ON THE COMPLEXITY OF FINDING UNIFORM EMULATIONS

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Abstract. Uniform emulations are a method to obtain efficient and structure preserving simulations of large networks on smaller (host-) networks. We show that the problem to decide whether a connected graph can be uniformly emulated on another connected graph is NP-complete, even if one requires that the graphs have bounded degrees or that the host network is a grid, a cube, or a shuffle-exchange graph, or is fixed to be a graph with 3 nodes and 2 edges.

1. Introduction. Parallel algorithms are normally designed for execution on a suitable network of N processors with N depending on the size of the problem to be solved. In practice N will be large and varying, whereas processor networks will be small and fixed. The resulting disparity between algorithm design and implementation must be resolved by simulating a network of some size N on a fixed and smaller size network of a similar or different kind, in a structure preserved manner. For this purpose, a notion of simulation, termed: emulation, was first proposed by Fishburn and Finkel[5]. Independently Berman[1] proposed a similar notion. An extensive analysis of emulations was made in [2,3].

Definition. Let $G=(V_G,E_G)$ and $H=(V_H,E_H)$ be networks of processors

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(graphs). We say that G can be emulated on H if there exists a function f: $V_G \to V_H$ such that for every edge (g, g') $\in E_G$: f(g)=f(g') or (f(g), f(g')) $\in E_H$. The function f is called an emulation function or, in short, an emulation of G on H. We call G the guest graph and H the host graph.

Let f be an emulation of G on H. Any processor h $\in V_H$ must actively emulate the processors belonging to $f^{-1}(h)$ in G. When g $\in$ $f^{-1}(h)$ communicates information to a neighbouring processor g', then h must communicate the corresponding information either "internally", when it emulates g' itself or to a neighbouring processor h'=f(g') otherwise. If all processors act synchronously in G, then the emulation will be slowed by a factor proportional to $\max_{h \in V_H} |f^{-1}(h)|$.

Definition. Let G, H be as above. The emulation f is said to be (computationally) uniform if for all h, h' $\in V_H$: $|f^{-1}(h)|=|f^{-1}(h')|$.

Every uniform emulation f has associated with it a fixed constant c, called: the computationfactor, such that for all h $\in V_H$: $|f^{-1}(h)|=c$. It means that every processor of H emulates the same number of processors of G.

In this paper we examine the following problem:

[UNIFORM EMULATION]

Instance: Connected graphs $G=(V_G,E_G)$, $H=(V_H,E_H)$

Question: Is there a uniform emulation of G on H?

The variant of this problem in which the computation factor $c=|V_G|/|V_H|$ is fixed will be called c-UNIFORM EMULATION. Note that the related question whether G can be emulated on H without the constraint of uniformity always yields the answer yes (provided H contains at least one node): every constant function is an emulation. We assume that the reader is familiar with the theory of NP-completeness (see Garey and Johnson[6]).

We will prove that c-UNIFORM EMULATION is NP-complete, for every $c \in N^+$. The problem remains NP-complete under several additional constraints on the guest and host networks. Realistic constraints are: the graphs are connected (or strongly connected in the case of directed graphs) and the nodes of the graph have a bounded degree. We will only consider connected and strongly connected graphs, respectively.

This paper is organized as follows. In section 2 we prove that c-UNIFORM EMULATION is NP-complete and consider undirected graphs of bounded degree. In section 3 we consider directed graphs of bounded degree. In section 4 we prove that c-UNIFORM EMULATION is NP-complete if the host graph is restricted to be a grid, for every $c \geq 2$. In section 5 we prove that c-UNIFORM EMULATION is NP-complete if the host graph is restricted to be a cube, for every $c \geq 1$. In section 6 we prove that c-UNIFORM EMULATION is NP-complete if the host graph is restricted to be a 4-pin shuffle or a shuffle-exchange graph, for every $c \geq 7$ and $c \geq 15$ respectively. (The 4-pin shuffle and the shuffle-exchange graph are two versions of Stone's original shuffle-exchange network. Cf.[9].) In section 7 we prove that for certain graphs H (for instance the graph with 3 nodes and 2 edges), one can fix the hostgraph to H, and the UNIFORM EMULATION problem remains NP-complete.

2. <u>c-UNIFORM EMULATION and undirected graphs of bounded degree</u>. The following result was observed in [2].

<u>Theorem 2.1</u>. [2] UNIFORM EMULATION is NP-complete.

<u>Proof</u>. It is easy to see that this problem is in NP. (Guess a function f: $V_G \rightarrow V_H$, and check whether it is an emulation and whether it is uniform.) To prove NP-completeness, observe that one can polynomially transform HAMILTONIAN CIRCUIT to the problem. Let $H=(V_H,E_H)$ be a connected, undirected graph and let $G=(V_G,E_G)$ be the undirected graph consisting of one cycle of $|V_H|$ nodes. G can be uniformly emulated on H if and only if H contains a Hamiltonian circuit. □

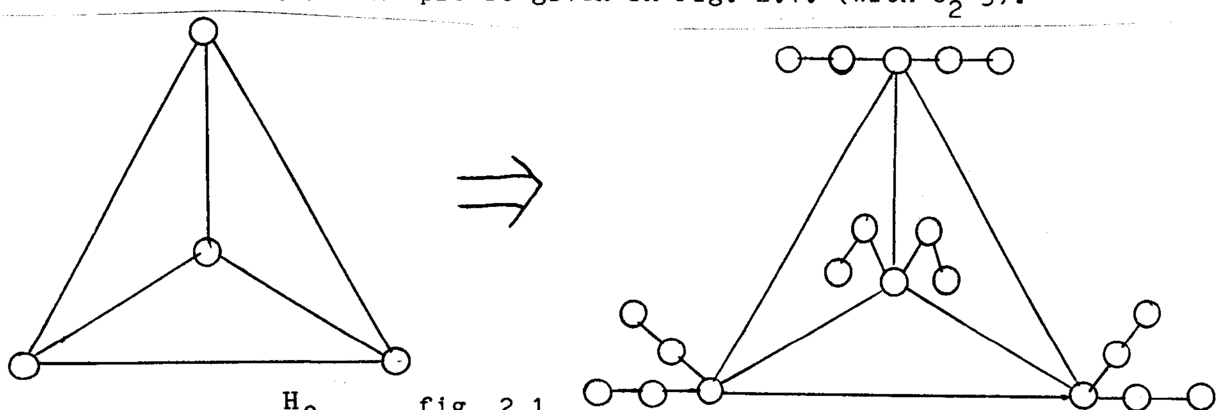
Theorem 2.2. For every c_1, $c_2 \in N^+$, the following problem is NP-complete:

Instance: Connected, undirected graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$, with each node of V_G of degree at most c_1+1 and each node of V_H of degree at most c_2+2, and $|V_G|=c_1.c_2.|V_H|$.

Question: Is there a uniform emulation of G on H?

Proof. Clearly the problem is in NP. To prove NP-completeness, we transform HAMILTONIAN CIRCUIT for undirected graphs with nodes of degree exactly 3 and no cycles of length 3 or 4, to this problem. This version of HAMILTONIAN CIRCUIT is NP-complete [6,7]. Note that the result in [7] did not state the last constraint, but the (planar) graphs resulting from the construction in [7] indeed do not have cycles of length 3 or 4.

Let an arbitrary graph $H_0=(V_0,E_0)$ be given with nodes of degree 3 and no cycles of length 3 or 4. Construct a graph $H=(V_H,E_H)$ with $|V_H|=|V_0|+|V_0|.2.(c_2-1)$ nodes by adding to each node of H_0 c_2-1 extra "branches" of 2 nodes. An example is given in fig. 2.1. (with $c_2=3$).



H_0 fig. 2.1. H

Let $n=c_2|V_H|=c_2(|V_0|+|V_0|.2.(c_2-1))$. Let $G=(V_G,E_G)$ be the graph defined by $V_G=\{(i,j) \mid 0\le i\le n-1 \wedge 1\le j\le c_1\}$, and $E_G=\{((i_1,j_1),(i_2,j_2)) \mid (i_1,j_1), (i_2,j_2) \in V_G \wedge ((i_1=i_2 \wedge j_1\ne j_2) \vee (j_1=j_2 \wedge i_1=i_2 \pm 1 \pmod n))\}$. So G consists of n cliques of c_1 nodes and thus has $c_1.c_2.|V_1|$ nodes; each node is also connected to a node in the "successor clique", and a node in the "predecessor clique". We assume for the sake of argument that G is oriented, say, counter-clockwise. An example of G, with n=6 and c_1=3 is given in fig. 2.2.

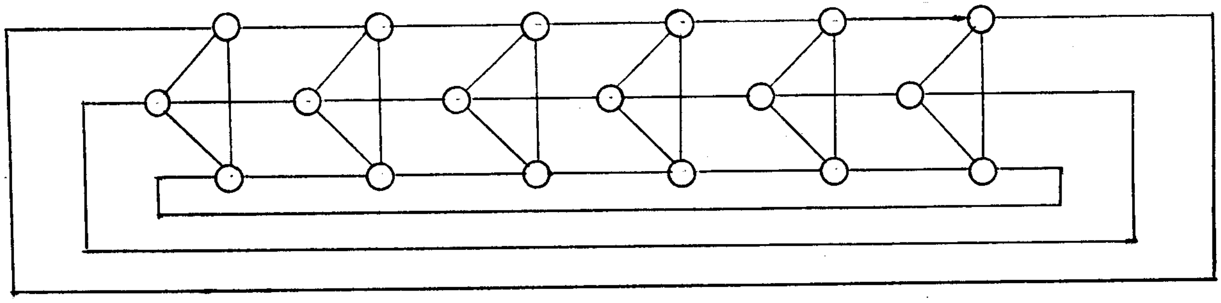


fig. 2.2.

The construction of G and H can be done in time polynomial in $|V_0|$, and G and H satisfy the conditions of the problem.

<u>Claim 2.2.1.</u> H_0 contains a Hamiltonian circuit if and only if there is a uniform emulation of G on H.

<u>Proof</u>. Suppose we have a uniform emulation f of G on H. Let A_j = $\{(i,j)|0\leq i\leq n-1\} \subseteq V_G$, for every j, $1\leq j\leq c_1$. Note that for every j, $1\leq j\leq c_1$, A_j is a cycle of G. We consider two cases:

<u>Case I</u> : $c_2\geq 2$. Let v $\in V_0 \subseteq V_H$ and let j be fixed, $1\leq j\leq c_1$. Now look at the set of successors of nodes (i,j) $\in$ G with f((i,j))=v. For every branch attached to v in H, there is at least one node of V_G that is mapped upon the "end of the branch", (that is, the node with degree 1). This node is connected to a node of the form (x,j), so the "cycle" $f(A_j)$ will visit this branch. So there are c_2-1 successors of nodes (i,j) with f((i,j))=v that are mapped upon branches of v. There must be also one such successor that is mapped upon another node v' $\in V_0 \subseteq V_H$. So there are at least c_2 nodes of A_j that are mapped upon v, for every j, $1\leq j\leq c_1$. Due to the uniformity of f, this number must be exactly c_2 for all j, $1\leq j<=c_1$. (This means essentially that the "cycle" $f(A_j)$ after visiting a node v $\in V_0$ for the first time, must visit successively all branches of v, and then leave v to another node v' $\in V_0$ and cannot return to v for a second time thereafter.) Now we have that for each node v $\in V_0 \subseteq V_H$ there is a unique v' $\in V_0 \subseteq V_H$ such that there is a i $\in$ {0,..,n-1} with f((i,1))=v and

$f((i+1) \bmod n)=v'$. We can call v' the successor of v. By successively visiting the successors of the nodes in $V_0 \subseteq V_H$ we have a Hamiltonian circuit of H_0.

Case II: $c_2=1$. First notice that $H=H_0$. We claim that there are no $i_1 \neq i_2$ with $f((i_1,1))=f((i_2,1))$. Suppose there are. In the following we let + be the addition modulo n and - the subtraction modulo n.

Let $J_1 = \{j \mid f((i_1,j)) = f((i_1,1))\}$, $J_2 = \{j \mid f((i_1,j)) \neq f((i_1,1))\}$, $v_1 = f((i_1,1))$, $j_2 \in J_2$, and $v_2 = f((i_1,j_2))$.

Claim: $f((i_1+1,J_1)) \neq v_1$ and $|f((i_1+1,J_1)| = 1$.

