Classes of graphs with bounded tree-width

H.L. Bodlaender

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Utrecht University

Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

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Abstract

In this paper we show a number of classes of graphs to be subclasses of the graphs with tree-width, bounded by some constant integer k, (also called the partial k-trees). These classes include all trees, forests, almost trees with parameter k_1 (k_1 a constant), graphs with bandwidth or cutwidth bounded by some constant, outerplanar graphs, series-parallel graphs, Halin graphs, k_2-outerplanar graphs (k_2 a constant), k_3-bounded tree-partite graphs (k_3 a constant), chordal graphs with maximum clique size k_4 (k_4 a constant) and circular arc graphs with maximum clique size k_5 (k_5 a constant). Some of these results were well-known, others are new. Also some other relations between the considered classes of graphs are shown. For many of the classes, it has been shown that many NP-complete problems can be solved in polynomial time, when restricted to graphs in the specific class. The results in this paper illustrate why this similarity occurs.

1 Introduction

NP-complete problems are generally believed not to be solvable in polynomial time. Hence there is much effort spent on finding subproblems of NP-complete problems for which polynomial time algorithms can be designed. For a number of classes of graphs, it has been shown that many NP-complete graph problems become solvable in polynomial time, when restricted to graphs in the specific class. An overview of some important

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NP-complete graph problems, and their (known) complexity when restricted to a number of important classes of graphs is given in [10].

In this paper we show that many of the classes, that yield polynomial time solutions for many problems that are NP-complete for general graphs, are contained in the class of graphs with tree-width bounded by some constant integer k, also called the partial k-graphs. Arnborg and Proskurowski [2] show that for many NP-complete graph problems linear time algorithms can be obtained when one restricts the instances to graphs with tree-width bounded by some constant k. These results illustrate the similarity in the complexity results that are known for the various discussed classes of graphs.

We consider the following classes of graphs, and show them to have tree-width $\leq k$, for some constant k:

- Trees and forests
- Almost trees with parameter k_1
- Graphs with bandwidth at most k_2
- Graphs with cutwidth at most k_3
- Outerplanar graphs
- Series-parallel graphs
- Halin graphs
- k_4-outerplanar graphs
- Chordal graphs with maximum cliquesize k_5
- Undirected path graphs with maximum clique size k_6
- Directed path graphs with maximum clique size k_7
- Interval graphs with maximum cliquesize k_8
- Proper interval graphs with maximum cliquesize k_9
- Circular arc graphs with maximum cliquesize k_{10}
- Proper circular arc graphs with maximum cliquesize k_{11}
- k_{12}-bounded tree-partite graphs

where $k_1, k_2, \cdots, k_{12}$ are fixed constants. Some of the inclusion-relations are already well known, but are included in this paper for completeness sake. A schematic overview of the results is given in fig. 1 and fig. 2.

Throughout this paper we will assume all graphs to be undirected and free from self-loops and parallel edges, unless mentioned otherwise.

Class of graphs	Upperbound for Maximum Tree-width	Reference
Trees, forests	1	
Almost trees with parameter k	$k+1$	
Graphs with bandwidth $\leq k$	k	
Graphs with cyclic bandwidth $\leq k$	$2k$	
Graphs with cutwidth $\leq k$	k	
Series-parallel graphs	2	[17]
Outerplanar graphs	2	[17]
Halin graphs	5	
k-outerplanar graphs	$3^k - 1$	
Chordal graphs with max. cliquesize k	$k-1$	[7,13]
Undirected pathgraphs with max. cliquesize k	$k-1$	[13]
Directed pathgraphs with max. cliquesize k	$k-1$	[13]
Interval graphs with max. cliquesize k	$k-1$	[13]
Proper interval graphs with max. cliquesize k	$k-1$	[13]
Circular arc graphs with max. cliquesize k	$2k-1$	
Proper circular arc graphs with max. cliquesize k	$2k-2$	
k-bounded tree-partite graphs	$2k-1$	[15]

Figure 1: Classes of graphs and upperbounds for the maximum tree-width of graphs in the classes.

2 Partial k-trees and the tree-width of a graph

Let C_k be the complete graph on k vertices. The (partial) k-trees are defined as follows:

Definition.

- C_k is a k-tree.

