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# DIAMETER INCREASE CAUSED BY EDGE DELETION

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Abstract. We consider the following problem: Given positive integers k and D, what is the maximum diameter of the graph obtained by deleting k edges from a graph G with diameter D, assuming that the resulting graph is still connected. For undirected graphs G we prove an upper bound of (k+1)D and a lower bound of (k+1)D-k for even D and of (k+1)D-2k+2 for odd D≥3. For the special cases of k=2 and k=3, we derive the exact bounds of 3D-1 and 4D-2, respectively. For the special case of D=1 we prove an exact bound on the resulting maximum diameter of order $\Theta(\sqrt{k})$. For directed graphs G, the bounds depend strongly on D: for D=1 and D=2 we derive exact bounds of $\Theta(\sqrt{k})$ and of 2k+2, respectively, while for D≥3 the resulting diameter is in general unbounded in terms of k and D. Finally, we prove several related problems NP-complete.

1. Introduction. Consider a communication network with a certain diameter D (the maximum number of links over which a message between two nodes must travel). In this paper we consider the question what maximum diameter can result if a certain number of links go down, assuming the network remains connected. The answer to this question is important if we want to kill broadcast messages in an unreliable network after they have traveled over a specific number of links. Clearly this number can be D when the network is completely reliable because every node in the network can be reached within D steps. By modeling the interconnection structure of the network by a graph, the

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question can be rephrased as follows: Given positive integers k and D, what is the maximum diameter of the graph obtained by deleting k edges from a graph G with diameter D, assuming that the resulting graph is still connected.

For the case of undirected graphs, Plesnik [3] was the first to note that the deletion of one edge from a graph can at most double the diameter of the graph, and that this bound is best possible. Chung and Garey [1] studied the problem in more detail. They proved a lower bound for the maximal resulting diameter of $(k+1)(D-3)$ and an upper bound of $(k+1)D+k$. In this paper we improve the bounds as follows. We derive an upper bound of $(k+1)D$. For even D, we prove a lower bound of $(k+1)D-k$, while for odd $D\geq 3$, we prove a lower bound of $(k+1)D-2k+2$. The results are proved in section 2.

In section 3 we discuss some special cases. For k=2 and k=3 we derive exact bounds of $3D-1$ and $4D-2$ respectively. For the case D=1, (i.e., G is a complete graph) we prove an exact bound of order $\Theta(\sqrt{k})$.

In section 4 we deal with the corresponding problem for directed graphs, now demanding that the resulting graph is strongly connected. The results now depend critically on D: for D=1 we prove an exact bound of $\Theta(\sqrt{k})$, for D=2 we prove an exact bound of $2k+2$ and for $D\geq 3$ one can bound the resulting diameter only by the number of vertices minus one.

In section 5 we prove that the following related problems are NP-complete: (a) Given k,D and an undirected graph G, determine whether there exists a connected subgraph of G, obtained by deleting k edges from G, that has diameter at least D; (b) Given k,D and an undirected graph G, determine whether there exists a supergraph of G, obtained by adding k edges to G, that has diameter at most D. We prove similar results for directed graphs.

2. General bounds on diameter increase for undirected graphs. For connected graphs $G=(V,E)$ let $d_G(x,y)$ denote the shortest distance from x to y (the smallest number of edges of any path from x to y). If the choice of G is clear from the context, we drop the subscript and write $d(x,y)$. The diameter of a (connected) graph $G=(V,E)$ is defined by

diameter(G) = $\max\{d(x,y) \mid x,y \in V\}$. Let $f(k,D)$ denote the maximum diameter of any connected graph $G'$ obtained after deleting $k$ edges from a (connected) graph $G$ with diameter $D$. We are interested in deriving precise bounds for $f(k,D)$. Let $k$ and $D$ be positive integers.

Theorem 2.1. $f(k,D) \le (k+1)D$.

Proof. Let $G$ be a connected graph with diameter $D$, and let $G'$ be a connected graph obtained by deleting $k$ edges from $G$. Assume diameter$(G') > (k+1)D$. Then there are vertices $x$ and $y$ in $G'$ such that $d_{G'}(x,y) > (k+1)D$. Let the shortest path from $x$ to $y$ be $x=x_0, x_1, x_2, \ldots, x_{d_{G'}(x,y)}=y$. We know $x_{(k+1)D+1}$ is on this path from $x$ to $y$ since $d_{G'}(x,y) \ge (k+1)D+1$. Now we have $d_{G'}(x_{iD}, x_{(i+1)D+1}) = D+1 > D$ for $0 \le i \le k$. Since the diameter of $G$ is $D$, $d_G(x_{iD}, x_{(i+1)D+1}) \le D$. Hence for each $i$, $0 \le i \le k$, there is a shorter path in $G$ from $x_{iD}$ to $x_{(i+1)D+1}$ which contains at least one of the $k$ deleted edges. Let the $k$ deleted edges be $(u_1,v_1), \ldots, (u_k,v_k)$. Define sets of deleted edges as follows: for $0 \le i \le k$ $S_i = \{(u_j,v_j) \mid (u_j,v_j)$ is contained in the shortest path from $x_{iD}$ to $x_{(i+1)D+1}$ in $G\}$. We know that for $0 \le i \le k$, $S_i \ne \emptyset$. We can represent the sets $S_i$ by a column of $k$ zero's and ones, which together form a $k \times (k+1)$ matrix $(\alpha_{ji})$ over the field GF(2): $\alpha_{ji}=1$ if $S_i$ contains $(u_j,v_j)$, and $\alpha_{ji}=0$ otherwise. Since there are more columns than rows, the columns are linearly dependent over GF(2) and there exists a non-trivial linear combination of columns over GF(2) that yields the zero vector. Since the only non-trivial coefficient in GF(2) equals 1, there are an $n \ge 1$ and indices $i_1 < i_2 < \ldots < i_n$ such that $\sum_{m=1}^{n} \alpha_{j i_m} = 0$ for all $j$ with $1 \le j \le k$. This means that in the sets $S_{i_1}, S_{i_2}, \ldots, S_{i_n}$ each deleted edge occurs an even number of times. Now we construct a graph $G''$ which intuitively condenses all the segments to and from the deleted edges of paths from $x_{iD}$ to $x_{(i+1)D+1}$ to single edges. Formally we define $G''=(V'',E'')$ with

$$V''=\{x_0, x_{i_1 D}, \ldots, x_{i_n D}, x_{(i_1+1)D+1}, \ldots, x_{(i_n+1)D+1}, x_{(k+1)D+1}\} \cup$$
$$\{u_j, v_j \mid (u_j,v_j) \text{ occurs in } S_{i_1}, \ldots, S_{i_n}\},$$

$E''=\{(x_0,x_{(i_1+1)D})$ if $i_1\neq 0$,

$(x_{(i_n+1)D+1},x_{(k+1)D+1})$ if $i_n\neq k$,

$(x_{(i_m+1)D+1},x_{i_{m+1}D})$ for $1\leq m\leq n-1$,

$(x_{i_mD},u_j)$ if the shortest path in G from $x_{i_mD}$ to $x_{(i_m+1)D+1}$ uses $(u_j,v_j)$, and the segment of that path from $x_{i_mD}$ to $u_j$ contains no deleted edges, (for $1\leq m\leq n$)

$(v_j,x_{(i_m+1)D+1})$ if the shortest path in G from $x_{i_mD}$ to $x_{(i_m+1)D+1}$ uses $(u_j,v_j)$, and the segment of that path from $v_j$ to $x_{(i_m+1)D+1}$ contains no deleted edges, (for $1\leq m\leq n$)

$(v_j,u_{j'})$ if for some m, $1\leq m\leq n$, the shortest path in G from $x_{i_mD}$ to $x_{(i_m+1)D+1}$ uses $(u_j,v_j)$ and $(u_{j'},v_{j'})$, and the segment of the path from $v_j$ to $u_{j'}$ contains no deleted edges.

$\}$.

(If in G $u_i=v_j$, for certain $i\neq j$, we define different vertices $u_i$ and $v_j$ in G''.) Hence the edges of G'' represent "deleted-edge-free" segments of shortest paths in G, combined with segments of the shortest path between x and y in G'. See for an example figure 2.1. In this example V'' is $\{x_0, x_6, x_7, x_{12}, x_{13}, x_{19}, x_{25}, u_1, v_1, u_3, v_3\}$ and $E''=\{(x_{19},x_{25}), (x_7,x_6), (x_{13},x_{12}), (x_0,u_1), (v_1,x_7), (x_6,v_1), (u_1,u_3), (v_2,x_{13}), (x_{12},u_3), (v_3,x_{13})\}$. We use these segments to find

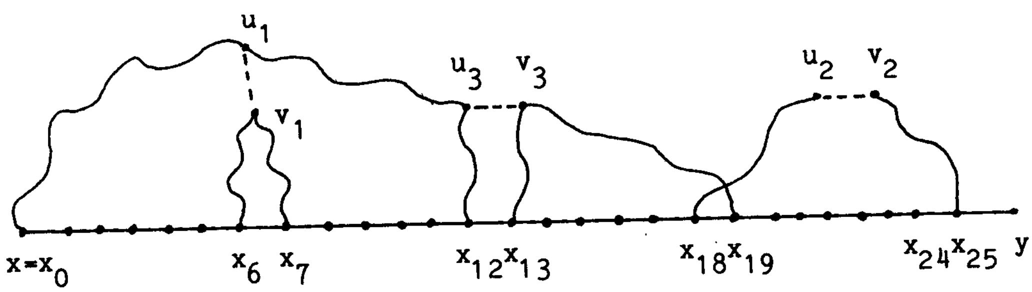



Figure 2.1. Example of a construction of a shorter path in G' from x to y, with D=6 and k=3.

a shorter path from x to y to arrive at a contradiction with the initial assumption. The idea will be that some deleted edges must have been "avoided" at least twice, and thus the path from $x_0$ to $x_{(k+1)D+1}$ in G' can be shortened by eliminating one of the bypasses.

<u>Claim</u> 2.1.1. The degree of all vertices in G'' is even, except for $x_0$ and $x_{(k+1)D+1}$ which have degree one.