Proof. First suppose $f((i_1+1, J_1))=v_1$. The nodes in $f((i_1+1, J_2))$ must be equal of adjacent to $f((i_1,J_2)) = v_2$ and to $f((i_1+1,J_1)) = v_1$. As $H=H_0$ does not have 3-cycles, we have $f((i_1+1,J_2)) \subseteq \{v_1,v_2\}$. So $f^{-1}(\{v_1,v_2\}) = \{(i_1,j^*) \mid 1 \leq j^* \leq c_1\} \cup \{(i_1+1,j^*) \mid 1 \leq j^* \leq c_1\}$. Now $f((i_1-1,1))$ must be adjacent to $v_1 = f((i_1,1))$ (it cannot be equal to it due to the uniformity of f), $f((i_1-1, j_2))$ must be adjacent to $v_2=f((i_1,j_2))$ (again it cannot be equal to it) and $f((i_1-1,1))$ and $f((i_1-1,j_2))$ must be equal or adjacent, so $H=H_0$ contains a 3-cycle or a 4-cycle. Contradiction. Suppose $|f((i_1+1,J_1))| \geq 2$. Notice that every node in $f((i_1+1,J_1))$ is equal or adjacent to v_1 and H does not contain 3-cycles, so $v_1 \in f((i_1+1,J_1))$. If $v_2 \in f((i_1+1,J_1))$ then $f^{-1}((v_1,v_2)) = \{(i_1,j^*) \mid 1 \leq j^* \leq c_1\} \cup \{(i_1+1,j^*) \mid 1 \leq j^* \leq c_1\}$ and a contradiction can be obtained as before. So there is a node $v_3 \notin \{v_1,v_2\}$ with $v_3 \in f((i_1+1,J_1))$. Every node in $f((i_1+1,J_2))$ must be equal or adjacent to v_2 and v_3, and v_2 is adjacent to v_3, so $f((i_1+1,J_2)) = v_1$ which contradicts uniformity. (We did suppose there was a i_2 with $f((i_1,1)) = f((i_2,1))$, so $f^{-1}(v_1) \supsetneq \{(i_1,J_1)\} \cup \{(i_1+1,J_2)\} \cup \{(i_2,1)\}$). □

Now every node in $f((i_1+1,J_2))$ must be adjacent or equal to v_2 and to $v_3 = f((i_1+1,J_1))$. Note that v_3 is adjacent to v_1 and, because H does not contain 3-cycles or 4-cycles we have: $f((i_1+1,J_2)) \doteq \{v_1\}$ and $f^{-1}(v_1) \supsetneq \{(i_1+1,J_2), (i_1,J_1), (i_2,1)\}$, hence $|f^{-1}(v_1)| \geq c_2+1$, which contradicts uniformity. This completes the proof of the claim that there are no $i_1 \neq i_2$ with $f((i_1,1)) = f((i_2,1))$. This shows that

f_1 restricted to the set of nodes $\{(i,1) | 1 \leq i \leq n\}$ is a graph isomorphism of a cycle with n nodes to a subgraph of $H=H_0$, so H_0 has a Hamiltonian circuit.

Suppose H_0 has a Hamiltonian circuit. First note that $f: V_G \rightarrow \{0,..,n-1\}$, given by $f((i,j)) = i$ is a uniform emulation of V_G on a cycle with n nodes. Because uniform emulation is transitive, it is sufficient to give a uniform emulation of a cycle with n nodes on H for showing that G can be uniformly emulated on H.

First we derive a cyclic path in H that visits each node at least once and at most c_2 times: add to the Hamiltonian circuit of H_0 extra path parts that visit the branches, added to the nodes of H. This path can be transformed to a uniform emulation of the cycle with n nodes to H, by mapping successive nodes on the cycle on the same node v in H, if the path visits v less than c_2 times. □

Corollary 2.3. For every $c \in N^+$, the following problem is NP-complete:

[c-UNIFORM EMULATION]

Instance: Connected graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$, with $|V_G|=c|V_H|$.

Question: Is there a uniform emulation of G on H?

By further refining the technique of the proof of theorem 2.2. slightly better results can be obtained.

Theorem 2.4. For every c_1, $c_2 \in N^+$ with $c_2 \geq 2$ the following problem is NP-complete:

Instance: Connected, undirected, planar graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$, with each node of V_G of degree at most 3, and each node of V_H of degree at most c_2+2, and $|V_G|=c_1.c_2.|V_H|$.

Question: Is there a uniform emulation of G on H?

Note: the theorem is stronger than theorem 2.2. in two ways: we can choose G and H planar, and the degree of the nodes in G is not dependent on c_1.

Proof. Clearly the problem is in NP. To prove NP-completeness we will again transform HAMILTONIAN CIRCUIT for undirected planar graphs with nodes of degree 3 to this problem [6,7]. As noted before, this version of HAMILTONIAN CIRCUIT is NP-complete.

Let $H_0=(V_0,E_0)$ be an arbitrary undirected planar graph with nodes of degree 3. We construct a graph $H=(V_H,E_H)$ by adding to each node of V_0 c_2-1 extra branches, each branch now consisting of $2c_1+1$ nodes. An example is given in fig. 2.3., with $c_1=3$ and $c_2=2$. Note that $|V_0|$ must be even (every node in H_0 has exacty 3 adjacent edges), hence $|V_H|$ is even. Let $n=|V_H|$.

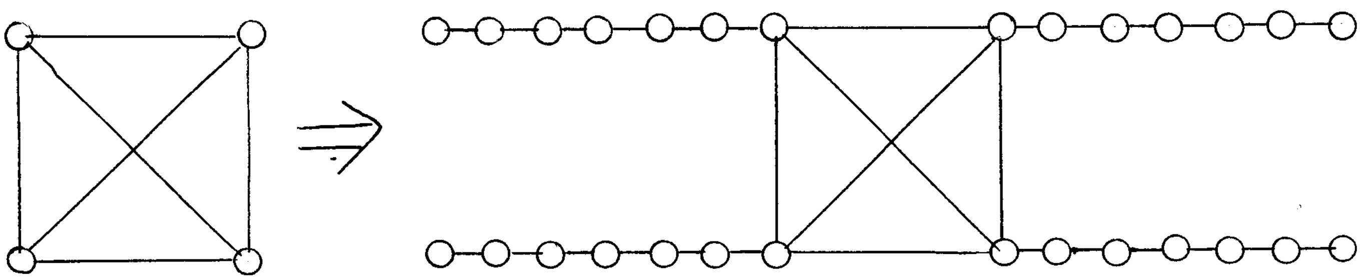


fig. 2.3. H_0 and H with $c_1=3$ and $c_2=2$.

Let $G=(V_G,E_G)$ be defined by $V_G = \{(i,j) | 0 \leq i \leq n-1 \wedge 1 \leq j \leq c_1\}$, and $E_G = \{((i_1,j_1), (i_2,j_2)) | (i_1,j_1),(i_2,j_2) \in V_G \wedge ((i_1=i_2 \wedge |j_1-j_2| =1 \wedge (\min(j_1,j_2) \text{ is odd} \Leftrightarrow i_1 = \text{odd})) \vee (j_1=j_2 \wedge i_1=i_2 \pm 1 \pmod{n}))\}$. Thus the cliques of the graph G of the proof of theorem 2.2. are replaced by structures that have a much smaller number of edges. The example of fig. 2.4. illustrates the construction. In this example n=6 and $c_1=5$.

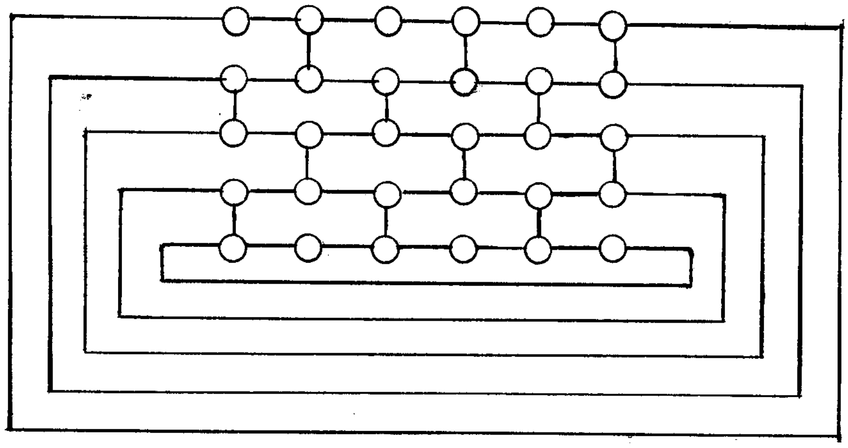


fig. 2.4. G with n=6 and c_1=5.

Again we claim that H_0 contains a Hamiltonian circuit if and only if there is a uniform emulation of G on H. The proof of claim 2.2.1. for the case $c_2 \geq 2$ can be followed to obtain a proof for this case, with the following observation: for every $v \in V_0$ and every branch of v there is at least one node that is mapped upon the last node of the branch (i.e. the node with degree 1). For every j, $1 \leq j \leq c_1$ there is a node $(i,j) \in V_G$ that has a distance $\leq 2c$ to w. This means that (i,j) must also be mapped upon a node of this branch, so the image of the cycle $A_j = \{(i,j) | 0 \leq i \leq n-1\}$ must visit this branch. Now the remaining part of this proof can be done in a similar way as in the proof of claim 2.2.1. Notice that for every n, c_1 G is planar. If H_0 has a Hamiltonian circuit, then one can construct a uniform emulation of G on H similar to the construction in claim 2.2.1. □

We mention the following corollaries of theorem 2.2 and 2.4:

<u>Corollary 2.5</u>. c-UNIFORM EMULATION FOR PLANAR GRAPHS is NP-complete.

<u>Corollary 2.6</u>. For every $c \in N^+$, c even, the following problem is NP-complete:

<u>Instance</u>: Connected, undirected, planar graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, with each node of G of degree at most 3, each node of H of degree at most 4, and $|V_G| = c.|V_H|$.

<u>Question</u>: Is there a uniform emulation of G on H?

Proof. This is a special case of theorem 2.4. with $c_1=c/2$ and $c_2=2$. □

Corollary 2.7. For every $c \in N^+$ the following problem is NP-complete:

Instance: Connected, undirected, planar graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$, with each node of G of degree at most 2 (i.e. G is a path of a cycle), each node of H of degree at most c+2, and $|V_G|=c.|V_H|$.

Question: Is there a uniform emulation of G on H?

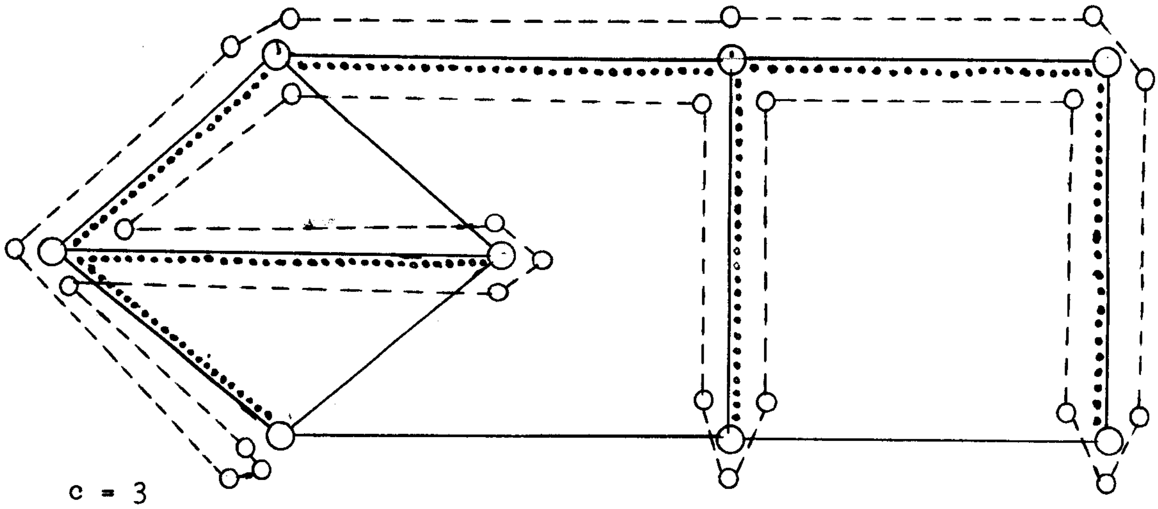
Proof. This is a special case of theorem 2.2. with $c_1=1$ and $c_2=c$. Note that the graphs H, resulting from the construction in the proof of theorem 2.2. are planar. □

The result of corollary 2.7. should be contrasted with the following proposition:

Proposition 2.8. For every $c \in N^+$, and connected, undirected graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$ with G a cycle or a path (i.e., each node of V_G has degree at most 2), each node of H of degree at most c and $|V_G|=c.|V_H|$, there exists a uniform emulation of G on H. The emulation function can be found in $O(|V_G|+|E_H|)$ time.