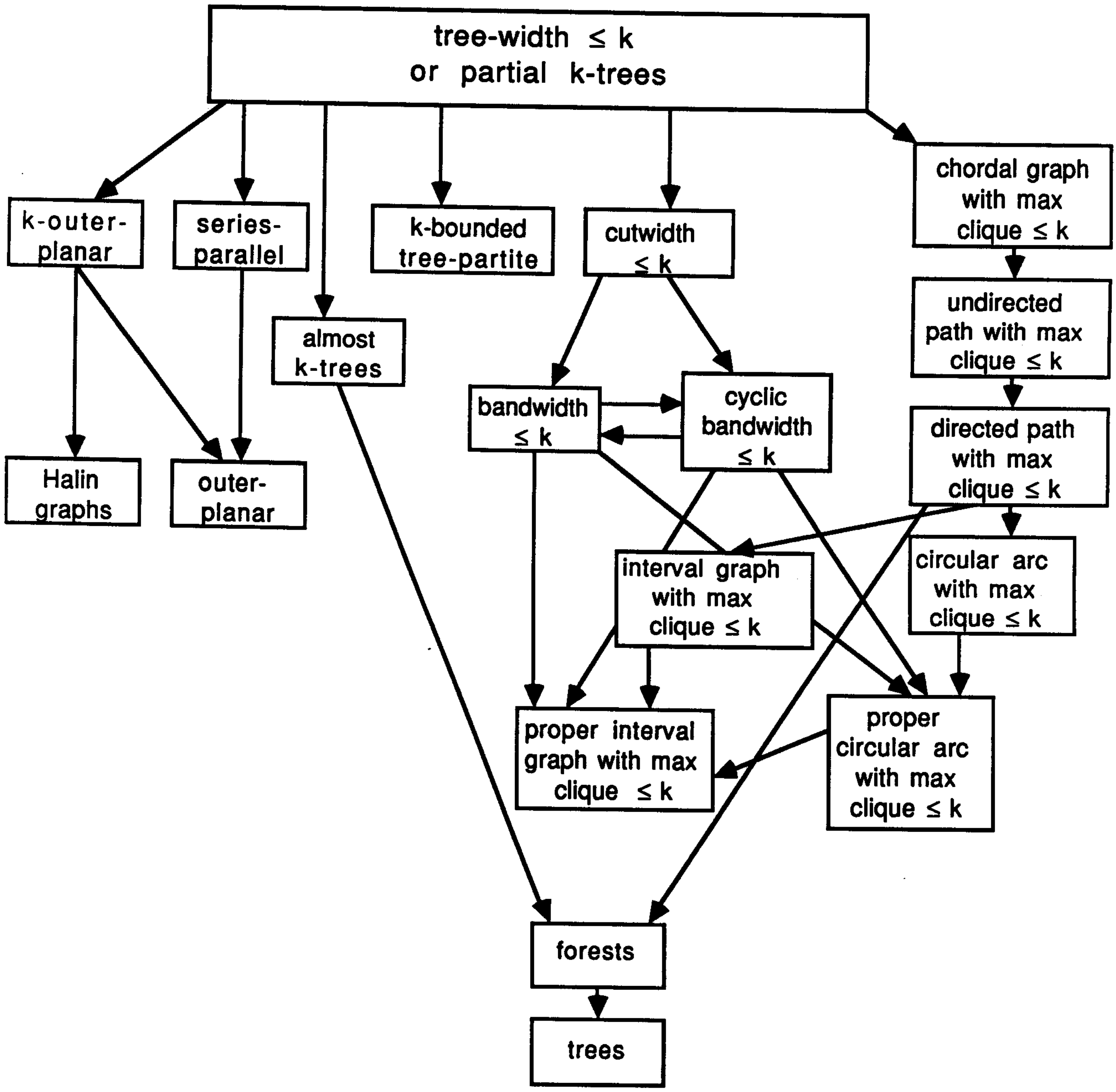


Figure 2. Schematic overview of containment relations between discussed classes of graphs. Note that the constants k in different boxes may denote different constants. E.g. not every graph with bandwidth $\leq k$ has cutwidth $\leq k$, but for every k, there is an m, such that every graph with bandwidth $\leq k$ has cutwidth $\leq m$,

- If $G = (V, E)$ is a k-tree, and $V' \leq V$ is a set of k vertices, inducing a complete subgraph in G, then $G' = (V \cup \{v\}, E \cup \{(w, v) \mid w \in V'\})$, with v a new vertex, i.e. $v \notin V$, is a k-tree.

I.e. a new k-tree G' can be obtained by taking a k-tree G and adding a new vertex v with edges to each vertex in a clique with k vertices in G.

Definition.
H is a partial k-tree, if H is a subgraph of a k-tree G.

Lemma 2.1
Let G be a partial k-tree. Then G does not contain a clique with $k+2$ vertices.

Proof.
It is sufficient to prove the lemma for G a k-tree. Use induction. If a k-tree $G = (V, E)$ does not contain a clique with $k+2$ vertices, then $G' = (V \cup \{v\}, E \cup \{(w, v) \mid w \in V'\})$, with $V' \subseteq V$, $|V'| \leq k$, $v \notin V$, will also not contain a clique with $k+2$ vertices. □

Another way of characterizing partial k-trees is with help of the notion of tree-width, introduced by Robertson and Seymour [13].

Definition.
Let $G = (V, E)$ be a graph. A tree-decomposition of G is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, with $\{X_i \mid i \in I\}$ a family of subsets of V, and T a tree, with the following properties:

- $\bigcup_{i \in I} X_i = V$
- For every edge $e = (v, w) \in E$, there is a subset X_i, $i \in I$ with $v \in X_i$ and $w \in X_i$.
- For all $i, j, k \in I$, if j lies on the path in T from i to k, then $X_i \cap X_k \subseteq X_j$.