<u>Proof</u>. Every vertex $x_{i_m D}$ is incident to two edges: one edge of the form $(x_{i_m D},u_j)$ and one of the form $(x_{i_m D},x_{(i_{m-1}+1)D+1})$ , except $x_0$ which has degree one. Every vertex $x_{(i_m+1)D+1}$ is incident to two edges: one edge of the form $(x_{(i_m+1)D+1},x_{i_{m+1}D})$, and one of the form $(v_j,x_{(i_m+1)D+1})$, except $x_{(k+1)D+1}$ which has degree one. The other vertices in G'' are vertices incident to a deleted edge in $S_{i_1}$, $\ldots,S_{i_n}$. Since each deleted edge occurs an even number of times in $S_{i_1}$, $\ldots,S_{i_n}$, and each occurence of $(u_j,v_j)$ gives rise to one edge adjacent to $u_j$ and one adjacent to $v_j$, the degree of these vertices is even too. Q.E.D.

Note that G'' is not necessarily connected. However, one easily sees from claim 2.1.1 that G'' must contain a connected component C which contains both vertices of odd degree $x_0$ and $x_{(k+1)D+1}$. As all other vertices have even degree, C contains an Eulerian path from $x_0$ to $x_{(k+1)D+1}$. Since each edge in C corresponds to a path in G', we have found an alternative path from $x_0$ to $x_{(k+1)D+1}$ in G', and hence an alternative path from x to y. We will now estimate the length of this path. For all $1\le m\le n$, $d_G(x_{i_m D},x_{(i_m+1)D+1})\le D$, hence $d_{G'}(x_{i_m D},u_{j_1})+ d_{G'}(v_{j_1},u_{j_2})+ \ldots +d_{G'}(v_{j_r},x_{(i_m+1)D+1})\le D-1$, where $(u_{j_1},v_{j_1})$, $(u_{j_2},v_{j_2})$, $\ldots$ $,(u_{j_r},v_{j_r})$ are the successive deleted edges on the shortest path in G from $x_{i_m D}$ to $x_{(i_m+1)D+1}$. Taking the sum over all m, $1\le m\le n$ gives

$d_{G'}(x_0,x_{i_1D}) + d_{G'}(x_{(i_n+1)D+1},x_{(k+1)D+1}) + \sum_{m=1}^{n-1} d_{G'}(x_{(i_m+1)D+1},x_{i_{m+1}D}) =$ $k+(k+1-n)(D-1) = (k+1)D-n(D-1)-1$. Thus the total length of this alternative path from x to y is at most $n(D-1)+(k+1)D-n(D-1) + d_{G'}(x_{(k+1)D+1},y) = (k+1)D-1 + d_{G'}(x_{(k+1)D+1},y) < (k+1)D + d_{G'}(x_{(k+1)D+1},y)$ which was the length of the original "shortest" path in G' between x and y. Hence we have a contradiction and conclude that diameter(G')$\leq$(k+1)D. Q.E.D.

For the lower bound on f(k,D), the results depend on whether D is even or odd.

Theorem 2.2. If D is even, $f(k,D) \geq (k+1)D-k$.

Proof. We construct a graph which attains this bound as follows. See figure 2.2 for an example. Let p = D/2 and n=(k+1)D-k. The vertices of G are $x_0$, $x_1$, ...,$x_n$. The edges of G are $(x_i,x_{i+1})$ for $0 \leq i \leq n-1$ plus the k to be deleted edges $(x_p,x_{p+1+i(D-1)})$ for $1 \leq i \leq k$. We now show that the diameter of G is D. For each $x_j$ with $j \geq 2p+1$ we can reach one of the $x_{p+1+i(D-1)}$, $1 \leq i \leq k$ in at most p-1 steps. The distance between $x_{p+1+k(D-1)}$ and $x_n$ is $(k+1)D-k-(p+1+k(D-1)) = kD+D-k-p-1-kD+k = D-1-p = p-1$. Hence we can reach $x_p$ within p steps from every vertex $x_j$, with $j \geq 2p+1$. Moreover, $x_0$ up to $x_{2p}$ are within p steps of $x_p$. Hence every pair of vertices of G is joined by a path of length at most 2p=D via $x_p$. Thus the diameter of G is D. Deleting the k edges $(x_p,x_{p+1+i(D-1)})$ for $1 \leq i \leq k$ leaves us with just the path $x_0$, $x_1$,

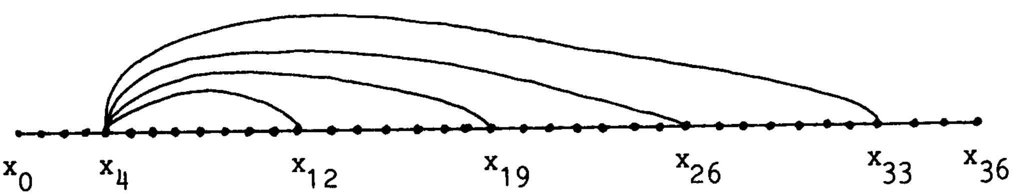



Figure 2.2.
Lower bound construction of theorem 2.2 for k=4 and D=8.

$\ldots, x_n$ which is $(k+1)D-k$ long. Hence $(k+1)D-k$ is a lowerbound for the maximal value the diameter can reach for graphs with even D. Q.E.D.

Theorem 2.3. For odd $D \geq 3$, $f(k,D) \geq (k+1)D-2k+2$.

Proof. For odd $D \geq 3$ we construct a graph similar to the one in the proof of theorem 2.2. See figure 2.3 for an example. Let $p=(D-1)/2$ and $n=(k+1)D-2k+2$. The vertices of G are $x_0, x_1, \ldots, x_n$. As edges we take $(x_i, x_{i+1})$ for $0 \leq i \leq n-1$, plus the k edges $(x_p, x_q)$ and $(x_q, x_{q+1+i(D-2)})$ for $1 \leq i \leq k-1$, to be deleted, where $q=3p+2$. For each $x_j$ with $j \geq q$ we can reach $x_q$ in p steps as in the construction of theorem 2.2, as the distance between $x_n$ and $x_{q+1+(k-1)(D-2)}$ is $(k+1)D-2k+2-(3p+2+1+(k-1)(D-2)) = 2D-2k+2-3p-3+2k-2 = 2D-3p-3 = 2(2p+1)-3p-3 = p-1$. The $x_j$ with $j \leq p$ are at most $p+1$ steps away from $x_q$, as are the $x_j$ with $p<j<q$. Hence each $x_j$ with $j \leq q$ is at most $2p+1$ steps distant from an $x_{j'}$, with $j' \geq q$. Since the $x_j$ with $p<j \leq q$ are within $p+1$ steps from $x_p$, all $x_j$, $j<q$ are within $2p+1$ steps from each other. Hence the diameter of G is $2p+1=D$, while the deletion of k edges leaves us with a path of length $(k+1)D-2k+2$. Q.E.D.

Note that this construction does not work for D=1. In the next section we derive a sharper bound for the special case that G is a complete graph (i.e., the case D=1).

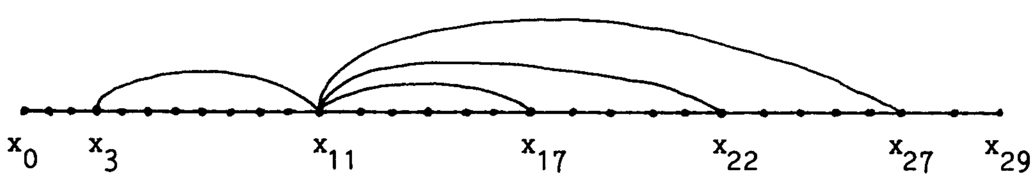



Figure 2.3.
Lower bound construction of theorem 2.3 for k=4 and D=7.

3. Bounding diameter increase for special values of k and D in undirected graphs. For the special case of k=1, Plesnik [3] already derived a best possible bound of 2D. We will derive best possible bounds for the case of k=2 and k=3, and also for the special case of D=1. For the proof we use two lemmas about the effect of adding two, respectively three, edges to a path of length n.

Lemma 3.1. Let the graph G be a path of length n. Let G' be a graph obtained by adding two edges to G, and let the diameter of G' be D. Then $n \leq 3D-1$.

Proof. Let the vertices of G be $x_0, x_1, \ldots, x_n$. Let the edges of G be $(x_i, x_{i+1})$ for $0 \leq i \leq n-1$, plus two edges $(x_i, x_{i'})$ and $(x_j, x_{j'})$. Without loss of generality let $i \leq j$, $i < i'$, $j < j'$. Now i, i', j and j' divide the path of length n into five segments of non-negative lengths a, b, c, d and e. Hence a+b+c+d+e=n. Assume $n \geq 3D$. Since the diameter of G' is D, we can derive several relations between a, b, c, d and e by computing the shortest path between several points in G'. We distinguish three cases.

Case 1. $i' \leq j$.

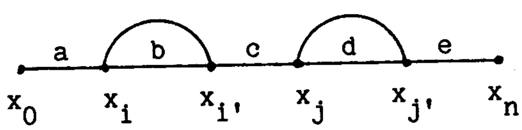



Figure 3.1.

Clearly a=i, b=i'-i, c=j-i', d=j'-j and e=n-j'. Also

$a+c+e+2 \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}b+c+\frac{1}{2}d \leq \lfloor \frac{1}{2}(b+1) \rfloor + c + \lfloor \frac{1}{2}(d+1) \rfloor \leq D$ (halfway b to halfway d),

and hence $a+b+3c+d+e \leq 3D-2$. Contradiction.

Case 2. $i \leq j < i' < j'$.

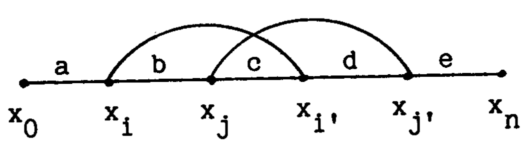



Figure 3.2.

In this case we have a=i, b=j-i, c=i'-j, d=j'-i' and e=n-j'. Now distinguish three subcases.

Case 2.1. $c+2=\min(b+1,c+2,d+1)$. Then

$a+e+c+2 \le D$ (the distance from $x_0$ to $x_n$),

$b+d \le 2\lfloor\frac{1}{2}(b+1)\rfloor+2\lfloor\frac{1}{2}(d+1)\rfloor \le 2D$ (halfway b to halfway d),

and hence $a+b+c+d+e \le 3D-2$. Contradiction.

Case 2.2. $b+1=\min(b+1,c+2,d+1)$. Then

$a+e+b+1 \le D$ (the distance from $x_0$ to $x_n$),

$c+d \le 2\lfloor\frac{1}{2}(c+1)\rfloor+2\lfloor\frac{1}{2}(d+1)\rfloor \le 2D$ (halfway c to halfway d),

and hence $a+b+c+d+e \le 3D-1$. Contradiction.