Proof. It is easy to construct a cyclic path that visits each node at least once and at most c times: construct a spanning tree T of H. Now visit the root of T; visit recursively the nodes in the subtree of the leftmost son of the root, starting and ending with the root of this subtree, then visit the root of T, then visit the nodes in the subtree of the one-but-leftmost son of the root, then again visit the root of T, etc. In this way a node v is visited at least once, and at most degree(v) $\leq$ c times.

We can map the nodes of the path or cycle G on the successive nodes visited by this algorithm. If the algorithm visits a node v less than c times we have to map some successive nodes of G on v. In this way a uniform emulation can be obtained.



c = 3

..... = spanning tree

----- = uniform emulation of a cycle on H

fig. 2.5

The time necessary to create the spanning tree is $O(|E_H|)$, the rest of the work can be done in time $O(|V_G|)$. □

Interesting open problems are: given a computation factor c, what is the complexity of the problem to determine whether a path or a cycle can be uniformly emulated on a graph with maximum node degree c+1; and is the c-UNIFORM EMULATION problem for G and H graphs with maximum node degree 3 NP-complete (for c>1)?

3. Directed graphs of bounded degree. For directed graphs the c-UNIFORM EMULATION problem remains NP-complete if we restrict G to be a cycle and H a graph with each node involved in at most 3 edges:

Theorem 3.1. For every $c \in N^+$ the following problem is NP-complete:

Instance: Directed, strongly connected, planar graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$, such that every node in G is involved in at most 2 edges (i.e., G is a cycle), every node in H is involved in at most 3 edges, and $|V_G|=c.|V_H|$.

Question: Is there a uniform emulation of G on H?

Proof. Clearly the problem is in NP. To prove NP-completeness we transform DIRECTED HAMILTONIAN CIRCUIT for directed, strongly connected, planar graphs with each node involved in exactly 3 edges to this problem. This version of DIRECTED HAMILTONIAN CIRCUIT is NP-complete [6].

Let a directed, strongly connected, planar graph $H_0=(V_0,E_0)$ be given, with each node $v \in V_0$ involved in exactly 3 edges. H_0 has two types of nodes. (See fig. 3.1.)

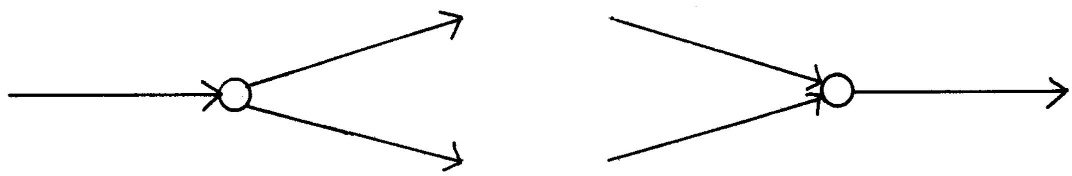


fig. 3.1.

We first replace H_0 by H_1 by replacing nodes as in fig. 3.2. (The digits 0, 1, 2 indicate the "type" of a new node for later reference.)

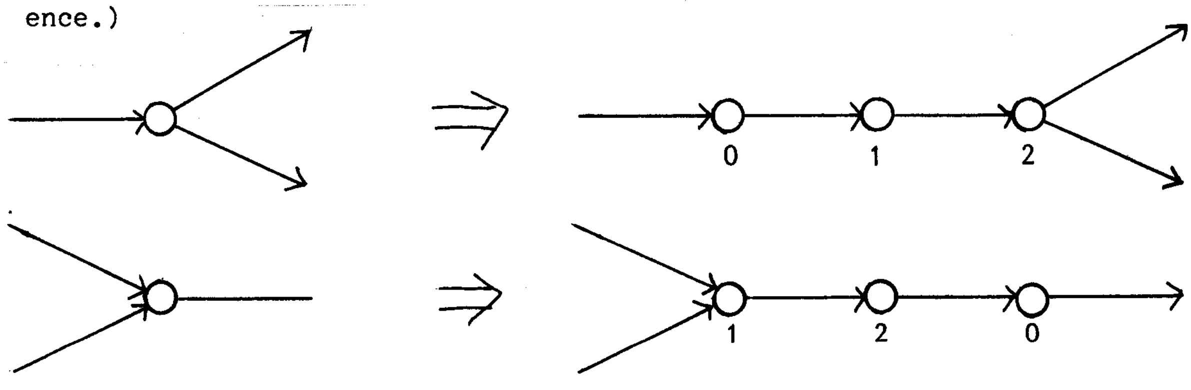


fig. 3.2.

To each node of type 2 in fig. 3.2. we add a binary tree with c-1 leaves, with the edges in the tree directed towards the leaves. Another tree with edges directed towards the root is placed on these leaves. The root of this tree is connected with an edge to the corresponding node of type 1. An example is given in fig. 3.3., with c=7. An example of the whole transformation is given in fig. 3.4. (with c=3).

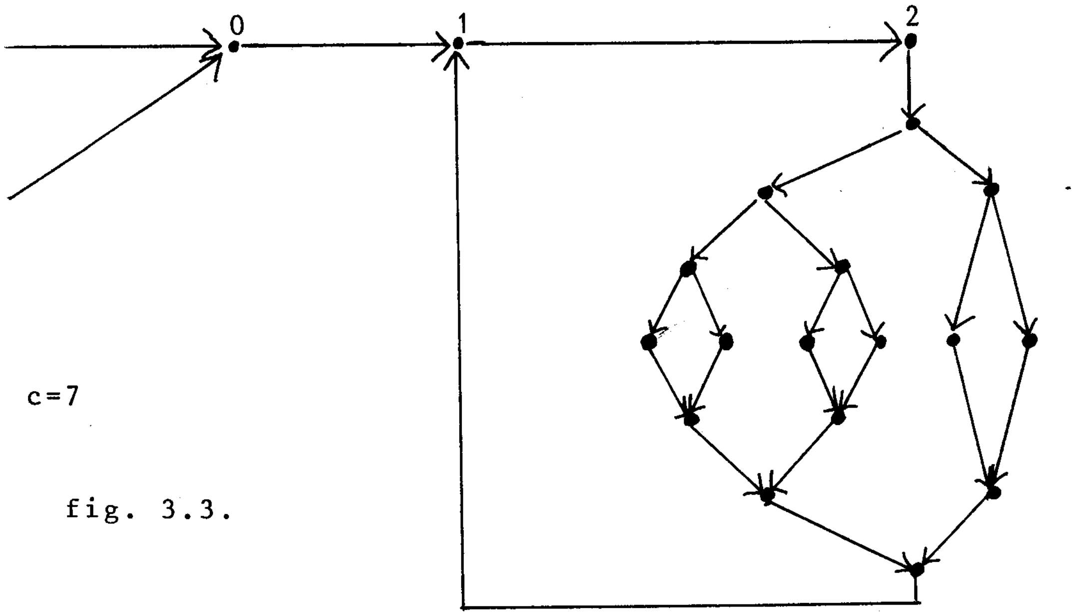


fig. 3.3.

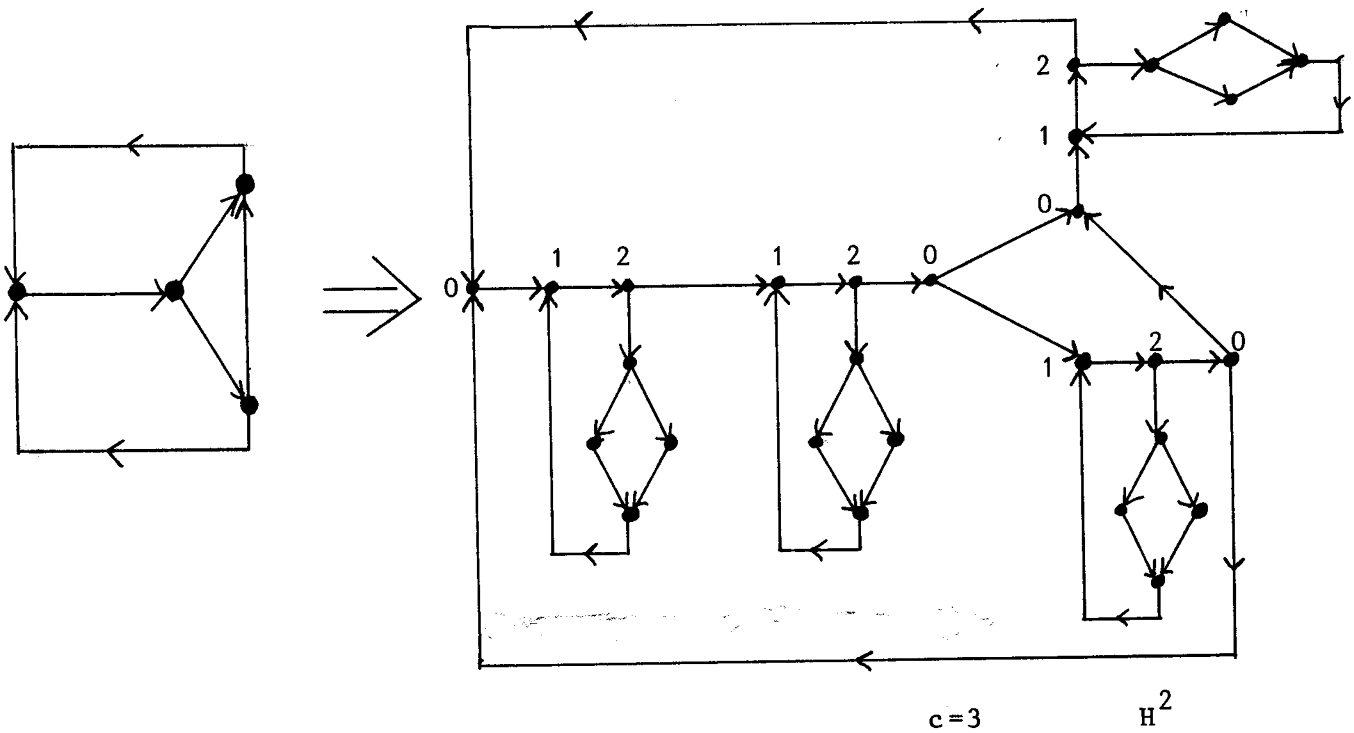


fig. 3.4.

Let $H=(V_H,E_H)$ be the graph that is obtained from H_1 in this way. If $c=1$ then one can take $H=H_0$. We let $n=|V_H|$, and G a directed cycle of c.n nodes.

Claim 3.1.1. H_0 contains a Hamiltonian circuit if and only if there is a uniform emulation of G on H.

Proof. If H_0 contains a Hamiltonian circuit then there exists a cyclic path in H that visits each node at least once and at most c times. We visit nodes of type 0 in the order of the Hamiltonian circuit and, when we arrive at a pair of nodes of type 1 and 2, we go c-1 times over the added structure of the two trees, each time visiting another leaf. This path can be transformed to a uniform emulation of G on H by mapping successive nodes of G on the same node v in H, if the path visits v less than c times.

Now suppose we have a uniform emulation f of G on H. For every node v* of type 1, look at the c predecessors of the c nodes that are mapped upon v* by f. Because every leaf in the two-tree structure added to v*, (and the neighbour of v* of type 2) must be visited by f(G), and there are c-1 such leaves, c-1 of these predecessors must be mapped upon the root of the tree with edges towards v* and exactly one is mapped upon a node of type 0 or 2. This means that every node cluster, consisting of a node of type 0, a node of type 1, a node of type 2 and the added two-tree structure can be visited at most once from another node cluster. It has also to be visited at least once from another node cluster. Look at the successive node clusters in H that are visited. The corresponding nodes in H_0 form a Hamiltonian circuit. □

The graphs G and H fulfill the conditions and can be obtained from H_0 in time polynomial in $|V_0|$ and c. This completes the proof of theorem 3.1. □

4. Uniform emulation on the two-dimensional grid network. The two-dimensional grid (or mesh) is often used as a processor interconnection network. We use a version of the grid with "wrap-around" connections along the boundaries. Let GR_n be the nxn grid network (with n^2 nodes), with wrap-around connections, and let GR_n' be the nxn grid network without wrap-around connections. The nodes of GR_n and GR_n' are

named by their plane coordinates (i,j) with $0 \le i,j \le n-1$ in the usual representation of the nxn grid.