The tree-width of a tree-decomposition $(\{X_i \mid i \in I\}, T)$ is $\max_{i \in I} |X_i| - 1$. The tree-width of G, denoted by tree-width(G) is the minimum tree-width of a tree-decomposition of G, taken over all possible tree-decomposition of G.

Theorem 2.2 [1,13]
G is a partial k-tree if and only if G has tree-width k or less.

Independently, Arnborg, Corneil and Proskurowski [1], and Robertson and Seymour [13] have shown that there exist polynomial algorithms to test whether a graph has tree-width $\leq k$, for any given, fixed k. Arnborg, Corneil and Proskurowski [1] have also shown that the problem to determine the tree-width of a graph is NP-complete. Arnborg and Proskurowski [2] have shown that several NP-complete graphs problems are solvable in linear time when restricted to graphs with tree-width bounded by some fixed k (or equivalently, to partial k-graphs, for fixed k). Similar results can be found in [5]. The following lemma's will be used in section 3.

Lemma 2.3
Let $G = (V, E)$ be a graph and let $k \in N^+$. Then treewidth$(G) \leq k$, if and only if for each biconnected component $G_i = (V_i, E_i)$ of G, treewidth$(G_i) \leq k$.

Proof.
$\Rightarrow$ Trivial.
$\Leftarrow$ Suppose we have tree-decompositions $(\{X_i^1 \mid i \in I^1\}, T_1), \cdots, (\{X_i^c \mid i \in I^c\}, T_c)$ of the biconnected components $G_1, \ldots, G_c$ of G with tree-width $\leq k$ each. Now one can obtain tree-decompositions of connected subgraphs of G, consisting of more and more biconnected components, each tree-decomposition having tree-width k or less, in the following manner. Suppose we have connected subgraphs $G_\alpha = (V_\alpha, E_\alpha)$, $G_\beta = (V_\beta, E_\beta)$, each consisting of one or more biconnected components of G. Let $(\{X_i^\alpha \mid i \in I^\alpha$, $T_\alpha = (I^\alpha, F^\alpha))$, $(\{X_i^\beta \mid i \in I^\beta\}$, $T_\beta = (I^\beta, F^\beta))$, be tree-decompositions of G_α and G_β, respectively, with tree-width $\leq k$ each. Further suppose $\{v\} = V_\alpha \cap V_\beta$. (I.e. G_α and G_β share exactly one vertex v). There are $i_0 \in I^\alpha$, $i_1 \in I^\beta$ with $v \in X_{i_0}^\alpha$, $v \in X_{i_1}^\beta$. Now let $T_\gamma = (I^\alpha \cup I^\beta$, $F^\alpha \cup F^\beta \cup \{(i_0, i_1)\})$. T_γ is a tree. Now it is easy to check that $(\{X_i^\alpha \mid i \in I^\alpha\} \cup \{X_i^\beta \mid i \in I^\beta\}$, $T_\gamma)$ is a tree-decomposition of $G_\gamma = (V_\alpha \cup V_\beta$, $E_\alpha \cup E_\beta)$ with tree-width $\leq k$.

We can repeat this construction, obtaining tree-decompositions of connected subgraphs of G, containing more and more biconnected components, each with tree-width $\leq k$. (If G is not connected, then a similar, but still easier construction can be used). Finally one obtains a tree-decomposition of G with width $\leq k$. □

Lemma 2.4
Let G be a subgraph of H. Then treewidth$(G) \leq$ treewidth (H).

Proof.
Trivial. □

3 Classes of graphs with bounded tree-width

In this section we will discuss a number of classes of graphs with the property that the maximum tree-width of all graphs in the class is bounded by some fixed number.

3.1 Trees and forests.

The following well-known propositions follow directly from the definitions of "k-tree" and " partial k-tree ".

Proposition 3.1
$G = (V, E)$ is a tree, if and only if G is a 1-tree.

Proposition 3.2
$G = (V, E)$ is a forest, if and only if treewidth$(G) \leq 1$.

3.2 Almost trees with parameter k.

Definition.
$G = (V, E)$ is an almost tree with parameter $\leq k$ iff for some spanning tree T of G, in each biconnected component of G there are at most k edges of G that are not in T.

With other words, $G = (V, E)$ is an almost tree with parameter $\leq k$ if and only if for each biconnected component $G_i = (V_i, E_i)$ of G one has $|E_i| - |V_i| + 1 \leq k$.

Theorem 3.3
Let $G = (V, E)$ be an almost tree with parameter k. Then treewidth$(G) \leq k + 1$.

Proof.
From lemma 2.3 it follows that is sufficient to prove the theorem for biconnected graphs $G = (V, E)$. Let $G = (V, E)$ be a biconnected almost tree with parameter k. Let $T_0 = (V, F)$ be a spanning tree of G. Note that $|E - F| \leq k$. Now let $(\{X_i \mid i \in I\}, T)$ be a tree-decomposition of T_0 with tree-width 1, i.e. for all $i \in I$: $|X_i| \leq 2$. We now write $E - F = \{(v_1, w_1), (v_2, w_2), \ldots, (v_l, w_l)\}$ $(l \leq k)$.