Case 2.3. $d+1=\min(b+1,c+2,d+1)$. This is symmetric to case 2.2.

Case 3. $i \le j < j' \le i'$.

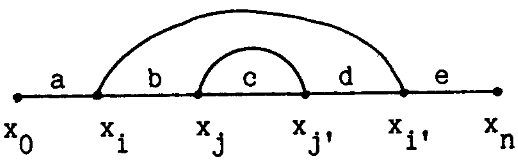



Figure 3.3.

Now we have $a=i$, $b=j-i$, $c=j'-j$, $d=i'-j'$ and $e=n-i'$. Then

$a+e+1 \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}(b+d) \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(b+d+1)\rfloor \le D$ (halfway c to halfway b+d),

and hence $a+b+c+d+e \le 3D-1$. Contradiction.

Since all three cases lead to a contradiction if we assume $n \ge 3D$, we conclude that $n \le 3D-1$. Q.E.D.

Lemma 3.2. Let G be a path of length n. Let G' be obtained by adding three edges to G, and let the diameter of G' be D. Then $n \le 4D-2$.

Since the proof of this lemma is completely analogous to the proof of the previous lemma, albeit considerably longer, it is defered to the appendix.

Theorem 3.1. $f(2,D)=3D-1$.

Proof. We first show that $3D-1$ is an upper bound. Let G be any graph with diameter D, and let G' be a connected graph obtained by deleting two edges from G. Let the diameter of G' be D', and let x and y be two vertices such that $d_{G'}(x,y)=D'$. Partition the vertices of G' into sets $X_i$ ($0 \le i \le D'$) by defining $X_i=\{u \mid d(x,u)=i\}$. Notice that all these sets are non empty. Let H and H' be the graphs obtained from G and

G', respectively, by contracting each set $X_i$ to a single vertex $x_i$ and removing any selfloops and duplicate edges. Let the diameter of H and H' be h and h', respectively. Then $h \le D$ and $h' \le D'$. Since H' simply consists of the path $x_0, x_1, \ldots, x_{D'}$, we have $h'=D'$. The graph H contains the path $x_0, x_1, \ldots, x_{D'}$ and at most two additional edges. From lemma 3.1 we know that $D' \le 3h-1$ and hence $D' \le 3D-1$.

The following constuction shows that this bound can be achieved. Define the graph G with vertices $x_0, x_1, \ldots, x_{3D-1}$ and edges $(x_i, x_{i+1})$ for $0 \le i \le 3D-2$ plus $(x_D, x_{2D})$ and $(x_{D-1}, x_{3D-1})$. It is easily seen that the diameter of this graph is D. Since deleting the two edges $(x_D, x_{2D})$ and $(x_{D-1}, x_{3D-1})$ from G results in a path of length 3D-1, we can conclude that $f(2,D) \ge 3D-1$. Q.E.D.

Theorem 3.2. $f(3,D)=4D-2$.

Proof. We use exactly the same argument as in theorem 3.1. We can use the same projection on a path in the resulting graph G' of D' long. With lemma 3.2 we now have $D' \le 4D-2$.

The following construction shows that this bound can be achieved. Define a graph G with vertices $x_0, x_1, \ldots, x_{4D-2}$ and edges $(x_i, x_{i+1})$ for $0 \le i \le 4D-3$ plus $(x_0, x_{2D-1})$, $(x_{D-1}, x_{3D-1})$ and $(x_{2D-1}, x_{4D-2})$. The diameter of this graph is D. (See figure 3.4.) Since deleting the three edges $(x_0$,x2d), $(x_{D-1}, x_{3D-1})$ and $(x_{2D-1}, x_{4D-2})$ from G results in a path of length 4D-2, we can conclude that $f(3,D) \ge 4D-2$. Q.E.D.

Finally we consider the case D=1. We prove that if we start out with a complete graph $G=K_n$ of n vertices and delete a number of edges, then the maximum diameter of the resulting graph is of a different order than suggested by theorem 2.1. We recall that f(k,1) denotes the

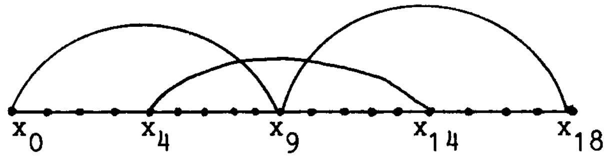



Figure 3.4. Lower bound construction of theorem 3.2 for D=5.

maximum diameter of a connected graph G' which is obtained by deleting k edges from a complete graph G. Let $f_n(k,1)$ denote the maximum diameter of a connected graph G' obtained by deleting k edges from $K_n$.

Lemma 3.3. If one can obtain a graph G' with diameter D, $1<D<n$, by deleting k edges from $K_n$, then $k \geq \frac{1}{2}D(D-1)+(D-2)(n-D-1)$.

Proof. Let x and y be vertices in G' with $d_{G'}(x,y)=D$. Let $x=x_0, x_1, \ldots, x_D=y$ be a shortest path from x to y. Hence $d(x_0,x_D)=D$. This means that the edges $(x_i,x_{i+1})$ for $0\leq i\leq D-1$ are the only edges between vertices $x_i$ and $x_j$ for $i,j\leq D$, otherwise there would have been a shorter path from $x_0$ to $x_D$. Hence all the other edges between these vertices must have been deleted. This accounts for $\frac{1}{2}D(D+1)-D=\frac{1}{2}D(D-1)$ deleted edges. Let the remaining vertices in G' be $y_1, y_2, \ldots, y_{n-D-1}$. If a vertex $y_i$ has edges to $x_j$ and $x_{j'}$, then $|j-j'|\leq 2$, because otherwise the path from $x_0$ to $x_D$ could have been shortened by going over $y_i$. Hence each $y_i$ can have edges to at most three (consecutive) x-vertices. Thus the edges to the other x-vertices must have been deleted. If, for each y-vertex, we delete the edges to the same x-vertices, we can leave all the edges between y-vertices in G', without having to fear for a shortcut between $x_0$ and $x_D$ over y-vertices. Hence $k \geq \frac{1}{2}D(D-1)+(D-2)(n-D-1)$. Q.E.D.

Lemma 3.4. $$f_n(k,1)=\begin{cases}\lfloor n+\frac{1}{2}-\sqrt{(n+\frac{1}{2})^2+4-4n-2k}\rfloor & \text{for } k\leq\frac{1}{2}(n-1)(n-2),\\ \text{undefined otherwise.}\end{cases}$$

Proof. Since the maximum number of edges we can delete from a complete graph on n vertices without necessarily disconnecting it is $\frac{1}{2}(n-1)(n-2)$, we have

$$f_n(k,1)=\max\{D\,|\,D\leq n-1,\ \tfrac{1}{2}D(D-1)+(D-2)(n-1-D)\leq k,\ \tfrac{1}{2}(n-1)(n-2)\geq k\}.$$

Let $g:\mathbb{R}\to\mathbb{R}$ be the function given by $g(D)=D(D-1)+(D-2)(n-1-D)$. $g'(D) = -D+n+\frac{1}{2}$ hence $g'(D)=0$ for $D=n+\frac{1}{2}$. Thus the function g is increasing for all $D\leq n-1$. Since $\frac{1}{2}(n-1)(n-2)=g(n-1)$, $f_n(k,1) = \max\{D|D\leq n-1,\ g(D)\leq k,\ \frac{1}{2}(n-1)(n-2)\geq k\} = \max\{D|g(D)\leq k\}$. $g(D)-k\leq 0$ implies $D\leq n+\frac{1}{2}-\sqrt{(n+\frac{1}{2})^2+4-4n-2k}$. Since the value of $f_n(k,1)$ is an integer, we have

$$f_n(k,1) = \begin{cases} \lfloor n+\tfrac{1}{2}-\sqrt{(n+\tfrac{1}{2})^2+4-4n-2k}\rfloor & \text{for } k\leq\tfrac{1}{2}(n-1)(n-2), \\ \text{undefined otherwise.} \end{cases}$$

Q.E.D.

Theorem 3.3.

$$f(k,1)=\left\lfloor \left\lceil\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}\right\rceil+\tfrac{1}{2}-\sqrt{\left(\left\lceil\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}\right\rceil+\tfrac{1}{2}\right)^2+4-4\left\lceil\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}\right\rceil-2k}\right\rfloor.$$

Proof. $f(k,1)=\max_n\{f_n(k,1)\,|\,k\leq\tfrac{1}{2}(n-1)(n-2)\}$. Since $k\leq\tfrac{1}{2}(n-1)(n-2)$ implies $2k+\tfrac{1}{4}\leq(n-1\tfrac{1}{2})^2$, we have $n\geq\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}$ and $f(k,1)=\max_n\{f_n(k,1)\,|\,n\geq\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}\}$. Let $h:R\rightarrow R$ be the function defined by $h(n)=n+\tfrac{1}{2}-\sqrt{(n+\tfrac{1}{2})^2+4-4n-2k}$.
Since $h'(n)=1-\dfrac{2n-3}{2\sqrt{(n+\tfrac{1}{2})^2+4-4n-2k}}$, we have $h'(n)\leq 0$ for $2n-3\geq 2\sqrt{(n+\tfrac{1}{2})^2+4-4n-2k}$, which inequality is true for $k\geq 1$. Hence the function h is decreasing in n, and $f(k,1) = \max_n\{\lfloor h(n)\rfloor\,|\,n\geq\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}\} = \lfloor h(\lceil\sqrt{2k+\tfrac{1}{4}}+1\tfrac{1}{2}\rceil)\rfloor$. Q.E.D.

Note that the function f(k,1) is neither monotone increasing nor monotone decreasing. For example f(6,1)=4 while f(7,1)=3. This is due to the fact that f(6,1) is obtained in a complete graph with 5 vertices, while we need to start with a complete graph with 6 vertices to ensure that the resulting graph is connected if we delete 7 edges.

4. General bounds on diameter increase for directed graphs. The problem of bounding the diameter of directed graphs after some edges are deleted turns out to be much simpler. Let g(k,D) denote the maximum diameter of a strongly connected directed graph G' which can be obtained by deleting k edges from a directed graph G with diameter D.