<u>Definition</u>. The two-dimensional grid network with wrap-around connections is the graph $GR_n=(V_n,E_n)$ with $V_n=\{(i,j) \mid i,j \in N \text{ and } 0 \le i,j \le n-1\}$ and $E_n=\{((i,j),\ (i',j')) \mid (i,j),\ (i',j') \in V_n \text{ and } (i=i' \wedge j=(j' \pm 1) \bmod n) \text{ or } (i=(i' \pm 1) \bmod n \wedge j=j')\}$. The two-dimensional grid without wrap-around connections is the graph $GR_n' = (V_n,E_n')$, with $E_n' = \{((i,j),\ (i',j')) \mid (i,j),\ (i',j') \in V_n \text{ and } (i=i' \wedge j=j' \pm 1) \text{ or } (i=i' \pm 1 \wedge j=j')\}$.

<u>Theorem 4.1</u>. For every $c \in N^+$, $c \ge 2$, the following problem is NP-complete:

[c-UNIFORM EMULATION ON A GRID]

<u>Instance</u>: A connected, undirected graph $G=(V_G,E_G)$, such that there is an $n \in N^+$ with $|V_G|=c.n^2$.

<u>Question</u>: Is there a uniform emulation of G on GR_n?

<u>Proof</u>. Clearly the problem is in NP. To prove NP-completeness we first consider the following problem.

Given a set of $V \subseteq V_n$ in a two-dimensional grid $GR_n=(V_n,E_n)$, the subgraph of GR_n induced by V is the graph $G_V=(V,E_V)$, with $E_V=\{(v_1,v_2) \mid v_1,v_2 \in V \text{ and } (v_1,v_2) \in E_n\}$, i.e. every two nodes in V that are adjacent in the grid are adjacent in the subgraph induced by V. The following problem is known to be NP-complete [8]:

[HAMILTONIAN CIRCUIT IN A GRID GRAPH]

<u>Instance</u>: $n \in N$ and a set of nodes $V \subseteq V_n$.

<u>Question</u>: Does the subgraph of GR_n, induced by V contain a Hamiltonian circuit?

We show that HAMILTONIAN CIRCUIT IN A GRID GRAPH can be polynomially transformed to c-UNIFORM EMULATION ON A GRID (for every $c \ge 2$). Let $n \in N^+$ and a set of nodes $V \subseteq V_n$ be given and let $c \ge 2$, $c \in N^+$. We will construct a connected undirected graph $G=(V_G,E_G)$, with $|V_G|=c.(2n)^2$, such that there is a uniform emulation of G on GR_{2n}, if and only if the subgraph of GR_n induced by V contains a Hamiltonian circuit.

We let $G=(V_G,E_G)$ consist of the following parts:

a) c-1 gridlayers of 2nx2n nodes:

$A = \{v^i_{j,k} | 1 \le i \le c-1,\ 0 \le j \le 2n-1,\ 0 \le k \le 2n-1\}$

b) one gridlayer of 2nx2n nodes with the nodes of V omitted:

$B = \{v^c_{j,k} | 0 \le j \le 2n-1,\ 0 \le k \le 2n-1,\ (j,k) \notin V\}$

c) a cycle of $|V|$ points:

$C = \{w_i | 0 \le i \le |V|-1\}$.

We connect points whose (j,k)-coordinates are adjacent in GR_{2n}, without regard to the layer, and one node of the cycle to some point $v^i_{j,k}$ with $(j,k) \in V$. Choose an arbitrary $(j^*,k^*) \in V$.

Now $V_G = A \cup B \cup C$

$$E_G = \{(v^{i_1}_{j_1,k_1}, v^{i_2}_{j_2,k_2}) | v^{i_1}_{j_1,k_1}, v^{i_2}_{j_2,k_2} \in A \cup B \wedge ((j_1,k_1),(j_2,k_2)) \in E_{2n}\} \cup \{(w_i, w_j) | w_i, w_j \in V \wedge i=(j\pm 1) \bmod |V|\} \cup \{(w_1, v^1_{j^*,k^*})\}.$$ (E_{2n} is the set of edges of GR_{2n}.)

Claim 4.1.1. The subgraph of GR_n induced by V contains a Hamiltonian circuit if and only if there is a uniform emulation of $G=(V_G,E_G)$ on GR_{2n}.

Proof. First suppose the subgraph of GR_n induced by V contains a Hamiltonian circuit. Then the subgraph of GR_{2n} induced by V contains a Hamiltonian circuit. Now let $f(v^i_{j,k}) = (j,k)$, for all $v^i_{j,k} \in A \cup B$. Then every node in $V_{2n}-V$ has c nodes of $A \cup B$ mapped upon it, every node in V has c-1 such nodes mapped upon it. Now we can map w_1 on (j^*,k^*), w_2 on the node that is visited by the Hamiltonian circuit after (j^*,k^*), etc., i.e., we let f map w_i on the i'th node on the Hamiltonian circuit, where (j^*,k^*) is considered to be the first node. In this way a correct uniform emulation of G on GR_{2n} is obtained.

Now suppose f is a uniform emulation of $G=(V_G,E_G)$ on GR_{2n}. We claim that for all $(j_1,k_1) \in \{0,..,2n-2\} \times \{0,..,2n-2\} - \{o,..,n\} \times \{0,..,n\}$, $(j_2,k_2) \in \{0,..,2n-1\} \times \{0,..,2n-1\}$, $1 \le i_1,i_2 \le c$, with $(j_1,k_1) \ne (j_2,k_2)$: $f(v^{i_1}_{j_1,k_1}) \ne f(v^{i_2}_{j_2,k_2})$. Suppose there do exist (j_1,k_1), (j_2,k_2), i_1,i_2 fulfilling the conditions, with $f(v^{i_1}_{j_1,k_1}) = f(v^{i_2}_{j_2,k_2})$.

Then we can reach a contradiction as follows: $V \subseteq \{0,..,n-1\} \times \{0,..,n-1\}$, so $(j_1,k_1) \notin V$ and (j_1,k_1) is not a neighbour of a node in V. So $v^{i_1}_{j_1,k_1}$ has 5c-1 neighbours in G*, and $v^{i_2}_{j_2,k_2}$ has at least one neighbour in G* that is not $v^{i_1}_{j_1,k_1}$ or a neighbour of $v^{i_1}_{j_1,k_1}$, so at least 5c+1 nodes must be mapped upon $f(v^{i_1}_{j_1,k_1})$ and its 4 neighbours in GR_{2n}. This contradicts uniformity. In the same way one can prove that for (j_1,k_1), i_1 as before, and $0 \le i \le |V|-1$, $f(w_i) \ne f(v^{i_1}_{j_1,k_1})$. So the only nodes that can be mapped to $f(v^{i_1}_{j_1,k_1})$ $((j_1,k_1) \in \{0..2n-2\} \times \{0..2n-2\} - \{0..n\} \times \{0..n\})$, are the nodes $v^{\alpha}_{j_1,k_1}$, $1 \le \alpha \le c$. There are exactly c such nodes, so for all $(j_1,k_1) \in \{0..2n-2\} \times \{0..2n-2\} - \{0..n\} \times \{0..n\}$, i_1, $i_2 \in \{1,..,c\}$: $f(v^{i_1}_{j_1,k_1}) = f(v^{i_2}_{j_2,k_2})$. Now we claim that the set $\{v^1_{i,2n-2} | i=0..2n-1\}$ must be mapped upon a cycle in GR_{2n} (that is a set of the form $\{(j,k) | 0 \le k \le 2n-1\}$ or $\{(k,j) | 0 \le k \le 2n-1\}$, with j fixed). Suppose this is not the case. Then the successive nodes $f(v^1_{i,2n-2})$ (i=0..2n-1) form a path that must make a bend on the grid. Suppose there is a bend in the form as shown in fig. 4.1. (The other cases are similar.) Then the nodes $\{f(v^1_{i^*,2n-3}) | i^* \in \{i,i+1,i+2\}\}$ cannot be mapped such that adjacencies are preserved. Contradiction.

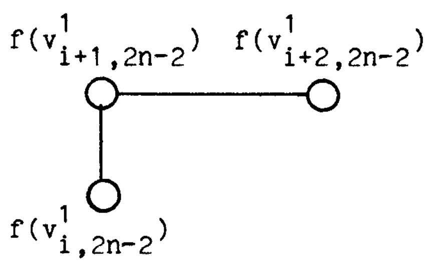


fig. 4.1.

In the same way one can prove that the set $\{v^1_{2n-2,i} | i=0..2n-1\}$ must be mapped upon a cycle in GR_{2n}. Without loss of generality one may suppose $f(v^1_{i,2n-2}) = (i,2n-2)$ and $f(v^1_{2n-2,i}) = (2n-2,i)$. With induction one can prove that for all (i,j) $f(v_{i,j}) = (i,j)$. Now $f(v^{\alpha}_{i,j}) = (i,j)$ for all $v^{\alpha}_{i,j} \in A \cup B$. This means that $f(w_i) \in V$ for all $w_i \in C$ (use uniformity of f). So $f(w_1), f(w_2),\ldots,f(w_{|V|})$ form a Hamiltonian circuit in G_V. □

Note that $|V_G| = c.(2n)^2$, and the construction of G can be carried out in time polynomial in $|V|$ and n. Hence c-UNIFORM EMULATION ON A GRID is NP-complete, for $c\geq2$. □

Theorem 4.2. For every $c \in N^+$, $c\geq2$ the following problem is NP-complete:

Instance: A connected, undirected graph $G=(V_G,E_G)$, such that there is an $n\in N^+$ with $|V_G|=c.n^2$.

Question: Is there a uniform emulation of G on GR_n?

Proof. Similar to that of theorem 4.1. □

5. Uniform emulation on the cube network. Let C_n denote the cube network with 2^n nodes. The nodes in the network are given n-bit addresses in the range $0..2^n-1$, and there is an edge from node b to node c if and only if c is obtained by flipping precisely one bit in b. The i'th bit of an address b is denoted by b_i $(1\leq i\leq n)$. We use x, y to denote segments of bits. For $|x|=|y|$, let $d(x,y)$ be the Hamming distance between the bitstrings x and y, i.e., the number of bitpositions in which x and y differ. We use $\frac{0}{1}$ to denote a single bit, that can be 0 or 1. For a bitstring $x=x_1..x_n$ let $(x_1..x_n)|_m=x_1..x_m$, i.e., the first m bits of x.

Definition. The cube network (or n-cube) is the graph $C_n=(V_n,E_n)$ with $V_n = \{(b_1..b_n)|\forall\ 1\leq i\leq n\ b_i=\frac{0}{1}\}$ and $E_n = \{(b,c)|b,c \in V_n \wedge d(b,c) = 1\}$.

We recall the following fact from [2]:

<u>Theorem 5.1</u>. [2] For $n\geq 1$ GR_{2^n} can be uniformly emulated on C_{2n}, i.e., G_{2^n} is isomorphic to a spanning subgraph of C_{2n}.

The main result in this section is:

<u>Theorem 5.2</u>. For every $c \in N^+$, the following problem is NP-complete:

[c-UNIFORM EMULATION ON A CUBE]

<u>Instance</u>: A connected, undirected graph $G=(V_G,E_G)$, such that there is a $k \in N^+$ with $|V_G|=c.2^k$.

<u>Question</u>: Is there a uniform emulation of G on C_k?

<u>Proof</u>. Clearly the problem is in NP. To prove NP-completeness we will transform the HAMILTONIAN CIRCUIT IN A GRID GRAPH problem to this problem. (For details on this version of HAMILTONIAN CIRCUIT see the proof of theorem 4.1.)

Let $n \in N^+$ and a set of nodes V in GR_n be given. We may suppose that $n=2^k$ for some $k \in N^+$. (If not we can reduce the problem to this case in polynomial time.)

Let f be a uniform emulation of GR_n on C_{2k} (as implied from theorem 5.1.) (Note that f is a bijection, i.e. a subgraph isomorphism.) Let g be the mapping $C_{2k} \to C_{6k+2}$, defined by $g(x_1 \ldots x_{2k}) = x_1x_1x_1x_2x_2x_2 \ldots x_{2k}x_{2k}x_{2k}00$.