We can now obtain a tree-decomposition of G with tree-width $\leq k+1$, by adding the vertices $v_1, \ldots, v_e$ to each set X_i, i.e. we have the tree-decomposition $(\{X_i \cup \{v_1, \ldots, v_e\} \mid i \in I\}, T$. One easily verifies that this is a correct tree-decomposition. For instance, for every edge $(v_j, w_j) \in E - F$, there is a $i \in I$, with $w_j \in X_i$. Hence $w_j \in X_i \cup \{v_1, \ldots, v_e\}$, and by definition $v_j \in X_i \cup \{v_1, \ldots, v_e\}$. The tree-width of this tree-decomposition is $\max_{i \in I} \mid X_i \cup \{v_1, \ldots, v_e\}| - 1 \leq 2 + l - 1 \leq k + 1$. □

Coppersmith and Viskin [6] and Gurevich, Stockmeyer and Viskin [9] have shown that a number of important NP-complete graph problems can be solved in linear time for graphs, that are almost trees with parameter k, for fixed k. The time, needed for these algorithms is exponential in k. Theorem 3.4 shows the relation of these results with the results of Arnborg and Proskurowski [2].

3.3 Graphs with bounded bandwidth or cutwidth.

In this section we consider graphs with bandwidth or cutwidth bounded by some fixed number. These graphs can be recognized in polynomial time (see [14],[8]), (but the time is exponential in the bandwidth or cutwidth of the graph). In [11] it is shown that several NP-complete problems can be solved in polynomial time for graphs $G = (V, E)$ with bandwidth bounded by $c \cdot \log(|V|)$ for some constant c. In [5] similar results are obtained for the larger class of graphs with treewidth, bounded by $c \log(|V|)$ for some constant c. (One must assume that the graphs are given together with the corresponding linear orderings or tree-decompositions.)

Definition.
Let $G = (V, E)$ be a graph, with $n = |V|$.

- A linear ordering of G is a bijective mapping $f \colon V \rightarrow \{1, \ldots, n\}$.
- A linear ordering f of G is said to have bandwidth k if $k = \max_{(u,v) \in E} |f(u) - f(v)|$.
- The bandwidth of G, denoted by bandwidth(G), is the minimum bandwidth of a linear ordering f, over all possible linear orderings of G.
- A linear ordering f of G is said to have cutwidth k if $k = \max_{1 \leq i \leq n} |\{(u, v) \in E \mid f(u) \leq i < f(v)\}|$.

- the cutwidth of G, denoted by cutwidth(G), is the minimum cutwidth of a linear ordering f over all possible linear orderings of G.

The following variant of the notion of bandwidth, called "cyclic bandwidth" was introduced in [11].

Definition.
Let $G = (V, E)$ be a graph, with $n = |V|$.

- A linear ordering f of G is said to have cyclic bandwidth k if $k = \max_{(u,v)\in E}(\min(|f(u)-f(v)|, n-|f(u)-f(v)|))$ (= the maximum distance of $f(u)$ and $f(v)$ in a ring with n vertices $R_n = (\{1, \ldots, n\}, \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\})$, taken over all $(u,v) \in E$).
- The cyclic bandwidth of G, denoted by cyclic bandwidth(G), is the minimum cyclic bandwidth of a linear ordering f over all possible linear orderings of G.

Lemma 3.4
Let $G = (V, E)$ be a graph, with bandwidth(G) = k. Then cutwidth (G) $\leq \frac{k(k+1)}{2}$.

Proof.
Consider a linear ordering f of G with bandwidth k. Let $n = |V|$. For all i, $1 \leq i \leq n$, $|\{(u,v) \in E \mid f(u) \leq i < f(v)\}| \leq |\{(j_1, j_2) \in \{1, \ldots, n\} * \{1, \ldots, n\} \mid j_1 \leq i < j_2 \wedge |j_1 - j_2| \leq k\}| \leq k(k+1)/2$. Hence cutwidth($G$) $\leq k(k+1)/2$. □

Showing that every graph with bandwidth $\leq k$ is a partial k-tree, i.e. has tree-width $\leq k$, is, in particular, very simple.

Definition.
The maximal graph on n vertices with bandwidth k is the graph $G_{k,n} = (V_n, E_{k,n})$, with $V_n = \{1, 2, \ldots, n\}$ and $E_{k,n} = \{(i,j) \mid i,j \in V_n \wedge |i-j| \leq k\}$. The following observation was made by Saxe [14].

Lemma 3.5 [14]
Let $G = (V, E)$ with $|V| = n$. Then bandwidth(G) $\leq k$, if and only if G is isomorphic to a subgraph of $G_{k,n}$.

Lemma 3.6

1. For all k, n, $n \geq k \geq 1$, $G_{k,n}$ is a k-tree.
2. For all k, $n \geq 1$, $G_{k,n}$ is a partial k-tree.

Proof.

1. Use induction to n.
2. Use (1) and the fact that every graph on $n \leq k$ vertices is a partial k-tree.

□

Corollary 3.7
For every graph $G = (V, E)$, bandwidth(G) $\geq$ treewidth(G).

