simulating.

<u>Proof</u>.

Suppose g is step-simulating. We use the notation $R^l(b)$ to denote the string obtained by rotating b l bits to the left, i.e. $R^l(b_1 \ldots b_n) = b_{l+1} \ldots b_n b_1 \ldots b_l$. Let $f(b) = o^{k-2}11$ for certain $b \in (\frac{o}{1})^n$.

With induction we prove : for all l $o \leq l \leq n$ $f(R^l(b)) = R^l(f(b))$. For $l = o$ this is trivially true. Suppose $f(R^l(b)) = R^l(f(b))$ for certain l. Then $f(R^l(b))|_{n-1}\psi(R^l(b))) = R^l(f(b))|_{k-1}\frac{o}{1}$ (notice that $R^l(f(b)) \in \{(o1)^*, (1o)^*\}$), hence $g(R^l(b)|_{n-1}) = R^l(f(b))|_{k-1}$. g is step-simulating, so $g({}_2|R^l(b)) = {}_2|(R^l(f(b))|_{k-1})\frac{o}{1}$ and $f({}_2|R^l(b)(\psi({}_2|R^l(b)))) = {}_2|(R^l(f(b))|_{k-1})\frac{o}{1}\frac{o}{1}$. Notice that for even k $f(R^{l+1}(b)) = f({}_2|R^l(b)R_1{}^l(b)) \notin \{(o1)^*, (1o)^*\}$, because $f(R^{l+1}(b))$ must be adjacent to $f(R^l(b)) = R^l(o^{k-2}11)$ in SE_k. So $f(R^{l+1}(b)) = f({}_2|R^l(b)R_1{}^l(b)) = {}_2|(R^l(f(b))|_{k-1})\frac{o}{1}\frac{o}{1}$ and, because $f(R^{l+1}(b))$ is adjacent to $f(R^l(b))$, $f(R^{l+1}(b)) = {}_2|(R^l(f(b))|_{k-1})R_k^l(f(b))R_1^l(f(b)) = R^{l+1}(f(b))$. This completes the inductive proof.

In particular we now have $f(R^n(b)) = R^n(f(b)) \Rightarrow f(b) = R^n(f(b)) \Rightarrow k|n$, contradiction. □

The lemma indicates that it is not very likely that for $k \nmid n$ and $n > k \geq 3$ there exist uniform emulations of SE_n on SE_k, and that if they do exist, they will probably not have a nice structure. Presently no uniform emulations of S_n on S_{n-k} are known that are not step-simulating. As a corollary we have:

<u>Theorem 4.8</u>. There exist no uniform emulations of SE_n on SE_{n-1}, for $n \geq 4$.

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* Note that every uniform emulation function is surjective.

Proof.

Suppose there exists a uniform emulation of SE_n on SE_{n-1}, for some $n \geq 4$. Then, by lemmas 4.5, 4.6 and 4.7 there exists a uniform emulation of S_{n-1} on S_{n-2}, that is not step-simulating. This contradicts theorem 2.5. □

5. Discussion. The 4-pin shuffle with 2^n nodes can be emulated on the 4-pin shuffle with 2^k nodes for all $k \in \{o, ..., n\}$, whereas the shuffle-exchange network with 2^n nodes cannot always be emulated on smaller shuffle-exchange networks. This indicates that from the (important) viewpoint of emulation the 4-pin shuffle is preferable over the (classical) shuffle-exchange network.

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