Let $V_1 = g \circ f(V)$, and

let $V_2 = \{x_1x_1x_1x_2x_2x_2 \ldots x_{i-1}x_{i-1}x_{i-1}001x_{i+1}x_{i+1}x_{i+1} \ldots x_{2k}x_{2k}x_{2k}00 \mid x_1 \ldots x_{i-1}0x_{i+1} \ldots x_{2k},\ x_1 \ldots x_{i-1}1x_{i+1} \ldots x_{2k} \in f(V)\}$,

$V_3 = \{x_1x_1x_1x_2x_2x_2 \ldots x_{i-1}x_{i-1}x_{i-1}011x_{i+1}x_{i+1}x_{i+1} \ldots x_{2k}x_{2k}x_{2k}00 \mid x_1 \ldots x_{i-1}0x_{i+1} \ldots x_{2k},\ x_1 \ldots x_{i-1}1x_{i+1} \ldots x_{2k} \in f(V)\}$,

$V_4 = \{x_1x_1x_1x_2x_2x_2 \ldots x_{2k}x_{2k}x_{2k}01 \mid f^{-1}(x_1 \ldots x_{2k}) \in V$ and has at most 3 neighbours in $V\}$

$V_5 = \{x_1x_1x_1x_2x_2x_2 \ldots x_{2k}x_{2k}x_{2k}10 \mid f^{-1}(x_1 \ldots x_{2k}) \in V$ and has at most 2 neighbours in $V\}$

We let $W = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \subseteq V_{6k+2}$ and G_W be the subgraph of

C_{6k+2} induced by W: $G_W=(W,E_W)$ and $E_W=\{(v,w) \mid v,w \in W \wedge (v,w) \in E_{6k+2}\}$ (where E_{6k+2} is the set of edges of C_{6k+2}). G_W resembles the subgraph of GR_n induced by V, $G_V=(V,E_V)$, with $E_V = \{(v,w) \mid v,w \in V$ and v,w adjacent in $GR_n\}$. One can obtain a graph isomorphic to G_W from G_V by adding on each edge 2 extra nodes (these correspond to the nodes of V_2 and V_3), by adding to each node in V with 3 neighbours one extra neighbour (these correspond to nodes of V_4) and by adding to each node in V with 2 neighbours 2 extra neighbours (nodes of V_4 and V_5). In fig. 5.1. we show an example of this transformation. Although G_W is a subgraph of a (6k+2)-cube and not of a grid, we draw it as a subgraph of a grid, for convenience.

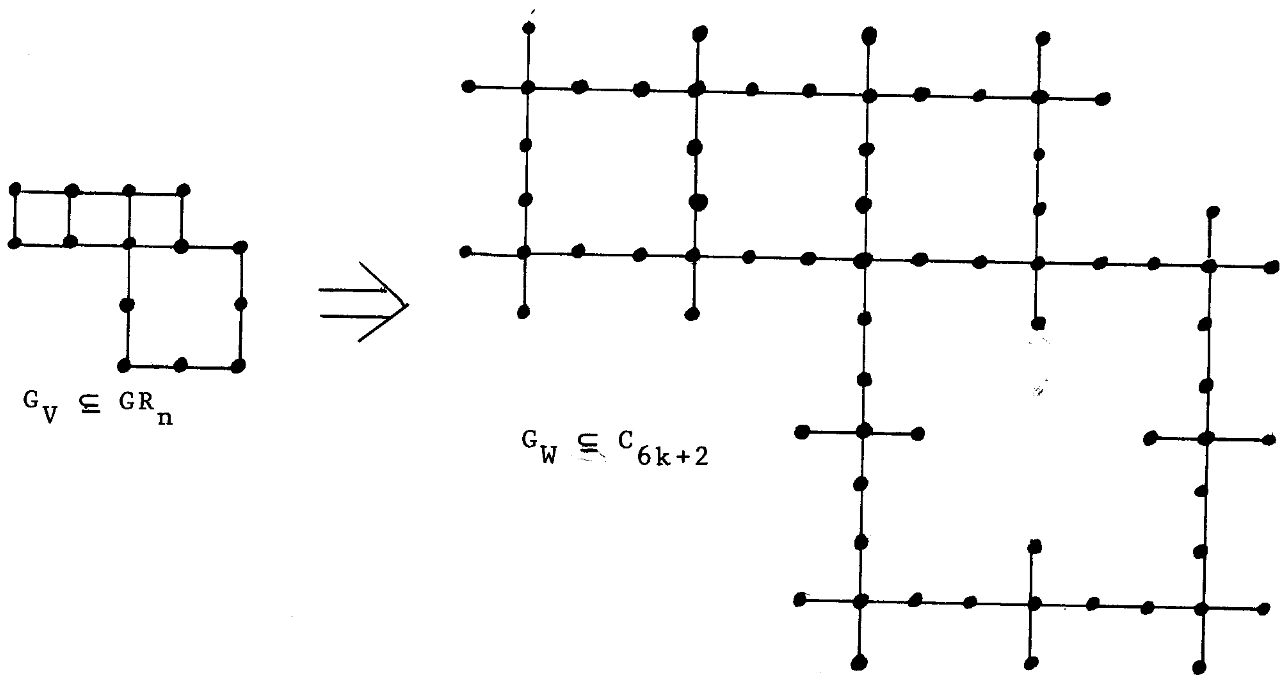
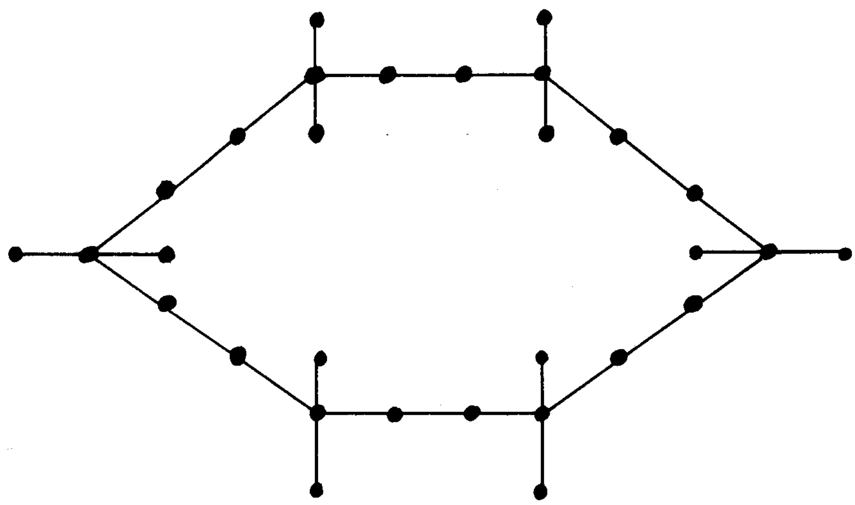


fig. 5.1.

Let $G=(V_G,E_G)$, with $V_G=\{v_{i,j} \mid 0 \le i \le |V|-1$ and $1 \le j \le 5\}$, and $E_G=\{(v_{i,1},v_{i,j}) \mid v_{i,1},v_{i,j} \in V_G$ and $2 \le j \le 5\} \cup \{(v_{i,2},v_{i+1,5}) \mid v_{i,2}, v_{i+1,5} \in V_G\}$, where + is the addition modulo $|V|$. For example, if $|V|=6$, then G is the graph of fig. 5.2. Notice $|V_G|=5.|V|=|W|$.



G, with $|V|=6$.

fig. 5.2.

Claim 5.1.1. G is isomorphic to a (spanning) subgraph of G_W if and only if G_V has a Hamiltonian circuit.

Proof. Let G_V have a Hamiltonian circuit. Number the successive nodes visited by this circuit $v_0, v_1, \ldots, v_{|V|-1}$. So $v_i \in V$ and v_i is adjacent to $v_{(i+1)\mathrm{mod}|V|}$. Now we can give a subgraph isomorphism $\tilde{f}$ of G into G_W. Let $\tilde{f}(v_{i,1}) = g \circ f(v_i)$, for all i, $0 \le i \le |V|-1$. We now map the nodes $v_{i,2}$ and $v_{i+1,5}$ on the two nodes between $g \circ f(v_i)$ and $g \circ f(v_{i+1})$. (This is a node in V_2 and a node in V_3.) There are two other nodes adjacent to $g \circ f(v_i)$ and we map $v_{i,3}$ and $v_{i,4}$ on these nodes. In this way an isomorphism of G to a (spanning) subgraph of G_W is obtained.

If $\tilde{f}$ is an isomorphism of G to a (spanning) subgraph of G_W then consider the row $\tilde{f}(v_{0,1}), \tilde{f}(v_{1,1}), \ldots, f(v_{|V|-1,1})$. Each of these nodes $\tilde{f}(v_{i,1})$ must have at least 4 neighbours in G_W, so is of the form $x_1x_1x_1x_2x_2x_2\ldots\ldots x_{2k}x_{2k}x_{2k}00$ and there is a node w of GR_{2k} with $\tilde{f}(v_{i,1}) = g \circ f(w)$. Let $w_0, \ldots, w_{|V|-1}$ be the nodes in GR_{2k} such that $g \circ f(w_i) = v_{i,1}$ for all i, $0 \le i \le |V|-1$. w_i is adjacent to $w_{(i+1)\mathrm{mod}|V|}$ (this is because $\tilde{f}(v_{i,2})$ and $\tilde{f}(v_{(i+1)\mathrm{mod}|V|,5})$ must be adjacent), so $w_0, w_1, \ldots, w_{|V|-1}$ form a Hamiltonian circuit in $V \subseteq GR_n$. □

Let m=6k+2. To complete the proof we will use basically the same technique as in section 4: we take a graph G' with the property that, for every emulation f of G' on C_{m-1}, if f maps at most c nodes of G' upon each node of C_{m-1}, f maps c-1 nodes on the nodes of W and c nodes on the nodes of $C_m \backslash W$. To this graph we add G, which will have to be mapped upon the nodes of W.

Choose a $x' \in \binom{0}{1}^{m+1}$, such that $x'_1 \ldots x'_m \in H$ and $x'_{m+1}=1$, and $x'_1 \ldots x'_m$ has degree 4 in G_H. Now let $G_*=(V_*,E_*)$ with $V_*=V_G \cup \{v^i_{x_1..x_{m+1}} \mid x_1..x_{m+1} \in \binom{0}{1}^{m+1}$ and $1 \leq i \leq c$ and ($i \neq c$ or $x_1..x_m \notin W$ or $x_{m+1} = 1$)} and $E_* = E_G \cup \{(v^i_x,v^j_y) \mid v^i_x,v^j_y \in V_*-V_G$ and x is adjacent to y in $C_{m+1}\} \cup \{(v_{0,1}, v^c_{x'})\}$.

Notice that G_* is connected and $|V_*|=c.2^{m+1}$.

<u>Claim 5.1.2.</u> G_* can be uniformly emulated on C_{m+1} if and only if G is isomorphic to a (spanning) subgraph of G_W.

<u>Proof</u>. If h is an isomorphism of G_* to a (spanning) subgraph of G_W, then we can suppose without loss of generality that $h(v_{0,1})=x'$. (Notice that $h^{-1}(x')$ must be a node $v_{i,1}$ for some i, $0 \leq i \leq |V|-1$, and use the symmetry of G.) Now let $f(v_{i,j}) = h(v_{i,j}).0$ for all $v_{i,j} \in V_G$ and $f(v^i_x) = x$ for all $v^i_x \in V_*-V_G$. One can easily check that f is a uniform emulation of G_* on C_{m+1}.

Now suppose f is a uniform emulation of G* on C_{m+1}. We will show that G is isomorphic to a subgraph of G_W. We first need:

<u>Claim 5.1.2.1.</u> $\forall i,j,\ 1 \leq i,j \leq c\ \forall x_1..x_m \in \binom{0}{1}^m\ f(v^i_{x_1...x_m1}) = f(v^j_{x_1..x_m1})$.

<u>Proof</u>. $f(v^i_{x_1...x_m1})$ and $f(v^j_{x_1..x_m1})$ are equal or adjacent. If they are adjacent then every node $f(v^\alpha_{x_1...\bar{x}_{i_0}...x_m1})$ ($1 \leq \alpha \leq c$, $1 \leq i_0 \leq m$) must be adjacent or equal to $f(v^i_{x_1...x_m1})$ and to $f(v^j_{x_1..x_m1})$. Because C_{m+1} does not contain triangles (cycles with length 3), each of these nodes

must be equal to $f(v^i_{x_1...x_m1})$ or to $f(v^j_{x_1..x_m1})$. So at least $mc+2$ nodes are mapped upon the 2 nodes $f(v^i_{x_1...x_m1})$, $f(v^j_{x_1..x_m1})$. This contradicts uniformity. Hence $f(v^i_{x_1...x_m1}) = f(v^j_{x_1..x_m1})$. □

Definition. For $0 \le p \le n$, a p-face of C_n is any subgraph of 2^p nodes of C_n that have identical bits in n-p corresponding positions (i.e.: a p-face of C_n is a subgraph of C_n, isomorphic to C_p).