Lemma 3.8
For every graph $G = (V, E)$:
cyclic bandwidth(G) $\leq$ bandwidth(G) $\leq 2\cdot$ cyclic bandwidth(G).

Proof. First we remark that cyclic bandwidth(G)$\leq$ bandwidth(G) follows directly from the definitions.

Now let $|V| = n$. Let $f : V \rightarrow \{1, \ldots, n\}$ be a linear ordering of G with cyclic bandwidth(f) $\leq k$. We suppose n is even. If n is odd, then a similar construction can be made. Let $g : V \rightarrow \{1, \ldots, n\}$ be defined by

$$g(v) = \begin{cases} 2 \cdot f(v), & \text{if } f(v) \leq n/2 \\ 2n + 1 - 2 \cdot f(v) & \text{if } f(v) > n/2 \end{cases}$$

It is easy to verify that f is a linear ordering of G with bandwidth(f) $\leq 2k$. Hence bandwidth(G)$\leq 2\cdot$ cyclic bandwidth(G). □

Corollary 3.9
For every graph $G = (V, E)$, cyclic bandwidth(G) $\geq \frac{1}{2}\cdot$ tree-width(G).

Theorem 3.10
For every graph $G = (V, E)$, cutwidth(G) $\geq$ treewidth (G).

Proof.
Suppose we have a linear ordering f of $G = (V, E)$ with cutwidth k. Let $n = |V|$. We now let $I = \{1, 2, \ldots, n\}$, for all $i \in I$; $X_i = \{w \mid f(w) > i \wedge \exists v \in V\colon (v, w) \in E \wedge f(v) \leq i\} \cup \{f^{-1}(i)\}$, and P is the pathgraph on n vertices, i.e. $P = (I, \{(i, i+1) \mid 1 \leq i \leq n-1\})$. Now we claim that $(\{X_i \mid i \in I\}, P)$ is a tree-decomposition of G with tree-width $\leq k$.

First note that for all $v \in V$, $v \in X_{f^{-1}(v)}$.

Secondly consider an edge $(v, w) \in E$. Either $f(v) < f(w)$ or $f(w) < f(v)$. Without loss of generality assume the former. Then $v \in X_{f^{-1}(v)}$, and $w \in X_{f^{-1}(v)}$, by definition.

Next suppose $i < j < k$ and $w \in X_i \cap X_k$. From $w \in X_k$ it follows that $f(w) > k \vee f(w) = k$, hence $f(w) \geq k > j > i$. Thus there must be a $v \in V$, with $(v, w) \in E \wedge f(v) \leq i$. So we have $f(w) > j \wedge \exists v \in V : (v, w) \in E \wedge f(v) \leq j$. Hence $w \in X_j$. It follows that $X_i \cap X_k \subseteq X_j$.

Finally note that for all i, $1 \leq i \leq n$, $|X_i| \leq |\{(v, w) \in E \mid f(v) \leq i < f(w)\}| + 1 \leq k + 1$. So the tree-width of the tree-decomposition is at most k. □

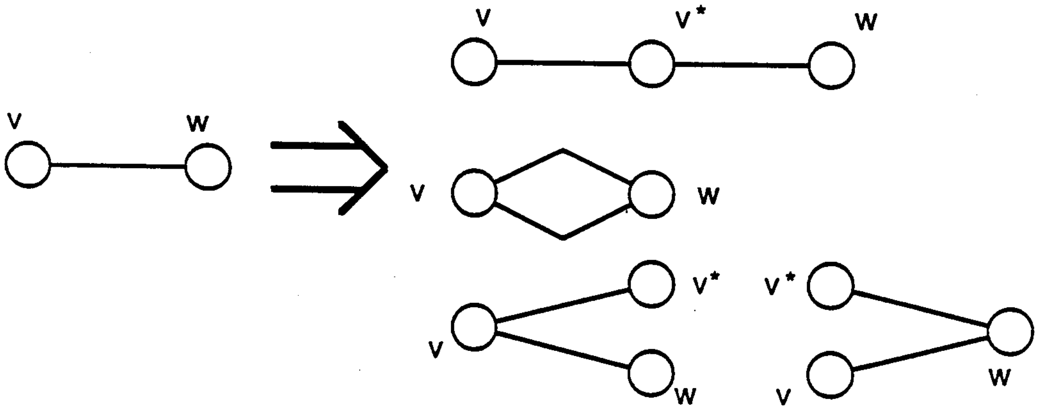
3.4 Classes of planar graphs.

In this section we consider a number of classes of planar graphs. Arbitrary planar graphs can have arbitrary large tree-width. For instance, an $n * n$ grid network has tree-width n [13]. For a number of classes it was already known that the tree-width of graphs in the class is bounded by some fixed number, e.g. the series-parallel graphs and the outerplanar graphs. We will review some of these results and will also show that every Halin graph has tree-width ≤ 5 and every k-outerplanar graph has tree-width $\leq 3^k - 1$.