Theorem 4.1. Let $D\geq 3$ and $k\geq 1$. There exists a strongly connected directed graph G', which is the result of deleting k edges from a directed graph G=(V,E) with diameter D, such that diameter(G')=|V|-1.

Proof. The largest diameter any strongly connected directed graph on n vertices can have, is n-1. This is clear by the example of a directed cycle on n vertices. The fact that this bound can be reached even for k=1 is shown by the following construction. Let the vertices of G be $x_0, \ldots, x_{(D-2)m}$, and the edges $(x_0, x_{(D-2)m})$, $(x_i, x_{i+1})$ for $0 \le i \le (D-2)m-1$, $(x_{(D-2)m}, x_i)$ for $0 \le i \le (D-2)m-1$ and $(x_{(D-2)i}, x_0)$ for $1 \le i \le m-1$. See figure 4.1 for an example. The diameter of G is D, since from each $x_i$ we can reach some vertex $x_{(D-2)j}$ in at most D-3 steps, from where we need one step to reach $x_0$, one more to reach $x_{(D-2)m}$ and finally one step more to reach any other $x_{i'}$. However, if we delete the edge $(x_0, x_{(D-2)m})$, the only way to get to $x_{(D-2)m}$ from $x_0$ is along the path $x_1, x_2, x_3, \ldots$ . Hence the diameter becomes (D-2)m. Q.E.D.

Theorem 4.1 implies that in general g(k,D) is not bounded in terms of k and D, for D≥3. We can derive better results for D=1 and D=2. As in the undirected case, we first count the number of edges we need to delete from a complete directed graph with n vertices to arrive at a graph with a diameter of n-1.

Lemma 4.1. In order to obtain a strongly connected graph G' with diameter n-1 by deleting k edges from a complete directed graph with n vertices, it is necessary that $\frac{1}{2}(n-1)(n-2) \le k \le n(n-2)$.

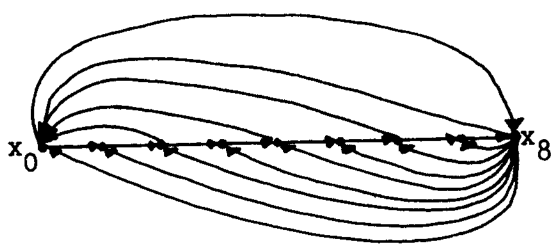



Figure 4.1. Lower bound construction of theorem 4.1 of a graph with 9 vertices with diameter 4.

Proof. Since a complete directed graph with n vertices contains $n(n-1)$ edges, we can delete at most $n(n-1)-n=n(n-2)$ of the edges without disconnecting the graph. Let $x_1$ and $x_n$ be two vertices in G' with $d(x_1,x_n)=n-1$, and let $x_1, x_2, \ldots, x_n$ be a shortest path from $x_1$ to $x_n$. Thus all edges $(x_i,x_j)$ with $j>i+1$ must have been deleted, otherwise $x_1, x_2, \ldots, x_n$ would not have been a shortest path. Hence we must at least delete $\frac{1}{2}(n-1)(n-2)$ edges. Q.E.D.

Theorem 4.2. $g(k,1) = \lfloor\sqrt{2k+\frac{1}{4}}+\frac{1}{2}\rfloor$.

Proof. Since $\frac{1}{2}(n-1)(n-2)<n(n-2)$ we can conclude from lemma 4.1 that $\frac{1}{2}(n-1)(n-2)\leq k<\frac{1}{2}n(n-1)$ implies $g(k,1)\geq n-1$. Since we can apply the proof of the lower bound of k in lemma 4.1 to any complete directed graph, and not only to a complete directed graph with n vertices, we conclude $g(k,1)\geq n-1$ implies $k\geq\frac{1}{2}(n-1)(n-2)$. Hence $k\geq\frac{1}{2}(n-1)(n-2) \Leftrightarrow g(k,1)\geq n-1$ and thus $g(k,1)=n-1$ for all k such that $\frac{1}{2}(n-1)(n-2)\leq k<\frac{1}{2}n(n-1)$. Hence $(n-1\frac{1}{2})^2 \leq 2k+\frac{1}{4} < (n-\frac{1}{2})^2$ and $n-1 \leq \sqrt{2k+\frac{1}{4}}+\frac{1}{2} < n$ so $n-1=\lfloor\sqrt{2k+\frac{1}{4}}+\frac{1}{2}\rfloor$. Thus $g(k,1)=\lfloor\sqrt{2k+\frac{1}{4}}+\frac{1}{2}\rfloor$. Q.E.D.

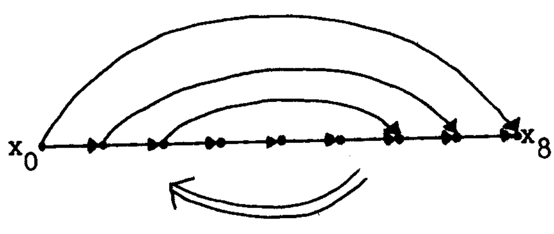
Next we consider the case D=2.

Lemma 4.2. Let G be a strongly connected directed graph on n+1 vertices with diameter n. Let G' be obtained by adding k edges to G, and let the diameter of G' be 2. Then $n\leq 2k+2$.

Proof. Let $x_0$ and $x_n$ be two vertices in G with $d(x_0,x_n)=n$, such that the shortest path from $x_0$ to $x_n$ is $x_0,x_1, \ldots,x_n$. Hence the only edges $(x_i,x_j)$ with $j>i$ that G can contain are $(x_i,x_{i+1})$. So all edges $(x_i,x_j)$ in G' with $j>i+1$ must be one of the k added edges. Let $x_i$ be the lowest numbered vertex that has no edge $(x_i,x_j)$ with $j>i+1$. Since the diameter of G' is two, we must be able to reach every other vertex in two steps from $x_i$. Hence we need edges $(x_{j'},x_j)$ with $j>j'+1$ for all j with $i+3\leq j\leq n$, since these vertices could not be reached in two steps from $x_i$ in G. Thus we have $k\geq i$ ($i'<i$ implies there is an edge $(x_{i'},x_j)$ with $j>i'+1$) and $k\geq n-(i+3)+1=n-2-i$ ($i'\geq i+3$ implies there is an edge $(x_j, x_{i'})$ with $i'>j+1$). So $2k \geq n-2$ and $n \leq 2k+2$. Q.E.D.

<u>Theorem</u> 4.3. $g(k,2) = 2k+2$.

<u>Proof</u>. We first show that 2k+2 is an upper bound. Let G be any strongly connected directed graph with diameter two, and let G' be a strongly connected directed graph obtained from G by deleting k edges. Let the diameter of G' be D', and let x and y be two vertices with $d_{G'}(x,y)=D'$. Partition the vertices in G' into sets $X_i$ $(0 \leq i \leq D')$ by setting $X_i=\{u | d_{G'}(x,u)=i\}$. Notice that all these sets are non empty. Let H and H' be the graphs obtained from G and G' respectively, by contracting each set $X_i$ to a single vertex $x_i$ and removing any selfloops and duplicate edges. Let the diameter of H and H' be h and h' respectively. Then $h \leq 2$ and $h' \leq D'$. Since H' consists of the path on $x_0, x_1, \ldots, x_{D'}$ and some edges $(x_i, x_j)$ with $j<i$, $h'=D'$. H consists of the path $x_0, x_1, \ldots, x_{D'}$, some edges $(x_i, x_j)$ with $j<i$ as in H', and at most k additional edges. From lemma 4.2 we know that if $h=2$, then $h' \leq 2k+2$. Since it is clear that we need to add even more edges to get $h=1$, we can conclude from $h \leq 2$, $h' \leq 2k+2$. Hence $D' \leq 2k+2$. The following construction shows that this bound can be achieved. See figure 4.2 for an example. Let the vertices of G be $x_0, \ldots, x_{2k+2}$, with edges $(x_i, x_{i+1})$ for $0 \leq i \leq 2k+1$, $(x_i, x_j)$ for $0 \leq j < i \leq 2k+2$ and the k to be deleted edges $(x_i, x_{2k+2-i})$ for $0 \leq i \leq k-1$. It is clear that if we delete those k edges, the diameter becomes 2k+2. That the diameter of the original graph is two, is clear if we note that a path between any




<u>Figure</u> 4.2. Lower bound graph for theorem 4.3 for 9 vertices and k=3.

two vertices can always be made by either one edge "back" eventually followed by one edge "forward", or by one edge "forward" eventually followed by one edge "back". (There is one exeption: from $x_k$ to $x_{k+2}$ we need two edges forward.) Q.E.D.

## 5. NP-complete problems related to edge deletion and diameter bounds.

We now consider several related problems and prove that they are NP-complete. For all preliminaries from the theory of NP-completeness we refer to Garey and Johnson [2].

Theorem 5.1. The following problem is NP-complete.

[MINIMUM DIAMETER EDGE DELETION]

Instance : $k,D\in N^+$, a connected graph $G=(V,E)$.

Question : Can we obtain a connected subgraph G' of G by deleting k edges from G, such that G' has a diameter of at least D?

Proof. It is easy to see that the problem is in NP, since we can guess the k edges to delete and compute the diameter of G' in polynomial time. To prove NP-completeness we use a polynomial transformation from the HAMILTONIAN PATH problem. Let a graph $G=(V,E)$ be given. G contains a Hamiltonian path if and only if G has a connected subgraph G' with $|V|-1$ edges and diameter $D=|V|-1$ (G' is a path). So by choosing $k=|E|-(|V|-1)$ and $D=|V|-1$ we have a reduction from HAMILTONIAN PATH to MINIMUM DIAMETER EDGE DELETION. Hence the latter problem is NP-complete. Q.E.D.

Theorem 5.2. The following problem is NP-complete.

[MAXIMUM DIAMETER EDGE ADDITION]

Instance : $k,D\in N^+$, a connected graph G.

Question : Can we obtain a supergraph G' of G by adding k edges to G, such that G' has a diameter of at most D?

Proof. This problem is in NP because we can guess which k edges to add to get G', and then compute the diameter of G' in polynomial time.

To prove NP-completeness we use a polynomial transformation from a variant of EXACT COVER BY 3-SETS (X3C). X3C is the following problem:

[X3C]

Instance : Set X with $|X|=3q$ and a collection C of 3-element subsets of X.

Question : Does C contain an exact cover for X, i.e., a subcollection $C' \subset C$ such that every element of X occurs in exactly one member of C'?