Consider $A = \{v^1_{x_1..x_m1} \mid x_1..x_m \in (\frac{0}{1})^m\}$. There do not exist $v,w \in A$, $v \ne w$ and $f(v)=f(w)$, else at least 2c nodes are mapped upon $f(v)=f(w)$ (use claim 5.1.2.1.) This means that f(A) is isomorphic to A, so is an m-face of C_{m+1}. Without loss of generality we may suppose $f(v^1_{x_1..x_m1}) = x_1..x_m1$ for all $x_1..x_m \in (\frac{0}{1})^m$. Now for all i, $1 \le i \le c$, $x_1..x_m \in (\frac{0}{1})^m$ $f(v^i_{x_1..x_m1}) = x_1..x_m1$. For all i, $1 \le i \le c$, $x_1..x_m \in (\frac{0}{1})^m$ with $v^i_{x_1..x_m0} \in V_*$ we have that $f(v^i_{x_1..x_m0})$ must be adjacent or equal to $f(v^i_{x_1..x_m1}) = x_1..x_m1$. Due to the uniformity of f, $f(v^i_{x_1..x_m0})$ cannot be mapped upon a node in $\{y_1..y_m1 \mid y_1..y_m \in (\frac{0}{1})^m\}$, so $f(v^1_{x_1..x_m0}) = x_1..x_m0$.

Let $B = \{x_1...x_m0 \mid x_1...x_m \in W\}$. We now have that each node in $V_{m+1} \backslash B$ has c nodes of V_*-V_G mapped upon it by f and each node in B has c-1 nodes of V_*-V_G mapped upon it by f. So the nodes of V_G must be mapped upon nodes of B; f, restricted to V_G is a bijection of V_G to B. Because f must preserve adjacencies and G_B (the subgraph of C_{m+1} induced by B) is graph-isomorphic to G_W, G is isomorphic to a (spanning) subgraph of G_W. The function ψ: $x \to f(x)|_m$ is a graphisomorphism $V_G \to W$. □

Combining claim 5.1.1. and 5.1.2. and noting that every construction can be done in time polynomial in $|V|$, we have a polynomial time transformation from the HAMILTONIAN CIRCUIT IN A GRID GRAPH problem to the c-UNIFORM EMULATION ON A CUBE-problem. Hence the latter is NP-complete. □

Corollary 5.2. The following problem is NP-complete:

[SUBGRAPH ISOMORPHISM IN A CUBE GRAPH]

Instance: A connected, undirected graph $G=(V_G,E_G)$ and a set of nodes $W \subseteq C_n$

Question: Is G isomorphic to a subgraph of G_W, the subgraph of C_n induced by W?

6. Uniform emulation on the shuffle-exchange network. The shuffle-exchange network was proposed initially by Stone [9] and has been successfully used as the interconnection network underlying a variety of parallel processing algorithms. There are two slightly different types of graphs, both realizing Stone's concept of a shuffle-exchange network. We will use the terminology of [5] and call these graphs the shuffle-exchange graph and the 4-pin shuffle. The nodes of the shuffle-exchange graph and the 4-pin shuffle are given n-bit addresses in the range $0..2^{n-1}$. In the shuffle-exchange graph there is an edge from node b to node c if and only if b can be "shuffled" (move the leading bit to tail position) or "exchanged" (flip the tail bit) into c. In the 4-pin shuffle there is an edge from node b to node c if and only if c can be reached from b by a shuffle or by a shuffle followed by an exchange. We use the following notations:

$\frac{0}{1}$: a bit that can be 0 or 1

$\bar{\alpha}$: the complement of bit α ($\bar{0}=1$, $\bar{1}=0$)

b : the n bit address $b_1..b_n$

$\bar{b}$: the address that is obtained by complementing each bit of b ($\overline{b_1..b_n} = \bar{b}_1...\bar{b}_n$)

[0] : zero or one occurrence of bit 0 (i.e. 'empty' or '0')

[1] : zero or one occurrence of bit 1 (i.e. 'empty' or '1')

(01)*: zero or more repetitions of the string 01 (as required)

(10)*: zero or more repetitions of the string 10 (as required)

The length (n) of a bitstring will always be clear from the context and is usually not given by separate indices. For example, the notation $(01)^*[0]$ for n odd will denote the string $(01)^{\lfloor n/2 \rfloor}0$. For n even it will denote the string $(01)^{\lfloor n/2 \rfloor}$.

<u>Definition</u>. The shuffle-exchange network is the graph $SE_n=(V_n,\tilde{E}_n)$ with $V_n = \{(b_1...b_n) | \forall\ 1 \le i \le n\ b_i = {0 \atop 1}\}$ and $\tilde{E}_n = \{(b,c) | b,c \in V_n$, and $(\forall\ 2 \le i \le n\ b_i = c_{i-1} \wedge b_1 = c_n)$ or $(\forall\ 1 \le i \le n-1\ b_i = c_i \wedge b_n = \overline{c_n})\}$. The 4-pin shuffle network is the graph $S_n=(V_n,E_n)$ with $E_n = \{(b,c) | b,c \in V_n \wedge (\forall\ 2 \le i \le n\ b_i = c_{i-1}\}$.

We recall the following facts from [2,3]:

<u>Lemma 6.1</u>. [2,3] For all $n \ge 1$ every graph isomorphism of S_n (or of SE_n) is one of the following list:

f: $f(b)=b$

$\overline{f}$: $\overline{f}(b)=\overline{b}$.

<u>Definition</u>. For directed graphs $G=(V,E)$ let G^R be the (directed) graph obtained by reversing the direction of the edges, i.e., $G^R=(V,E^R)$ with $E^R = \{(g',g) | (g,g') \in E\}$.

<u>Definition</u>. The inverse shuffle-exchange network is the graph $ISE_n=SE_n^R$. The inverse 4-pin shuffle is the graph $IS_n=S_n^R$.

<u>Lemma 6.2</u>. [2] f is a (uniform) emulation of G on H if and only if f is a (uniform) emulation of G^R on H^R.

<u>Theorem 6.3</u>. For every $c \in N^+$, with $c \ge 7$ the following problem is NP-complete:

[c-UNIFORM EMULATION ON A 4-PIN SHUFFLE]

<u>Instance:</u> A directed strongly connected graph $G=(V_G,E_G)$, such that there is an $n \in N^+$ with $|V_G|=c.2^n$.

<u>Question:</u> Is there a uniform emulation of G on S_n?

<u>Proof</u>. Clearly the problem is in NP. To prove NP-completeness we will transform HAMILTONIAN CIRCUIT for directed graphs with each node

involved in exactly 3 edges to this problem. As noted in section 3, this version of HAMILTONIAN CIRCUIT is NP-complete [6]. Let $c \geq 7$ be given. Note that c is the computationfactor of the uniform emulation.

Let a directed graph $G=(V,E)$ be given with each node involved in exactly 3 edges. We will construct a strongly connected graph $G^*=(V^*,E^*)$ such that G^* can be uniformly emulated on S^n (where $c.2^n=|V^*|$), if and only if G contains a Hamiltonian circuit. Without loss of generality we may suppose that G is strongly connected with no selfloops or parallel edges.

Let α be the smallest integer such that $2^{\alpha}-1 \geq |V|$. Let f_1 be an injection of V to the set $\{x_1 \ldots x_{\alpha} | \forall i \; x_i=0 \vee x_i=1 \wedge \exists i \; x_i=0\} = (\frac{0}{1})^{\alpha} \setminus \{1^{\alpha}\}$. Such an injection can easily be found in time polynomial in $|V|$. Let $n=5\alpha+7$. Let f_2 be the mapping $(\frac{0}{1})^{\alpha} \to (\frac{0}{1})^n$, given by $f_2(x) = 01^{\alpha}0x01^{\alpha+1}0x01^{\alpha}0$.

Let $W \subseteq (\frac{0}{1})^n$ be given by $W = \{y |$ y is a substring of length n of a string $01^{\alpha}0x01^{\alpha+1}0x01^{\alpha}0y01^{\alpha+1}0y01^{\alpha}0$, where there are $v, w \in V$, with $(v,w) \in E$ and $x=f_1(v)$ and $y \in f_1(w)\}$. With the subgraph of S_n induced by W we denote the graph $G_W=(W,E_W)$, with $E_W = \{(v,w) | (v,w) \in E_n, v,w \in W\}$.

Now we claim:

<u>Claim 6.3.1</u>. G has a Hamiltonian circuit, if and only if there is a cycle of $|V|.(4\alpha+5)$ nodes in G_W, that visits each node of W at most once.

<u>Proof</u>. First suppose V has a Hamiltonian circuit. Let $v_1, \ldots, v_{|V|}$ be the successive nodes on this circuit.

For every pair of nodes v_i, v_{i+1} (+ is addition modulo $|V|$) notice that every substring of length n of the string $01^{\alpha}0f_1(v_i)01^{\alpha+1}0f_1(v_i)01^{\alpha}0f_1(v_{i+1})01^{\alpha+1}0f_1(v_{i+1})01^{\alpha}0$ is an element of W. So there is a path of length $4\alpha+5$ of $f_2 \circ f_1(v_i)$ to $f_2 \circ f_1(v_{i+1})$. By adding these paths together one obtains a cycle in G_W of length

$|V|.(4\alpha+5)$. It is not difficult to check from the construction of f_1, f_2 and W that no node of W can be visited more than once, in this cycle.

Now suppose there is a cycle of $|V|.(4\alpha+5)$ nodes in G_W that visits each node of W at most once.

Note that every address of a node of W has exactly one substring $1^{\alpha+1}$. When we follow the cycle this substring moves in the addresses of the nodes we visit one place to the left, i.e. we visit successively nodes of the form $(\frac{0}{1})^{n-(\alpha+1)-m}1^{\alpha+1}(\frac{0}{1})^m$, $(\frac{0}{1})^{n-(\alpha+1)-m-1}1^{\alpha+1}(\frac{0}{1})^{m+1}$, etc. After a node of the form $1^{\alpha+1}(\frac{0}{1})^{n-(\alpha+1)}$ a node of the form $(\frac{0}{1})^{n-(\alpha+1)}1^{\alpha+1}$ is visited. This means that every $(4\alpha+5)$ steps on the cycle a node of the form $(\frac{0}{1})^{2\alpha+3}1^{\alpha+1}(\frac{0}{1})^{2\alpha+3}$ is visited. Necessarily this node is of the form $01^{\alpha}0x01^{\alpha+1}0x01^{\alpha}0 \in f_2 \circ f_1(V)$.

Now let $v_1,\ldots,v_{|V|}$ be the nodes of V, such that the cycle successively visits $f_2 \circ f_1(v_1),\ldots,f_2 \circ f_1(v_{|V|})$ after every $4\alpha+5$ steps. There is a path of length $4\alpha+5$ from $f_2 \circ f_1(v_i)$ to $f_2 \circ f_1(v_{i+1})$; due to the construction of f_1, f_2, and W there now must be an edge $(v_i, v_{i+1}) \in E$. (+ is the addition modulo $|V|$.) So $v_1,\ldots,v_{|V|}$ form a Hamiltonian circuit. □

Now we will give the definition of $G^*=(V^*,E^*)$. G^* consists of the following parts: we take $c-1$ layers, each consisting of a copy of S_n. Between the layers there are connections between copies of the same nodes and copies of adjacent nodes. From the upper two layers we leave out the nodes that are a member of W. To this we add $R_{|V|(4\alpha+5)}[K_2]$: that is two cycles with length $|V|(4\alpha+5)$, with connections between copies of the same and adjacent nodes; and one cycle with just the right number of nodes to fill up all the remaining free places.

<u>Definitions.</u>

$V_1 = \{v_{x,i} \mid x \in (\frac{0}{1})^n,\ 1 \le i \le c-3\}$
$\quad \cup \{v_{x,i} \mid x \in (\frac{0}{1})^n,\ x \notin W,\ i=c-2 \vee i=c-1\}$

$V_2 = \{w_{i,j} \mid 0 \le i \le (4\alpha+5).|V|-1,\ j=1 \vee j=2\}$

$r = c.2^n - |V_1| - |V_2|$
$V_3 = \{z_i \mid 0 \le i \le r-1\}$
$V^* = V_1 \cup V_2 \cup V_3$. Let an arbitrary $v^* \in V$ be given.

$$E^* = \{(v_{x,i}, v_{y,j}) \mid v_{x,i}, v_{y,j} \in V_1 \wedge ((x,y) \in E_n \vee (x=y \wedge i \neq j))\}$$
$$\cup \{(w_{i_1,j_1}, w_{i_2,j_2}) \mid (i_1=i_2 \wedge j_1 \neq j_2) \vee (i_2=i_1+1 \bmod (4\alpha+5)|V|) \text{ and } w_{i_1,j_1}, w_{i_2,j_2} \in V_2\}$$
$$\cup \{(z_i, z_{(i+1) \bmod r}) \mid z_i, z_{(i+1) \bmod r} \in V_3\}$$
$$\cup \{(v_{f_2 \circ f_1(v^*),1}, w_{0,1}), (w_{0,1}, v_{f_2 \circ f_1(v^*),1}), (v_{0^n,1}, z_0), (z_0, v_{0^n,1})\}$$

One may notice that $G^*=(V^*,E^*)$ is strongly connected, and $|V^*|=c.2^n$.