First we consider series-parallel graphs. A series-parallel graph can have parallel edges, i.e. it is a multigraph. One can define a series-parallel graph recursively as follows:

- The graph with 2 vertices and one edge is a series-parallel graph:

- Let $G = (V, E)$ be a series parallel graph. One obtains a new series parallel graph G', by replacing any edge $(v, w) \in E$ in one of the following 3 manners:



where $v^* \notin V$, i.e. v^* is a new vertex.

From this definition and the definition of the tree-width of a graph, one easily can proof, with induction, that every series-parallel graph has tree-width ≤ 2. To be precise, the class of graphs with tree-width ≤ 2 equals the class of series-parallel graphs [17].

Next we consider the outerplanar graphs. A graph G is outerplanar, if it is planar and it can be drawn in such a way in the plane that all vertices lie on the exterior face. It can be shown that every outerplanar graph is a series-parallel graph [10]. Thus outerplanar graphs have tree-width 2. This can also shown in the following way.

For every outerplanar graph $G = (V, E)$, there must be a vertex v with $degree(v) = 1 \vee degree(v) = 2$. Suppose $degree(v) = 2$. Let $(v, w) \in E$, $(v, x) \in E$, $w \neq x$. Now $G' = (V - \{v\}, (E - \{(v, w), (v, x)\}) \cup \{(w, x)\})$ is an outerplanar graph, and we may assume, with induction, that we have a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of G with tree-width ≤ 2. There must be an $i \in I$, with $w \in X_i \wedge x \in X_i$. Now let $i^* \notin I$, $I^* = I \cup \{i^*\}$, $X_{i^*} = \{v, w, x\}$ and $T^* = (I^*, F \cup \{(i, i^*)\})$. One easily verifies that $(\{X_i \mid i \in I^*\}, T^*)$ is a tree-decomposition of G with tree-width at most 2.

A generalization of the outerplanar graphs are the k-outerplanar graphs.

Definition.

- A graph $G = (V, E)$ is 1-outerplanar if and only if it is outerplanar.
- For $k \geq 2$, a graph $G = (V, E)$ is k-outerplanar if and only if it is planar and it has a planar embedding such that if all vertices on the exterior face (and all adjacent edges) are deleted, then the connected components of the remaining graph are all $(k-1)$-outerplanar.

The notion of k-outerplanar graphs was introduced by Baker [3], who also showed that several NP-complete graph problems can be solved in polynomial time, when restricted to k-outerplanar graphs.

Theorem 3.11
Let $G = (V, E)$ be a k-outerplanar graph. Then treewidth$(G) \leq 3^k - 1$.

Proof.
We use induction to k. For $k = 1$, the theorem holds because every outerplanar graph has tree-width ≤ 2, as discussed above.

Now let $k \geq 2$ and assume that the theorem holds for all $k' \leq k - 1$ and let $G = (V, E)$ be a k-outerplanar graph, embedded in the plane, such that if all vertices on the exterior face are removed, then each connected component of the remaining plane is $k - 1$-outerplaner. We first prove the following lemma.

Lemma 3.11.1
There is a k-outerplanar graph $H = (V, F)$, with $E \subseteq F$ (i.e. G is a subgraph of H), and there is an embedding of H in the plane such that

- if we remove all vertices on the exterior face, then each of the remaining connected components is $(k-1)$-outerplanar.
- every interior face which contains at least one vertex on the exterior face of H has exactly 3 sides.

Proof.
For every interior face of G, containing at least one vertex v on the exterior face, we add edges from v to every other vertex in the face (if not already present). The resulting graph H fulfills the stated conditions. □

Let $H = (V, F)$ be given, as indicated by the previous lemma. From lemma 2.4 it follows that it is sufficient to prove that treewidth$(H) \leq 3^k - 1$. By lemma 2.3 we may suppose that H is biconnected.

Now we define $V_{ex} = \{v \in V \mid v$ is on the exterior face of $H\}$; $V_{in} = V - V_{ex}$; H_{ex} is the subgraph of H, induced by V_{ex} and H_{in} is the subgraph of H induced by V_{in}. It follows that H_{in} is $(k-1)$-outerplanar, and thus treewidth$(H_{in}) \leq 3^{k-2} - 1$.

Lemma 3.11.2
Suppose H_{ex} is a cycle. Then H_{in} is connected.