X3C is NP-complete (see Garey and Johnson [2]). We use a variant of X3C which is clearly equivalent with X3C, and hence also NP-complete.

Instance : Set X with $|X|=3q$ and a collection C of 3-element subsets of X such that each element of X occurs in at least one member of C.

Question : Does C contain a cover for X, i.e., a subcollection $C' \subset C$ with $|C'|=q$, such that every element of X occurs in at least one member of C'?

Let an instance of this latter problem be given. We will construct a graph G such that we can obtain a supergraph of G with diameter at most three by adding q edges, if and only if C contains a subset $C' \subset C$ with $|C'|=q$ such that every element of X occurs in at least one member of C'. So we take D=3 and k=q. Let G=(V,E) be as follows (see figure 5.1). Let $|C|=n$.

$V=\{x_1, \ldots, x_{3q}, y_1, \ldots, y_{2q+1}, c_1, \ldots, c_n,$ a and b$\}$.

$E=\{(c_i,c_j)$ for $1 \le i < j \le n$, $(x_i,c_j)$ if $x_i \in C_j$,

$(b,y_i)$ for $1 \le i \le 2q+1$, $(c_1,a)$ and $(a,b)\}$.

Claim 5.2.1. C contains a subset $C' \subset C$ with $|C'|=q$ such that every element of X occurs in at least one member of C', if and only if G has a supergraph with diameter at most three obtained by adding q edges.

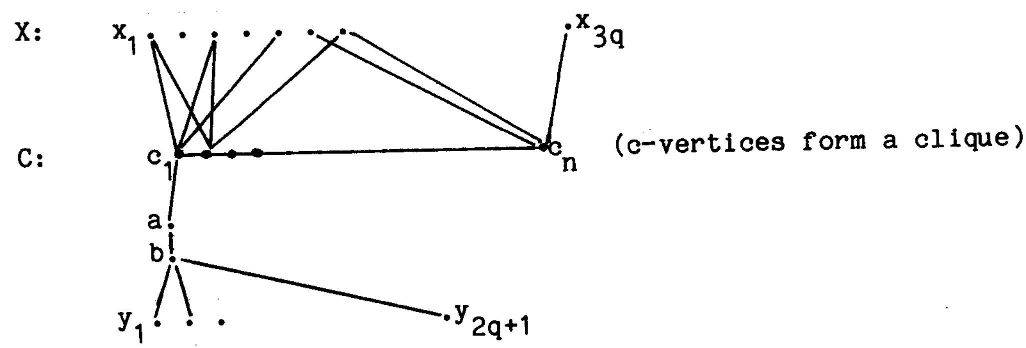



Figure 5.1. Graph used in proof of theorem 5.2.

Proof. Suppose C contains such a subset C'. Now we add the following edges to G: $(b,c_i)$ if $c_i \epsilon C'$. This are exactly q edges. The diameter of the resulting graph is three, since the distance from b to any $x_j$ is two via the proper $(b,c_i)$. Conversely, suppose G has such a supergraph. Since there are 2q+1 y-vertices, there is at least one $y_i$ which is not incident with one of the q added edges. Hence the shortest path from $y_i$ to any $x_j$ contains b. Thus the distance from b to any $x_j$ is at most two. Hence for every $x_j$ we must either have an edge $(b,x_j)$ or an edge $(b,c_{j'})$ for a $c_{j'}$ such that $x_j \epsilon C_{j'}$. Suppose there are s edges of the type $(b,x_j)$ and t edges of the type $(b,c_j)$. This gives us at most s+3t x-vertices at distance two from b. Since $|X|=3q$ and $s+t \leq q$, we have $s=0$. Define $C'=\{c \epsilon C \mid \text{there is an edge } (b,c)\}$. $|C'|=t \leq q$. Furthermore, since every x-vertex has distance two to b, it must have an edge to a $c \epsilon C'$. Hence $|C'|=q$ and C' is the desired cover of X. Q.E.D.

Finally note that G can be constructed in polynomial time in the size of X, given an instance of the "cover by 3-sets" problem. Hence the MAXIMUM DIAMETER EDGE ADDITION problem is NP-complete. Q.E.D.

Note that in the results of theorem 5.2 we can even take D=3, fixed. However, if we fix k, then the problem is polynomially solvable in time exponential in k, but polynomial in the size of G).

Theorem 5.3. The following problem is NP-complete.

[DIRECTED MINIMUM DIAMETER EDGE DELETION]

Instance : $k,d \in N^+$, a strongly connected directed graph G.

Question : Can we obtain a strongly connected directed subgraph G' of G by deleting k edges from G such that G' has a diameter of at least D?

Proof. The proof is very similar to the proof of theorem 5.1, and uses a polynomial transformation from DIRECTED HAMILTONIAN CIRCUIT. Q.E.D.

Theorem 5.4. The following problem is NP-complete.

[DIRECTED MAXIMUM DIAMETER EDGE DELETION]

Instance : $k,D \in N^+$, a strongly connected directed graph G=(V,E).

Question : Can we obtain a strongly connected subgraph G' of G by deleting k edges from G, such that G' has a diameter of at most D?

Proof. Similar to the proof of theorem 5.2. Q.E.D.

Theorem 5.5. The following problem is NP-complete.

[DIRECTED MAXIMUM DIAMETER EDGE ADDITION]

Instance : $k,D \in N^+$, a strongly connected directed graph G.

Question : Can we obtain a supergraph G' of G by adding k edges to G, such that G' has a diameter of at most D?

Proof. Analogous to the proof of theorem 5.2. Let $|C|$=n. The directed graph G=(V,E) is now defined as:

$V=\{x_1, \ldots, x_{3q}, y_1, \ldots, y_{2q+1}, c_1, \ldots, c_n$, a and b$\}$.

$E=\{(c_i,c_j)$ for $1 \le i,j \le n$, $i \ne j$, $(c_i,b)$ for $1 \le i \le n$,

$(x_i,c_j)$ and $(c_j,x_i)$ if $x_i \in C_j$, $(c_1,a)$, $(a,c_1)$,

$(b,y_i)$ and $(y_i,b)$ for $1 \le i \le 2q+1$, (a,b) and (b,a)$\}$.

Q.E.D.

## 6. References.

[1] Chung, F.R.K., and M.R.Garey, Diameter bounds for altered graphs, J. Graph Theory 8 (1984) 511-534.

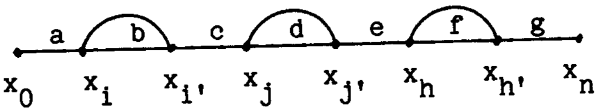
[2] Garey, M.R., and D.S.Johnson, Computers and Intractability, a guide to the theory of NP-completeness, W.H.Freeman, San Francisco, Calif. 1979.

[3] Plesník, J., Note on diametrically critical graphs, Recent Advances in Graph Theory, Proc. 2nd Czechoslovak Symp. (Prague 1974), Academia, Prague (1975) 455-465.

<u>Appendix</u>: the detailed proof of lemma 3.2.

Let G and G' have as vertices $x_0$, $x_1$, ..., $x_n$. Let G have as edges $(x_i, x_{i+1})$ for $0 \le i \le n-1$. Let G' have the edges of G plus three edges $(x_i, x_{i'})$, $(x_j, x_{j'})$ and $(x_h, x_{h'})$. Without loss of generality let $i<i'$, $j<j'$, $h<h'$ and $i \le j \le h$. Now i, i', j, j', h and h' divide the path of length n into seven segments of non-negative lengths a, b, c, d, e, f and g. Hence $a+b+c+d+e+f=n$. Assume $n \ge 4D-1$. As in the proof of lemma 3.1 we derive several relations between a, b, c, d, e, f and g, based on the fact that the diameter of G' is D. Now we distinguish fifteen cases.

<u>Case</u> 1. $i<i' \le j<j' \le h<h'$.




Clearly $a=i$, $b=i'-i$, $c=j-i'$, $d=j'-j$, $e=h-j'$, $f=h'-h$ and $g=n-h'$. Also

$a+c+e+g+3 \le D$ (the distance from $x_0$ to $x_n$),

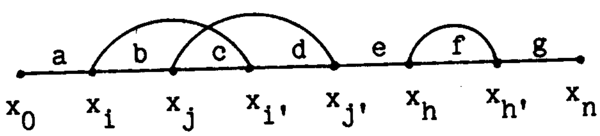
$\frac{1}{2}b+c+\frac{1}{2}d \le \lfloor\frac{1}{2}(b+1)\rfloor+c+\lfloor\frac{1}{2}(d+1)\rfloor \le D$ (halfway b to halfway d),

$\frac{1}{2}d+e+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

$\frac{1}{2}b+c+1+e+\frac{1}{2}f \le \lfloor\frac{1}{2}(b+1)\rfloor+c+1+e+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway b to halfway f),

and hence $a+b+3c+d+3e+f+g \le 4D-4$. Contradiction.

<u>Case</u> 2. $i \le j<i' \le j' \le h<h'$.




Now we have $a=i$, $b=j-i$, $c=i'-j$, $d=j'-i'$, $e=h-j'$, $f=h'-h$ and $g=n-h'$. We distinguish three subcases.

<u>Case</u> 2.1. $c+2=\min(b+1,c+2,d+1)$. Then

$a+c+2+e+1+g \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}b+\frac{1}{2}d \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \le D$ (halfway b to halfway d),

$\frac{1}{2}b+e+\frac{1}{2}f \le \lfloor\frac{1}{2}(b+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway b to halfway f),

$\frac{1}{2}d+e+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

and hence $a+b+c+d+3e+f+g \le 4D-3$. Contradiction.

Case 2.2. $b+1=\min(b+1,c+2,d+1)$. Then

$a+b+1+e+1+g \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}d \leq \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ (halfway c to halfway d),

$\frac{1}{2}c+e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(c+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway c to halfway f),

$\frac{1}{2}d+e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(d+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway d to halfway f),

and hence $a+b+c+d+3e+f+g \leq 4D-2$. Contradiction.