Claim 6.3.2. There exists a uniform emulation of G^* on S_n, if and only if there is a cycle of length $|V|(4\alpha+5)$ in G_W, that visits each node of W at most once.

Proof. First suppose there is a cycle of length $|V|.(4\alpha+5)$ in G_W, that visits each node of W at most once. We will now construct a uniform emulation f of G^* on S_n. For $v_{x,i} \in V_1$ we let $f(v_{x,i})=x$. In this way we have mapped $c-1$ nodes of V_1 on every node in $V_1 \backslash W$ and $c-3$ nodes on every node in W. From our previous observations it is clear that the cycle must use every node of the form $f_2 \circ f_1(v)$, for some $v \in V$, so it must use also $f_2 \circ f_1(v^*)$. We can now map the nodes of V_2 on the nodes of W, in the following manner: $f(w_{0,1}) = f(w_{0,2}) = f_2 \circ f_1(v^*)$. Let w^i be the i'th node after $f_2 \circ f_1(v^*)$ on the cycle. Then $f(w_{i,1}) = f(w_{i,2}) = w^i$. In this way adjacencies are preserved and every node in $(\frac{0}{1})^n$ has $c-1$ or $c-3$ nodes of $V_1 \uplus V_2$ mapped upon it. We now use that S_n has a Hamiltonian circuit (this follows from the existence of binary the Bruyn sequences [4]). The successive nodes of V_3 are mapped on the successive nodes of this Hamiltonian circuit, starting with $f(z_0)= 0^n$. However when we arrive in a node that has $c-3$ nodes of $V_1 \uplus V_2$ mapped upon it, we map 3 successive nodes of V_3 on this node. In this way a uniform emulation of G^* on S_n is obtained. □

Now suppose there exists a uniform emulation f of G* on S_n.

<u>Claim 6.3.2.1</u>. For all $v_{x,i}$, $v_{x,j} \in V_1$ $f(v_{x,i}) = f(v_{x,j})$.

<u>Proof</u>. Suppose there are $v_{x,i}$, $v_{x,j} \in V_1$ with $f(v_{x,i}) \neq f(v_{x,j})$. Note that $(v_{x,i}, v_{x,j}) \in E^*$ and $(v_{x,j}, v_{x,i}) \in E^*$. So $(f(v_{x,i}), f(v_{x,j})) \in E_n$ and $(f(v_{x,j}), f(v_{x,i})) \in E_n$, hence $\{f(v_{x,i}), f(v_{x,j})\} = \{(01)^*[0], (10)^*[1]\}$.

Every node $v_{\frac{0}{1}x_1...x_{n-1},k} \in V_1$ must be mapped adjacent or equal to $f(v_{x,i})$ and to $f(v_{x,j})$, so $f(v_{\frac{0}{1}x_1...x_{n-1},k}) \in \{(01)^*[0], (10)^*[1]\}$. In the same way one shows that for all $v_{x_2..x_n\frac{0}{1},k} \in V_1$ $f(v_{x_2..x_n\frac{0}{1},k}) \in \{(01)^*[0], (10)^*[1]\}$. If $x \in \{0^n, 1^n, (01)^*[0], (10)^*[1]\}$, then notice that neither these nodes and neither their neighbours are members of W, so we have at least $3(c-1)>2c$ nodes mapped upon $\{(01)^*[0], (10)^*[1]\}$, which contradicts uniformity. If $x \notin \{0^n,1^n,(01)^*(0),(10)^*[1]\}$, then all 5 nodes $\{\frac{0}{1}x_1...x_{n-1}, x_1...x_n, x_2...x_{n-1}\frac{0}{1}\}$ are different, so there are at least $5(c-3)$ nodes mapped upon the 2 nodes $\{(01)^*[0], (10)^*[1]\}$, again contradicting uniformity. □

<u>Claim 6.3.2.2</u>. For all $v_{x,i} \in V_1$ $f(v_{x,i}) = x$
or for all $v_{x,i} \in V_1$ $f(v_{x,i}) = \bar{x}$.

<u>Proof</u>. If $x \neq y$ then $f(v_{x,1}) \neq f(v_{y,1})$. (Suppose $f(v_{x,1}) = f(v_{y,1})$. Then at least $2(c-3)>c$ nodes are mapped upon one node, contradicting uniformity. Here we use $c \geq 7$, and claim 6.3.2.1.). So the mapping $x \to f(v_{x,1})$ is an isomorphism of S_n onto itself. Using lemma 6.1. we now have that either for all $x \in (\frac{0}{1})^n$ $f(v_{x,1})=x$ or for all $x \in (\frac{0}{1})^n$ $f(v_{x,1})=\bar{x}$. With claim 6.3.2.1. the result follows. □

Without loss of generality we may suppose $f(v_{x,i})=x$ for all $v_{x,i} \in V_1$. Now we have shown that on nodes of $(\frac{0}{1})^n \backslash W$ there are $c-1$ nodes of V_1 that are mapped upon that node, and on nodes of W there are $c-3$

nodes that are mapped upon that node by f.

Claim 6.3.2.3. For all i, $0 \le i \le (4\alpha+5).n-1$ $f(w_{i,1}) = f(w_{i,2})$.

Proof. $f(w_{i,1})$ and $f(w_{i,2})$ must be adjacent to each other. It is easy to check that $f(w_{i,1}), f(w_{i,2}) \notin \{(01)^*[0], (10)^*[1]\}$ ($(01)^*[0]$, $(10)^*[1]$ are not elements of W and neither are their neighbours; now use uniformity). Hence $f(w_{i,1}) = f(w_{i,2})$. □

Now we have that uniformity forces that every node of V_2 is mapped upon a node of W. Furthermore, if $0 \le i,j \le (4\alpha+5)|V|-1$, $i \ne j$ and $f(w_{i,1}) = f(w_{j,1})$ then a contradiction with uniformity arises. This shows that the successive nodes $f(w_{i,1})$, $i=0...(4\alpha+5)|V|-1$ are mapped upon the successive nodes of a cycle of length $(4\alpha+5)|V|$ in G_W, that visits each node of W at most once. □

Corollary 6.3.3. G* can be uniformly emulated on S_n if and only if G has a Hamiltonian circuit.

The construction of G* can be done in polynomial time in $|V|$, hence the problem stated in theorem 6.3. is NP-complete. □

Corollary 6.4. For every $c \in N^+$, $c \ge 7$ the following problem is NP-complete:

[c-UNIFORM EMULATION ON AN INVERSE 4-PIN SHUFFLE]

Instance: A directed, strongly connected graph $G=(V_G,E_G)$, such that there is an $n \in N^+$ with $|V_G|=c.2^n$.

Question: Is there a uniform emulation of G on IS_n?

Theorem 6.5. For every $c \in N$, $c \ge 15$ the following problem is NP-complete:

[c-UNIFORM EMULATION ON A SHUFFLE-EXCHANGE GRAPH]

Instance: A directed, strongly connected graph $G=(V_G,E_G)$, such that there is an $n \in N^+$ with $|V_G|=c.2^n$.

Question: Is there a uniform emulation of G on SE_n?

Proof. The proof is more or less similar to that of theorem 6.3. Clearly the problem is in NP. To prove NP-completeness we transform HAMILTONIAN CIRCUIT for directed graphs with each node involved in exactly 3 edges to this problem.

Let $c \geq 15$ be given and let a directed graph $G=(V,E)$ with each node involved in exactly 3 edges be given. We may suppose that G is strongly connected. Again let α be the smallest integer such that $2^\alpha - 1 \geq |V|$ and let f_1 be an injection of V to the set $\binom{0}{1}^\alpha \setminus \{1^\alpha\}$. Let $n=12\alpha+6$. Let f_2 be the mapping $\binom{0}{1}^\alpha \rightarrow \binom{0}{1}^n$ given by $f_2(x) = 01^{2\alpha}0x\bar{x}\,01^{4\alpha+1}0x\bar{x}01^{2\alpha}$. Let $W_1 = \{y \mid$ y is a substring of length n of a string $01^{2\alpha}0x\bar{x}01^{4\alpha+1}0x\bar{x}01^{2\alpha}0y\bar{y}\ 01^{4\alpha+1}0y\bar{y}01^{2\alpha}$, where there are $v,w \in V$ with $(v,w) \in E$ and $x=f_1(v)$ and $y \in f_1(w)\}$. Let $W = W_1 \cup \{y \mid y_1 \ldots y_{n-1}\bar{y}_n \in W_1\}$. Let the subgraph of SE_n, induced by W be the graph $G_W=(W,E_W)$, with $E_w = \{(v,w) \mid (v,w) \in \bar{E}_n, v,w \in W\}$. G_W has the property that between every pair of nodes $f_2 \circ f_1(w_1)$ and $f_2 \circ f_1(w_2)$, with $(w_1,w_2) \in E$ there is a path that uses exactly $10\alpha+5$ shuffle-edges and $4\alpha+2$ exchange edges of SE_n. One can prove in a way similar to claim 6.3.1.:

Claim 6.5.1. G contains a Hamiltonian circuit if and only if there is a cycle of $|V|.(14\alpha+7)$ nodes in G_W, that visits each node of W at most once.

We will now define $G^*=(V^*,E^*)$, such that G^* can be emulated on SE_n, if and only if G has a Hamiltonian circuit. G^* consists of the following parts: we take $c-2$ layers, each consisting of a copy of SE_n, with connections between the layers between copies of the same node and copies of adjacent nodes, but from the upper five layers we leave out the nodes that are a member of W. To this we add $R_{|V|(14\alpha+7)}[K_5]$, (that is: five cycles with connections between the layers between copies of the same node and between copies of adjacent nodes), and -again- one cycle with just the right number of nodes to fill up all the remaining free places.

Definitions.

$V_1 = \{v_{x,i} \mid x \in \binom{0}{1}^n, 1 \leq i \leq c-7\}$

$\cup \{v_{x,i} \mid x \in (\frac{0}{1})^n,\ x \notin W,\ c-6 \le i \le c-2\}$

$V_2 = \{w_{i,j} \mid 0 \le i \le (14\alpha+7).|V|-1,\ 1 \le j \le 5\}$

$r = c.2^n - |V_1| - |V_2|$

$V_3 = \{z_i \mid 0 \le i \le r-1\}$

$V^* = V_1 \cup V_2 \cup V_3$. Let an arbitrary $v^* \in V$ be given.

$$E^* = \{(v_{x,i}, v_{y,j}) \mid v_{x,i}, v_{y,j} \in V_1 \wedge (x,y \in \tilde{E}_n \vee (x=y \wedge i \neq j))\}$$
$$\cup \{(w_{i_1,j_1}, w_{i_2,j_2} \mid (i_1=i_2 \wedge j_1 \neq j_2) \vee (i_2=i_1+1 \bmod (14\alpha+7)|V|)$$
$$\wedge (w_{i_1,j_1}, w_{i_2,j_2} \in V_2)\}$$
$$\cup \{(z_i, z_{(i+1) \bmod r}) \mid z_i,\ z_{(i+1) \bmod r} \in V_3\}$$
$$\cup \{(v_{f_2 \circ f_1(v^*),1}, w_{0,1}), (w_{0,1}, v_{f_2 \circ f_1(v^*),1}), (v_{0^n,1}, z_0), (z_0, v_{0^n,1})\}$$

Again $G^*=(V^*,E^*)$ is strongly connected and $|V^*|=c.2^n$.

<u>Claim 6.5.2</u>. There exists a cyclic path in SE_n that visits each node of SE_n at least once and at most twice.

<u>Proof</u>. We use the fact that there exists a Hamiltonian circuit in S_{n-1}. Let $b^0, b^1, b^2, \ldots, b^{2^{n-1}-1}$ be the successive nodes on this circuit. The following algorithm generates the desired path: (let + and − be addition and subtraction modulo 2^{n-1}).