Proof.
Suppose not. Then there must be vertices v, $w \in V_{ex}$ with v and w adjacent to vertices in different components of H_{in}. We can choose v and w such that $(v, w) \in F$. Consider the vertices in V_{in} adjacent to v. We can order these $x_1, \ldots, x_r$, such that $\forall i \leq r$: $(x_i, x_{i+1}) \in F$. (Use the property that each face containing v has exactly 3 sides.) Likewise, let $y_1, \ldots, y_s$ be the vertices in V_{in}, adjacent to w, with $\forall i \leq s$, $(y_i, y_{i+1}) \in F$. (See fig. 3.) Now,

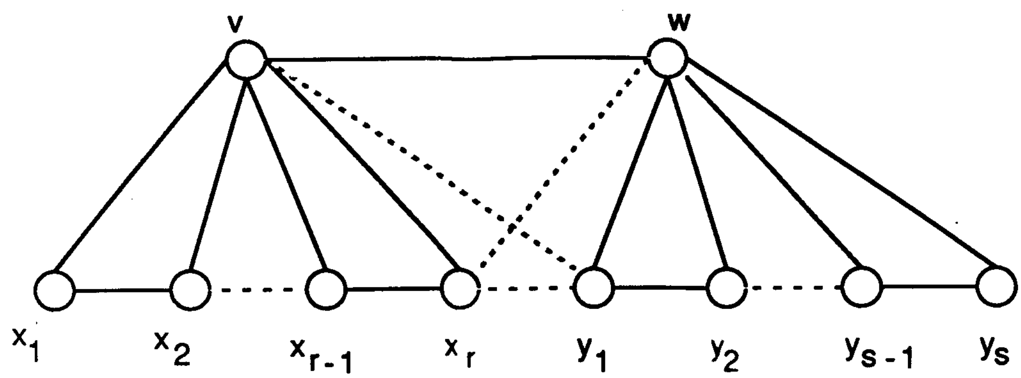


Figure 3:

because of the property that each face containing v and/or w, must have 3 sides, either $\{v, w, y_1\}$ and $\{v, y_1, x_r\}$ are faces of H, or $\{v, w, x_r\}$ and $\{w, y_1, x_r\}$ are faces of H. In both cases $(x_r, y_1) \in F$ and $x_1, x_2, \ldots, x_r, y_1, \ldots, y_s$ are in the same connected component in H_{in}. Contradiction. □

Now suppose H_{in} is connected. Let V_{inex} denote the set of vertices on the exterior face of H_{in}. It follows that each vertex $v \in V_{ex}$ only is adjacent to vertices in $V_{ex} \cup V_{inex}$.

Lemma 3.11.3
Suppose H_{ex} is a cycle and suppose H_{in} is biconnected. Then treewidth(H) $\leq 3^k - 1$.

Proof.
Note that the vertices in V_{inex} form a cycle. For all $x \in V_{inex}$ there are one and more *consecutive* vertices $x_1, \ldots, x_{r(x)} \in V_{ex}$ that are adjacent to x. (If not, then one can find a face with 4 sides, containing at least one vertex $v \in V_{ex}$). See fig. 4. H_{in} is $(k-1)$-outerplanar so we have, with induction,

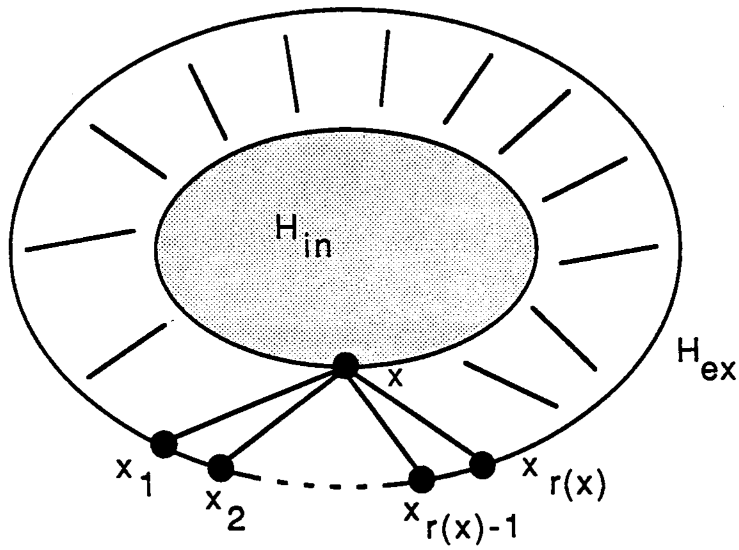


Figure 4

that there is a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, D))$ of H_{in} with tree-width at most $3^{k-1} - 1$.

For all $i \in I$, let $X_i' = X_i \cup \{x_1 \mid x \in X_i \cap V_{inex}\} \cup \{x_{r(x)} \mid x \in X_i \cap V_{inex}\}$. Recall that x_1 and $x_{r(x)}$ are the first and last vertex in V_{ex}, adjacent to x. (If $r(x) = 1$ then $x_1 = x_{r(x)}$).

Further, for all $x \in V_{inex}$ with $r(x) > 2$ we choose an $i(x) \in I$, with $x \in X_{i(x)}$. Let $I' = I \cup \{i_{x,j} \mid x \in V_{inex} \wedge r(x) > 2 \wedge 1 \leq j \leq r(x) - 2\}$, where all $i_{x,j} \notin I$, i.e. are new elements. Let $X'_{i_{x,j}} = \{x, x_j, x_{j+1}, x_{r(x)}\}$. Let $T' = (I', D')$, with $D' = D \cup \{(i(x), i_{x,1}) \mid x \in V_{inex} \wedge r(x) > 2\} \cup \{i_{x,j}, i_{x,j+1}) \mid x \in V_{inex} \wedge r(x) > 2 \wedge 1 \leq j \leq r(x) - 2\}$.