Case 2.3. $d+1=\min(b+1,c+2,d+1)$. Then

$a+d+1+e+1+g \leq D$ (the distance from $x_0$ to $x_n$),

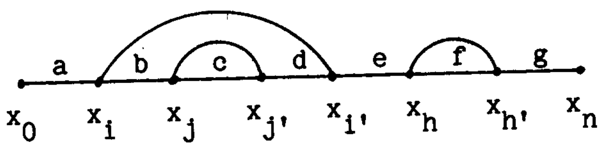
$\frac{1}{2}b+\frac{1}{2}c \leq \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(c+1)\rfloor \leq D$ (halfway b to halfway c),

$\frac{1}{2}b+e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(b+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway b to halfway f),

$\frac{1}{2}c+e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(c+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway c to halfway f),

and hence $a+b+c+d+3e+f+g \leq 4D-2$. Contradiction.

Case 3. $i \leq j < j' < i' \leq h < h'$.




Now $a=i$, $b=j-i$, $c=j'-j$, $d=i'-j'$, $e=h-i'$, $f=h'-h$ and $g=n-h'$. Then

$a+e+g+2 \leq D$ (the distance from $x_0$ to $x_n$),

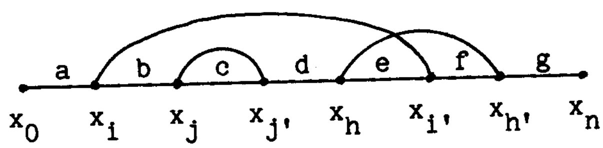
$\frac{1}{2}(b+d+1)+\frac{1}{2}c \leq \lfloor\frac{1}{2}(b+d+2)\rfloor+\lfloor\frac{1}{2}(c+1)\rfloor \leq D$ (halfway b+d to halfway c),

$\frac{1}{2}(b+d+1)+e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(b+d+2)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway b+d to halfway f),

$\frac{1}{2}c+e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(c+1)\rfloor+e+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway c to halfway f),

and hence $a+b+c+d+3e+f+g \leq 4D-3$. Contradiction.

Case 4. $i \leq j < j' \leq h < i' \leq h'$.




Now $a=i$, $b=j-i$, $c=j'-j$, $d=h-j'$, $e=i'-h$, $f=h'-i'$ and $g=n-h'$. We distinguish three subcases.

Case 4.1. $e+2=\min(b+d+2,e+2,f+1)$. Then

$a+e+2+g \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}(b+d+e+1)+\frac{1}{2}c \leq \lfloor\frac{1}{2}(b+d+e+2)\rfloor+\lfloor\frac{1}{2}(c+1)\rfloor \leq D$
(halfway b+d+e to halfway c),

$\frac{1}{2}c+\frac{1}{2}f \leq \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway c to halfway f),

$\frac{1}{2}f+\frac{1}{2}(b+d+1) \leq \lfloor\frac{1}{2}(f+1)\rfloor+\lfloor\frac{1}{2}(b+d+2)\rfloor \leq D$ (halfway f to halfway b+d),

and hence $a+b+c+d+1\frac{1}{2}e+f+g \leq 4D-3$. Contradiction.

<u>Case</u> 4.2. $b+d+2=\min(b+d+2,e+2,f+1)$. Then

$a+b+d+2+g \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}f \leq \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway c to halfway f),

$\frac{1}{2}c+\frac{1}{2}e \leq \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \leq D$ (halfway c to halfway e),

$\frac{1}{2}e+\frac{1}{2}f \leq \lfloor\frac{1}{2}(e+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway e to halfway f),

and hence $a+b+c+d+e+f \leq 4D-2$. Contradiction.

<u>Case</u> 4.3. $f+1=\min(b+d+2,e+2,f+1)$. Then

$a+f+1+g \leq D$ (the distance from $x_0$ to $x_n$),

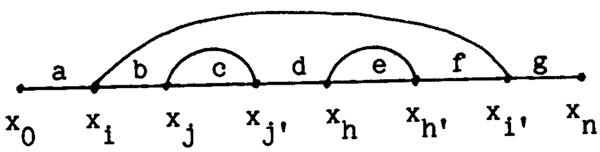
$\frac{1}{2}(b+d+1)+\frac{1}{2}c \leq \lfloor\frac{1}{2}(b+d+2)\rfloor+\lfloor\frac{1}{2}(c+1)\rfloor \leq D$ (halfway b+d to halfway c),

$\frac{1}{2}c+\frac{1}{2}e \leq \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \leq D$ (halfway c to halfway e),

$\frac{1}{2}(b+d+1)+\frac{1}{2}e \leq \lfloor\frac{1}{2}(b+d+2)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \leq D$ (halfway b+d to halfway e),

and hence $a+b+c+d+e+f \leq 4D-2$. Contradiction.

<u>Case</u> 5. $i \leq j < j' \leq h < h' < i'$.




Now $a=i$, $b=j-i$, $c=j'-j$, $d=h-j'$, $e=h'-h$, $f=i'-h'$ and $g=n-i'$. Then

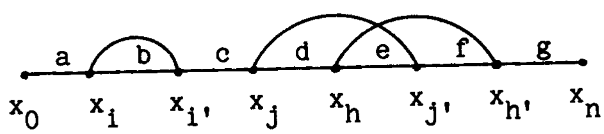
$a+\frac{1}{2}(b+d+f+2) \leq a+\lfloor\frac{1}{2}(b+d+f+3)\rfloor \leq D$ ($x_0$ to halfway b+d+f),

$g+\frac{1}{2}(b+d+f+2) \leq g+\lfloor\frac{1}{2}(b+d+f+3)\rfloor \leq D$ ($x_n$ to halfway b+d+f),

$c+e \leq 2\lfloor\frac{1}{2}(c+1)\rfloor+2\lfloor\frac{1}{2}(e+1)\rfloor \leq 2D$ (halfway c to halfway e),

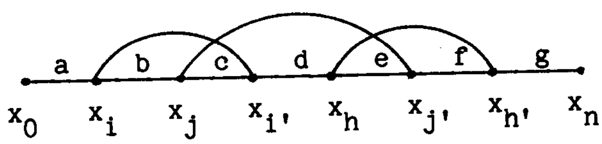
and hence $a+b+c+d+e+f \leq 4D-2$. Contradiction.

<u>Case</u> 6. $i < i' \leq j \leq h < j' < h'$.




Symmetric to case 2.

<u>Case</u> 7. $i \leq j < i' \leq h < j' \leq h'$.




Now $a=i$, $b=j-i$, $c=i'-j$, $d=h-i'$, $e=j'-h$, $f=h'-j'$ and $g=n-h'$. We distinguish five subcases.

Case 7.1. $b+f+1=\min(b+f+1,c+e+3,b+e+2,c+f+2,d+2)$. Then

$a+b+f+1+g \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}e \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \le D$ (halfway c to halfway e),

$\frac{1}{2}(d+1)+\frac{1}{2}c \le \lfloor\frac{1}{2}(d+2)\rfloor+\lfloor\frac{1}{2}(c+1)\rfloor \le D$ (halfway c to halfway d ($e\ge1$)),

$\frac{1}{2}(d+1)+\frac{1}{2}e \le \lfloor\frac{1}{2}(d+2)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \le D$ (halfway e to halfway d ($c\ge1$)),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 7.2. $c+e+3=\min(b+f+1,c+e+3,b+e+2,c+f+2,d+2)$. Then

$a+c+e+3+g \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}b+\frac{1}{2}d \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \le D$ (halfway b to halfway d),

$\frac{1}{2}b+\frac{1}{2}f \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway b to halfway f),

$\frac{1}{2}d+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

and hence $a+b+c+d+e+f \le 4D-3$. Contradiction.

Case 7.3. $b+e+2=\min(b+f+1,c+e+3,b+e+2,c+f+2,d+2)$. Then

$a+b+e+2+g \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}d \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \le D$ (halfway c to halfway d),

$\frac{1}{2}c+\frac{1}{2}f \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway c to halfway f),

$\frac{1}{2}d+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 7.4. $c+f+2=\min(b+f+1,c+e+3,b+e+2,c+f+2,d+2)$.

Symmetric to case 7.3.

Case 7.5. $d+2=\min(b+f+1,c+e+3,b+e+2,c+f+2,d+2)$. We distinguish four subcases.

Case 7.5.1. $b+f+1=\min(b+f+1,c+e+3,b+e+2,c+f+2)$. Then

$a+\frac{1}{2}(b+f+d+2) \le a+\lfloor\frac{1}{2}(b+f+d+3)\rfloor \le D$ ($x_0$ to halfway $b+f+d$),

$g+\frac{1}{2}(b+f+d+2) \le g+\lfloor\frac{1}{2}(b+f+d+3)\rfloor \le D$ ($x_n$ to halfway $b+f+d$),

$c+e \le 2\lfloor\frac{1}{2}(c+1)\rfloor+2\lfloor\frac{1}{2}(e+1)\rfloor \le 2D$ (halfway c to halfway e),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 7.5.2. $c+e+3=\min(b+f+1,c+e+3,b+e+2,c+f+2)$. Then

$a+1+\frac{1}{2}(c+e+d) \le a+1+\lfloor\frac{1}{2}(c+e+d+1)\rfloor \le D$ ($x_0$ to halfway $c+d+e$),

$g+1+\frac{1}{2}(c+e+d) \le g+1+\lfloor\frac{1}{2}(c+e+d+1)\rfloor \le D$ ($x_n$ to halfway $c+d+e$),

$b+f+2 \le 2\lfloor\frac{1}{2}(b+1)\rfloor+2+2\lfloor\frac{1}{2}(f+1)\rfloor \le 2D$ (halfway b to halfway f),

and hence $a+b+c+d+e+f \le 4D-4$. Contradiction.

<u>Case</u> 7.5.3. $b+e+2=\min(b+f+1,c+e+3,b+e+2,c+f+2)$. Then

$a+\frac{1}{2}(b+e+d+1) \leq a+\lfloor\frac{1}{2}(b+e+d+2)\rfloor \leq D$ ($x_0$ to halfway b+d+e),

$g+1+\frac{1}{2}(b+e+d+1) \leq g+1+\lfloor\frac{1}{2}(b+d+e+2)\rfloor \leq D$ ($x_n$ to halfway b+d+e),

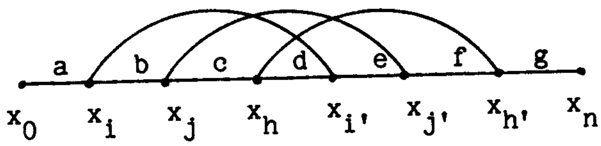
$c+f \leq 2\lfloor\frac{1}{2}(c+1)\rfloor+2\lfloor\frac{1}{2}(f+1)\rfloor \leq 2D$ (halfway c to halfway f),

and hence $a+b+c+d+e+f \leq 4D-2$. Contradiction.