<u>For</u> i := 0 <u>to</u> 2^{n-1}
<u>do</u> <u>begin</u> visit $b^i\ b_1^{i-1}$
  <u>if</u> $b_1^{i-1} = b_{n-1}^{i+1}$ <u>then</u> visit $b^i\ \overline{b_1^{i-1}} = b^i \overline{b_{n-1}^{i+1}}$
  visit $b^i\ b_{n-1}^{i+1}$
<u>end</u>

It is clear that this algorithm visits each node at least once and at most twice. Furthermore note that $b^{i+1}b_1^{i-1}$ can be obtained from $b^i b_{n-1}^{i+1}$ from one cyclic shift, so the successive nodes indeed form a path in SE_n. □

<u>Claim 6.5.3</u>. There exists a uniform emulation of G^* on SE_n if and only

if there is a cycle of length $|V|.(14\alpha+7)$ in G_W that visits each node of W at most once.

Proof. If there is a cycle of length $|V|.(14\alpha+7)$ in G_W that visits each node of W at most once, then we can map the nodes of V_1 and V_2 on V_n in the same way as in the proof of claim 6.3.3. Notice that in this way each node has at most c-2 nodes of V_1 V_2 mapped upon it. We can use the nodes of V_3 to fill up the remaining free places, using the circular path of claim 6.5.2.

Now suppose there exist a uniform emulation f of G* on S_n. We first need:

Claim 6.5.3.1. For all $v_{x,i}$, $v_{x,j} \in V_1$ $f(v_{x,i}) = f(v_{x,j})$.

Proof. Suppose there are $v_{x,i}$, $v_{x,j} \in V_1$ with $f(v_{x,i}) \neq f(v_{x,j})$. Then $f(v_{x,i})$ and $f(v_{x,j})$ must be mapped upon mutually adjacent nodes. By observing adjacencies one shows that every node of the form $v_{x_1...x_n,k}$, $v_{x_2...x_nx_1,k}$, $v_{x_nx_1...x_{n-1},k}$, or of the form $v_{x_1...x_{n-1}\bar{x}_n,k}$ in V_1 must be mapped upon $f(v_{x,i})$ or $f(v_{x,j})$. So at least $4(c-7) > 2c$ nodes are mapped upon 2 nodes, contradicting uniformity. □

Claim 6.5.3.2. Either for all $v_{x,i} \in V_1$ $f(v,i)=x$ or for all $v_{x,i} \in V_1$ $f(v_{x,i})=\bar{x}$.

Proof. Similar to 6.3.2.2. □

Claim 6.5.3.3. For all i, $0 \leq i \leq (14\alpha+7).|V|-1$, k_1,k_2, $1 \leq k_1,k_2 \leq 5$ $f(w_{i,k_1}) = f(w_{i,k_2})$.

Proof. Suppose there exists an i, $0 \leq i \leq (14\alpha+7).|V|-1$, such that $f(w_{i,k_1}) \neq f(w_{i,k_2})$ (for some k_1,k_2, $1 \leq k_1,k_2 \leq 5$). Then $f(w_{i,k_1})$ and $f(w_{i,k_2})$ are mutually adjacent nodes. Since there do not exist pairs of mutually adjacent nodes b,c in SE_m, such that there is a node d in

SE_m with $(b,d) \in \tilde{E}_m$ and $(c,d) \in \tilde{E}_m$ or $(d,b) \in \tilde{E}_m$ and $(d,c) \in \tilde{E}_m$ we have that $f(w_{i-1,j}) \in \{f(w_{i,k_1}), f(w_{i,k_2})\}$ $(1 \le j \le 5)$, $f(w_{i+1,j}) \in \{f(w_{i,k_1}), f(w_{i,k_2})\}$ $(1 \le j \le 5)$, and $f(w_{i,j}) \in \{f(w_{i,k_1}), f(w_{i,k_2})\}$ $(1 \le j \le 5)$. (+ and - are addition and substraction modulo $(14\alpha+7).|V|$.) There are also at least c-7 nodes of V_1 mapped upon each node of SE_m (see claim 6.5.3.2.), so $|f^{-1}(\{w_{i,k_1}, w_{i,k_2}\})| \ge 2(c-7)+15 > 2c$. This contradicts uniformity. □

In the same way as in the proof of claim 6.3.2. we can show that the successive nodes $f(w_{i,1})$ $(0 \le i \le (14\alpha+7)|V|-1)$ form a cycle in G_W that visits each node of W at most once, of length $(14\alpha+7).|V|$. □

So G* can be uniformly emulated on SE_n if and only of G contains a Hamiltonian circuit. The construction of G* can be done in time, polynomial in $|V|$, hence c-UNIFORM EMULATION ON A SHUFFLE EXCHANGE GRAPH is NP-complete, for $c \ge 15$. □

<u>Corollary 6.6</u>. For every $c \in N$, $c \ge 15$ the following problem is NP-complete:

[c-UNIFORM EMULATION ON AN INVERSE SHUFFLE-EXCHANGE GRAPH]

<u>Instance</u>: A directed, strongly connected graph $G=(V_G,E_G)$, such that there is an $n \in N^+$ with $|V_G|=c.2^n$.

<u>Question</u>: Is there a uniform emulation of G on ISE_n?

We conjecture that also for smaller computation factors c the problems stay NP-complete.

7. <u>Uniform emulation on fixed graphs</u>. Instead of fixing the computation factor c, one can also fix the hostgraph H. For certain (types of) hostgraphs H, the UNIFORM EMULATION problem still remains NP-complete. For instance, H can be fixed to the graph with 3 nodes and 2 edges.

<u>Theorem 7.1</u>. Let $n \in N^+$, $n \ge 3$. Let H be the graph obtained by removing

one edge from a totally connected graph with n nodes. The following problem is NP-complete:

Instance: A connected, undirected graph $G=(V,E)$.

Question: Is there a uniform emulation of G on H?

(Note that H is no longer part of the instance.)

Proof. Obviously the problem is in NP. To prove NP-completeness we first need the following lemma:

Lemma 7.1.1. Let $n \in N^+$, $n \geq 3$. The following problem is NP-complete:

Instance: Bipartite graph $G=(V,E)$ with $n \mid |V|$, and at least one node of V is isolated (i.e. is of degree 0).

Question: Are there two disjoint subsets $V_1, V_2 \subseteq V_1$, such that $u \in V_1$, $v \in V_2$ implies that $\{u,v\} \in E$ and $|V_1|=|V_2|=\frac{1}{n}\cdot|V|$?

Proof. Clearly the problem is in NP. To prove NP-completeness we transform the (strongly related) BALANCED COMPLETE BIPARTITE SUBGRAPH problem (hereafter abbreviated as BCBS) to it. This problem is known to be NP-complete [6, p. 196], and has the following form:

[BCBS]

Instance: Bipartite graph $G=(V,E)$, positive integer $K \leq |V|$.

Question: Are there two disjoint subsets $V_1, V_2 \subseteq V$, such that $u \in V_1$, $v \in V_2$ implies that $\{u,v\} \in E$ and $|V_1|=|V_2|=K$?

We use the following terminology: if a graph $G=(V,E)$ contains two disjoint subsets V_1, $V_2 \subseteq V$ such that $u \in V_1$, $v \in V_2$ implies that $\{u,v\} \in E$ and $|V_1|=|V_2|=K$, we say that G contains a BCBS with 2*K nodes.

Let an instance of BCBS be given. We consider three cases:

Case I: $cK>|V|$.

In this case add $cK-|V|$ extra nodes of degree 0 to G. The graph G', in this way obtained, is bipartite, has at least one node of degree 0, and contains a BCBS with $2*K = 2*\frac{1}{c}\cdot|V'|$ nodes if and only if G contains a BCBS with 2*K nodes.

Case II: $cK<|V|$.

We transform this case to case I. In polynomial time we find disjoint sets V_a, V_b such that $V = V_a \cup V_b$, and each edge $\in E$ goes between a node of V_a and a node of V_b. (V_a, V_b are the two "halves of the bipartition".)

Now add $|V|$ extra nodes to V_a and $|V|$ extra nodes to V_b and connect each of the extra nodes to each of the nodes in the other half: we have extra nodes $v_1^a, \ldots, v_{|V|}^a$, $v_1^b, \ldots, v_{|V|}^b$ and edges (v_i^a, v_j^b) $(1 \leq i,j \leq |V|)$; $(v_i^a, w)(1 \leq i \leq |V|,\ w \in V_b)$ and $(v_i^b, w')(1 \leq i \leq |V|, w' \in V_a)$. The graph $G_1 = (V_1, E_1)$ so obtained has the following properties:

- G_1 is bipartite
- G_1 contains a BCBS of $2*(K+|V|)$ nodes if and only if G contains a BCBS of $2*K$ nodes.
- $c.(K+|V|) > |V'|$

So now we have an instance of case I, and handle the graph as described there.

<u>Case III</u>: $4K = |V|$. If G does not contain isolated nodes, add one, and we are in case II.

We finally obtain in this way a graph, that has a BCBS of $2*\frac{1}{c}|V'|$ nodes if and only if G contains a BCBS of $2*K$ nodes. Note that the constructions can be carried out in time, polynomial in the size of G. □

<u>Definition</u>. The complement of the graph $G=(V,E)$ is the graph $G^c = (V, E^c)$, with $E^c = \{(v,w) | v \neq w \text{ and } (v,w) \notin E\}$.

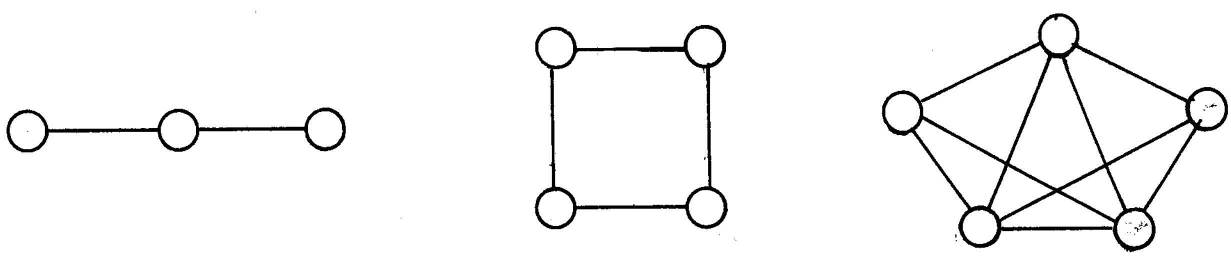
<u>Lemma 7.1.2</u>. Let $n \in N^+$, $n \geq 3$. Let $H = (V_H, E_H)$ be the graph obtained by removing one edge from a totally connected graph with n nodes. Then there is a uniform emulation of $G^c = (V, E^c)$ on H if and only if G contains a BCBS with $2*\frac{1}{c}|V|$ nodes.

<u>Proof</u>. First suppose f is a uniform emulation of G^c on H. There are two nodes $v_1, v_2 \in H$ with $(v_1, v_2) \notin E_H$. Choose $V_1 = f^{-1}(v_1)$ and $V_2 = f^{-1}(v_2)$. $v \in V_1$, $w \in V_2$ implies $(f(v), f(w)) \notin E_H \Rightarrow (v,w) \notin E^c \Rightarrow (v,w) \in E$. Hence G contains a BCBS with $2*\frac{1}{c}.|V|$ nodes.

Now suppose G contains a BCBS with $2*\frac{1}{c}.|V|$ nodes, i.e. there are sets V_1, V_2 with $|V_1|=|V_2|=\frac{1}{c}|V|$ and $v \in V_1$, $w \in V_2 \Rightarrow (v,w) \in E$. Let v_1, v_2 be the nodes in H with $(v_1,v_2) \notin E_H$. Now let $f(V_1)=v_1$, $f(V_2)=v_2$ and map the other nodes of V in equal portions on the remaining nodes of V_H. The function f so obtained is a uniform emulation: suppose $(v,w) \in E^c$ and $f(v) \neq f(w)$ and $(f(v),f(w)) \notin E_H$. Then either $f(v)=v_1$ and $f(w)=v_2$ or $f(v)=v_2$ and $f(w)=v_1$, hence $v \in V_1$ and $w \in V_2$ or $w \in V_1$ and $v \in V_2$, so $(v,w) \in E$. This contradicts $(v,w) \in E^c$. □

Lemma 7.1.2. gives a simple transformation, (that can be carried out in polynomial time), of the problem stated in lemma 7.1.1. to the considered version of UNIFORM EMULATION. Hence the latter is NP-complete. □

In figure 7.1. some examples of possible choices for H are given. We conjecture that UNIFORM EMULATION is NP-complete for every fixed, connected but not totally connected graph H. (If H is a totally connected graph with n nodes, then every graph with c.n nodes can be uniformly emulated on H.)



n=3 n=4 n=5

fig. 7.1. Possible choices for H in theorem 7.1.

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