<u>Case</u> 7.5.4. $c+f+2=\min(b+f+1,c+e+3,b+e+2,c+f+2)$.

Symmetric to case 7.5.3.

<u>Case</u> 8. $i \leq j \leq h < i' \leq j' \leq h'$.




Now $a=i$, $b=j-i$, $c=h-j$, $d=i'-h$, $e=j'-i'$, $f=h'-j'$ and $g=n-h'$. We distinguish five subcases.

<u>Case</u> 8.1. $b+f+1=\min(b+f+1,c+e+3,b+c+1,e+f+1,d+2)$. Then

$a+b+f+g+1 \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}d \leq \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ (halfway c to halfway d),

$\frac{1}{2}(c+e+1) \leq \lfloor\frac{1}{2}(c+e+2)\rfloor \leq D$ (halfway c to halfway e ($d \geq 1$)),

$\frac{1}{2}d+\frac{1}{2}e \leq \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \leq D$ (halfway d to halfway e),

and hence $a+b+c+d+e+f \leq 4D-1\frac{1}{2}$. Contradiction.

<u>Case</u> 8.2. $c+e+3=\min(b+f+1,c+e+3,b+c+1,e+f+1,d+2)$. Then

$a+c+e+g+3 \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}b+\frac{1}{2}d \leq \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ (halfway b to halfway d),

$\frac{1}{2}b+\frac{1}{2}f \leq \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway b to halfway f),

$\frac{1}{2}d+\frac{1}{2}f \leq \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway d to halfway f),

and hence $a+b+c+d+e+f \leq 4D-3$. Contradiction.

<u>Case</u> 8.3. $b+c+1=\min(b+f+1,c+e+3,b+c+1,e+f+1,d+2)$. Then

$a+b+c+g+1 \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}d+\frac{1}{2}e \leq \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \leq D$ (halfway d to halfway e),

$\frac{1}{2}d+\frac{1}{2}f \leq \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \leq D$ (halfway d to halfway f),

$\frac{1}{2}(e+f+1) \leq \lfloor\frac{1}{2}(e+f+2)\rfloor \leq D$ (halfway e to halfway f ($d \geq 1$)),

and hence $a+b+c+d+e+f \leq 4D-1\frac{1}{2}$. Contradiction.

<u>Case</u> 8.4. $e+f+1=\min(b+f+1,c+e+3,b+c+1,e+f+1,d+2)$.

Symmetric to case 8.3.

Case 8.5. $d+2=\min(b+f+1,c+e+3,b+c+1,e+f+1,d+2)$. We distinguish four subcases.

Case 8.5.1. $b+f+1=\min(b+f+1,c+e+3,b+c+1,e+f+1)$. Then

$a+\frac{1}{2}(b+f+d+2) \le a+\lfloor\frac{1}{2}(b+f+d+3)\rfloor \le D$ ($x_0$ to halfway $b+d+f$),

$g+\frac{1}{2}(b+f+d+2) \le g+\lfloor\frac{1}{2}(b+f+d+3)\rfloor \le D$ ($x_n$ to halfway $b+d+f$),

$c+e+1 \le 2\lfloor\frac{1}{2}(c+e+2)\rfloor \le 2D$ (halfway $c$ to halfway $e$ ($d\ge1$)),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 8.5.2. $c+e+3=\min(b+f+1,c+e+3,b+c+1,e+f+1)$. Then

$a+1+\frac{1}{2}(c+d+e) \le a+1+\lfloor\frac{1}{2}(c+d+e+1)\rfloor \le D$ ($x_0$ to halfway $c+d+e$),

$g+1+\frac{1}{2}(c+d+e) \le g+1+\lfloor\frac{1}{2}(c+d+e+1)\rfloor \le D$ ($x_n$ to halfway $c+d+e$),

$b+f \le 2\lfloor\frac{1}{2}(b+1)\rfloor+2\lfloor\frac{1}{2}(f+1)\rfloor \le 2D$ (halfway $b$ to halfway $f$),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 8.5.3. $b+c+1=\min(b+f+1,c+e+3,b+c+1,e+f+1)$. Then

$a+\frac{1}{2}(b+c+d) \le a+\lfloor\frac{1}{2}(b+c+d+1)\rfloor \le D$ ($x_0$ to halfway $b+c+d$),

$g+1+\frac{1}{2}(b+c+d) \le g+1+\lfloor\frac{1}{2}(b+c+d+1)\rfloor \le D$ ($x_n$ to halfway $b+c+d$),

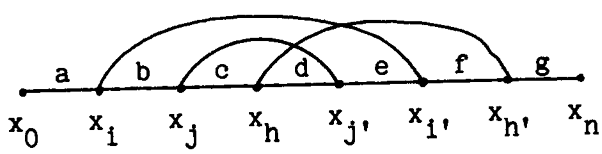
$e+f+1 \le 2\lfloor\frac{1}{2}(e+f+2)\rfloor \le 2D$ (halfway $e$ to halfway $f$ ($d\ge1$)),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 8.5.4. $e+f+1=\min(b+f+1,c+e+3,b+c+1,e+f+1)$.

Symmetric to case 8.5.4.

Case 9. $i\le j\le h<j'<i'<h'$.




Now $a=i$, $b=j-i$, $c=h-j$, $d=j'-h$, $e=i'-j'$, $f=h'-i'$ and $g=n-h'$. We distinguish five subcases.

Case 9.1. $b+c+1=\min(b+c+1,b+d+2,e+d+2,e+c+3,f+1)$. Then

$a+b+c+g+1 \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}d+\frac{1}{2}e \le \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \le D$ (halfway $d$ to halfway $e$),

$\frac{1}{2}(d+f+1) \le \lfloor\frac{1}{2}(d+f+2)\rfloor \le D$ (halfway $d$ to halfway $f$ ($e\ge1$)),

$\frac{1}{2}(e+f+1) \le \lfloor\frac{1}{2}(e+f+2)\rfloor \le D$ (halfway $e$ to halfway $f$ ($d\ge1$)),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 9.2. $b+d+2=\min(b+c+1,b+d+2,e+d+2,e+c+3,f+1)$. Then

$a+b+d+g+2 \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}c+\frac{1}{2}e \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \le D$ (halfway c to halfway e),

$\frac{1}{2}c+\frac{1}{2}f \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway c to halfway f),

$\frac{1}{2}(e+f+1) \le \lfloor\frac{1}{2}(e+f+2)\rfloor \le D$ (halfway e to halfway f ($d\ge 1$)),

and hence $a+b+c+d+e+f \le 4D-2\frac{1}{2}$. Contradiction.

Case 9.3. $e+d+2=\min(b+c+1,b+d+2,e+d+2,e+c+3,f+1)$. Then

$a+e+d+g+2 \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}b+\frac{1}{2}c \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(c+1)\rfloor \le D$ (halfway b to halfway c),

$\frac{1}{2}b+\frac{1}{2}f \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway b to halfway f),

$\frac{1}{2}c+\frac{1}{2}f \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway c to halfway f),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 9.4. $e+c+3=\min(b+c+1,b+d+2,e+d+2,e+c+3,f+1)$. Then

$a+e+c+g+3 \le D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}b+\frac{1}{2}d \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \le D$ (halfway b to halfway d ($e\ge 1$)),

$\frac{1}{2}b+\frac{1}{2}f \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway b to halfway f),

$\frac{1}{2}d+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

and hence $a+b+c+d+e+f \le 4D-3$. Contradiction.

Case 9.5. $f+1=\min(b+c+1,b+d+2,e+d+2,e+c+3,f+1)$. We distinguish four subcases.

Case 9.5.1. $b+c+1=\min(b+c+1,b+d+2,e+d+2,e+c+3)$. Then

$a+\frac{1}{2}(b+c+f+1) \le a+\lfloor\frac{1}{2}(b+c+f+2)\rfloor \le D$ ($x_0$ to halfway b+c+f),

$g+\frac{1}{2}(b+c+f+1) \le g+\lfloor\frac{1}{2}(b+c+f+2)\rfloor \le D$ ($x_n$ to halfway b+c+f),

$e+d+1 \le 2\lfloor\frac{1}{2}(e+d+2)\rfloor \le 2D$ (halfway e to halfway d ($f\ge 1$, $b+c\ge 1$)),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 9.5.2. $b+d+2=\min(b+c+1,b+d+2,e+d+2,e+c+3)$. Then

$a+\frac{1}{2}(b+d+f+2) \le a+\lfloor\frac{1}{2}(b+d+f+3)\rfloor \le D$ ($x_0$ to halfway b+d+f),

$g+\frac{1}{2}(b+d+f+2) \le g+\lfloor\frac{1}{2}(b+d+f+3)\rfloor \le D$ ($x_n$ to halfway b+d+f),

$c+e \le 2\lfloor\frac{1}{2}(c+1)\rfloor+2\lfloor\frac{1}{2}(e+1)\rfloor \le 2D$ (halfway c to halfway e),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 9.5.3. $e+d+2=\min(b+c+1,b+d+2,e+d+2,e+c+3)$. Then

$a+1+\frac{1}{2}(d+e+f) \le a+1+\lfloor\frac{1}{2}(d+e+f+1)\rfloor \le D$ ($x_0$ to halfway d+e+f),

$g+\frac{1}{2}(d+e+f) \le g+\lfloor\frac{1}{2}(d+e+f+1)\rfloor \le D$ ($x_n$ to halfway d+e+f+1),

$b+c+1 \le 2\lfloor\frac{1}{2}(b+c+2)\rfloor \le 2D$ (halfway b to halfway c),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 9.5.4. $e+c+3=\min(b+c+1, b+d+2, e+d+2, e+c+3)$. Then

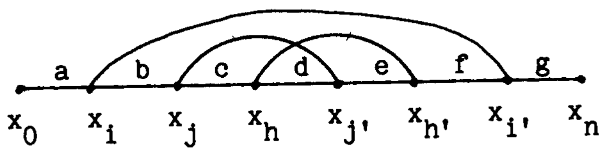
$a+1+\frac{1}{2}(c+e+f+1) \le a+1+\lfloor\frac{1}{2}(c+e+f+2)\rfloor \le D$ ($x_0$ to halfway $c+e+f$),

$g+\frac{1}{2}(c+e+f+1) \le g+\lfloor\frac{1}{2}(c+e+f+2)\rfloor \le D$ ($x_n$ to halfway $c+e+f$),

$b+d \le 2\lfloor\frac{1}{2}(b+1)\rfloor+2\lfloor\frac{1}{2}(d+1)\rfloor \le 2D$ (halfway $b$ to halfway $d$),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 10. $i\le j\le h<j'\le h'\le i'$.




Now we have $a=i$, $b=j-i$, $c=h-j$, $d=j'-h$, $e=h'-j'$, $f=i'-h'$ and $g=n-i'$.

We distinguish three subcases.

Case 10.1. $d+2=\min(d+2, c+1, e+1)$. Then

$a+\frac{1}{2}(b+d+f+2) \le a+\lfloor\frac{1}{2}(b+d+f+3)\rfloor \le D$ ($x_0$ to halfway $b+d+f$),

$g+\frac{1}{2}(b+d+f+2) \le g+\lfloor\frac{1}{2}(b+d+f+3)\rfloor \le D$ ($x_n$ to halfway $b+d+f$),

$c+e \le 2\lfloor\frac{1}{2}(c+1)\rfloor+2\lfloor\frac{1}{2}(e+1)\rfloor \le 2D$ (halfway $c$ to halfway $e$),

and hence $a+b+c+d+e+f \le 4D-2$. Contradiction.

Case 10.2. $c+1=\min(d+2, c+1, e+1)$. Then

$a+\max_{0\le k\le d}\min(b+1+d-k, f+2+k, b+c+k, f+1+e+d-k) \le D$ ($x_0$ to somewhere in $d$):

We distinguish six subcases.

Case 10.2.1. $f+e\le b$, $f+2\le b+c$, $d+2\le e+1$. Then

$a+f+d+2 = a+\max_{0\le k\le d}\min(f+2+k, f+1+e+d-k) \le D$,

$g+f+c+1 \le D$ (the distance from $x_n$ to $x_j$),

$b+e \le 2\lfloor\frac{1}{2}(b+1)\rfloor+2\lfloor\frac{1}{2}(e+1)\rfloor \le 2D$ (halfway $b$ to halfway $e$),

and hence $a+b+c+d+e+2f+g \le 2D-3$. Contradiction.

Case 10.2.2. $f+e\le b$, $f+2\le b+c$, $e+1\le d+2$. Then

$a+f+\frac{1}{2}(e+d) \le a+\max_{0\le k\le d}\min(f+2+k, f+1+e+d-k) \le D$,

$g+f+c+1 \le D$ (the distance from $x_n$ to $x_j$),

$\frac{1}{2}b+\frac{1}{2}d \le \lfloor\frac{1}{2}(b+1)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor \le D$ (halfway $b$ to halfway $d$),

$\frac{1}{2}(b+e+1) \le \lfloor\frac{1}{2}(b+e+2)\rfloor \le D$ (halfway $b$ to halfway $e$ ($d\ge 1$)),

and hence $a+b+c+d+e+2f+g \le 4D-1\frac{1}{2}$. Contradiction.

Case 10.2.3. $f+e\le b$, $f+2>b+c$.

Thus $f+e+c\le b+c<f+2$ and hence $c=0$ which gives case 15.

Case 10.2.4. $f+e>b$, $f+2\le b+c$. Then

$a+1+\frac{1}{2}(b+d+f) \le a+\max_{0\le k\le d} \min(f+2+k, b+1+d-k) \le D$,

$g+\frac{1}{2}(f+e+b+1) \le g+\max_{0\le k\le e} \min(b+2+k, f+e-k) \le D$ ($x_n$ to somewhere in e),

$\frac{1}{2}c+\frac{1}{2}e \le \lfloor\frac{1}{2}(c+1)\rfloor+e1 \le D$ (halfway c to halfway e ($d\ge 1$)),

$\frac{1}{2}d+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

and hence $a+b+c+d+e+f \le 4D-1\frac{1}{2}$. Contradiction.

Case 10.2.5. $f+e>b$, $f\le b+c+2<f+4$. Then

$a+b+\frac{1}{2}(d+c) \le a+\max_{0\le k\le d} \min(b+c+k, b+1+d-k) \le D$,

$g+\frac{1}{2}(f+e+b+1) \le g+\max_{0\le k\le e} \min(b+2+k, f+e-k) \le D$ ($x_n$ to somewhere in e),

$\frac{1}{2}(b+f+d+2) \le \lfloor\frac{1}{2}(b+f+d+3)\rfloor \le D$ (halfway d to halfway b+f),

$\frac{1}{2}c+\frac{1}{2}e \le \lfloor\frac{1}{2}(c+1)\rfloor+\lfloor\frac{1}{2}(e+1)\rfloor \le D$ (halfway c to halfway e ($d\ge 1$)),

and hence $a+1\frac{1}{2}b+c+d+e+f+g \le 4D-1\frac{1}{2}$. Contradiction.

Case 10.2.6. $f+e>b$, $f>b+c+2$. Then

$a+b+\frac{1}{2}(d+c) \le D$,

$g+b+1+\frac{1}{2}(c+e+1) \le g+\max_{0\le k\le e} \min(b+2+k, b+c+2+e-k) \le D$

($x_n$ to somewhere in e),

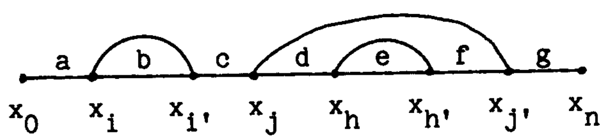
$\frac{1}{2}d+\frac{1}{2}f \le \lfloor\frac{1}{2}(d+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway d to halfway f),

$\frac{1}{2}e+\frac{1}{2}f \le \lfloor\frac{1}{2}(e+1)\rfloor+\lfloor\frac{1}{2}(f+1)\rfloor \le D$ (halfway e to halfway f),

and hence $a+2b+c+d+e+f+g \le 4D-1\frac{1}{2}$. Contradiction.

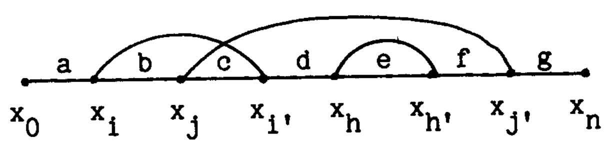
Case 10.3. $e+1=\min(d+2, c+1, e+1)$. Symmetric to case 10.2.

Case 11. $i<i'\le j\le h<h'<j'$.



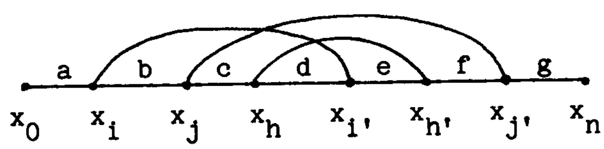

Symmetric to case 3.

Case 12. $i\le j<i'\le h<h'<j'$.



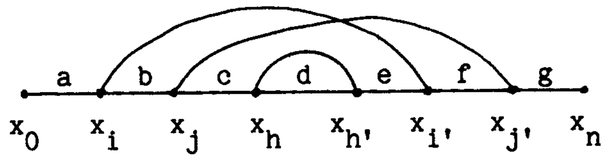

Symmetric to case 4.

Case 13. $i\le j\le h<i'\le h'<j'$.




Symmetric to case 9.

Case 14. $i \leq j \leq h < h' < i' \leq j'$.




Now $a=i$, $b=j-i$, $c=h-j$, $d=h'-h$, $e=i'-h'$, $f=j'-i'$ and $g=n-j'$. We distinguish four subcases.

Case 14.1. $e+1 \leq b+c$, $c+1 \leq f+e$. Then

$a+e+1+\frac{1}{2}d \leq a+e+1+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ ($x_0$ to halfway d),

$g+c+1+\frac{1}{2}d \leq g+c+1+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ ($x_n$ to halfway d),

$b+f \leq 2\lfloor\frac{1}{2}(b+1)\rfloor+2\lfloor\frac{1}{2}(f+1)\rfloor \leq 2D$ (halfway b to halfway f),

and hence $a+b+c+d+e+f \leq 4D-2$. Contradiction.

Case 14.2. $e+1 \leq b+c$, $c+1 > f+e$. Then

$a+e+1+\frac{1}{2}d \leq a+e+1+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ ($x_0$ to halfway d),

$g+f+e+\frac{1}{2}d \leq g+f+e+\lfloor\frac{1}{2}(d+1)\rfloor \leq D$ ($x_n$ to halfway d),

$b+c+1 \leq 2\lfloor\frac{1}{2}(b+1)\rfloor+2\lfloor\frac{1}{2}(c+2)\rfloor \leq 2D$ (halfway b to halfway c),

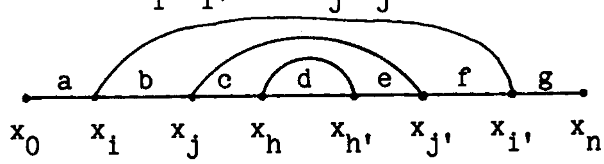
and hence $a+b+c+d+2e+f+g \leq 4D-2$. Contradiction.

Case 14.3. $e+1 > b+c$, $c+1 \leq f+e$. Symmetric to case 14.2.

Case 14.4. $e+1 > b+c$, $c+1 > f+e$.

Thus $f+e+b < b+c+1 < e+2$, so $f+b=0$ and $(x_i,x_{i'}) = (x_j,x_{j'})$.

Case 15. $i \leq j \leq h < h' < j' < i'$.




Now $a=i$, $b=j-i$, $c=h-j$, $d=h'-h$, $e=j'-h'$, $f=i'-j'$ and $g=n-i'$. Then

$a+g+1 \leq D$ (the distance from $x_0$ to $x_n$),

$\frac{1}{2}(b+f+c+e+1) \leq \lfloor\frac{1}{2}(b+f+c+e+2)\rfloor \leq D$ (halfway b+f to halfway c+e),

$\frac{1}{2}(c+e+d) \leq \lfloor\frac{1}{2}(c+e+d+1)\rfloor \leq D$ (halfway c+e to halfway d),

$\frac{1}{2}(b+f+1)+\frac{1}{2}d \leq \lfloor\frac{1}{2}(b+f+2)\rfloor+\lfloor\frac{1}{2}(d+1)\rfloor+\min(c,e) \leq D$
(halfway b+f to halfway d),

and hence $a+b+c+d+e+f \leq 4D-2$. Contradiction.

Q.E.D.