In other words, to each X_i of the old decomposition we add for each $x \in V_{inex}$ the first and last of the neighbors of x, that are in V_{ex}. For $x \in V_{inex}$ that are adjacent to more than 2 vertices, we choose a vertex $i \in I$

and add an extra branch with $r(x) - 2$ vertices to i in the tree, in order to represent $x_2, \ldots, x_{r(x)-1}$, as illustrated in fig. 5.

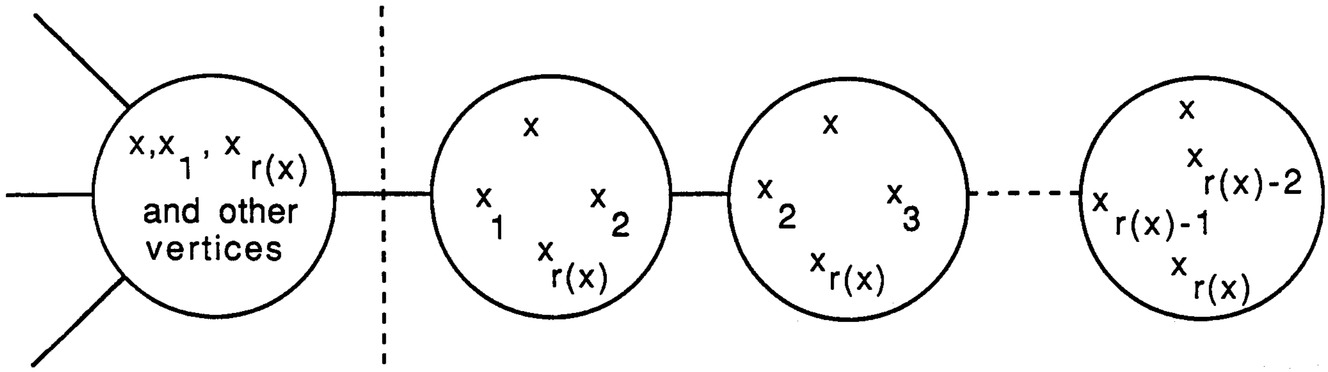


Figure 5

We now claim that $(\{X'_i \mid i \in I'\}, T' = (I', F'))$ is a tree-decomposition of H with tree-width $\leq 3^k - 1$.

It is easy to see that for all $v \in V$, there is a $i \in I'$ with $v \in X'_i$. Next we claim that $(v, w) \in F \Rightarrow \exists i \in I'$ with $v \in X'_i \wedge w \in X'_i$. We consider the following cases.

Case 1. $v \in V_{in} \wedge w \in V_{in}$. Then $\exists i \in I$ with $v \in X_i \wedge w \in X_i$. Hence also $v \in X'_i \supseteq X_i$ and $w \in X'_i$.

Case 2. $v \in V_{in} \wedge w \in V_{ex}$. Then $v \in V_{inex}$, and we can write $w = v_j$ with $1 \leq j \leq r(v)$. If $j = 1$ or $j = r(v)$, then for all $i \in I$ with $v \in X_i$ one has $v \in X'_i \wedge w = v_j \in X'_i$. If $1 < j < r(v)$, then $v \in X_{i_{v,j+1}}$ and $w = v_j \in X_{i_{v,j+1}}$.

Case 3. $v \in V_{ex} \wedge w \in V_{in}$. Similar to case 2.

Case 4. $v \in V_{ex} \wedge w \in V_{ex}$. Now there is a $x \in V_{inex}$, with $(v, x) \in F \wedge (w, x) \in F$, else we have a face in H, containing v and w and at least 4 sides. Hence, we can write $(v = x_j \wedge w = x_{j+1}, 1 \leq j \leq r(x) - 1)$ or $(v = x_{j+1} \wedge w = x_j, 1 \leq j \leq r(x) - 1)$. Without loss of generality, suppose the former. We have: if $r(x) \leq 2$ then for all $i \in I$ with $x \in X_i$:

$v = x_1 \in X'_i \wedge w = x_2 \in X'_i$, and if $r(x) > 2$, then, if $j < r(x) - 1$ then $v = x_j \in X_{i_{x,j}} \wedge w = x_j \in X_{i_{x,j}}$, and if $j = r(x) - 1$ then $v = x_{r(x)-1} \in V_{i_{x,r(x)-2}} \wedge w = x_{r(x)} \in X_{i_{x,r(x)-2}}$.

So we have that in all cases $\exists i \in I' : v \in X'_i \wedge w \in X'_i$.

Now we show that if $j \in I'$ is on the path in T' from $i \in I'$ to $k \in I'$ then $X'_j \supseteq X'_i \cap X'_k$. First consider the case that $i, k \in I$. It follows that $j \in I$ and if $z \in X'_i \cap X'_k$, then either $z \in V_{in}$, and hence $z \in X_i \cap X_k \subseteq X_j \subseteq X'_j$ or $z \in V_{ex}$, and hence z can be written $z = x_1$ or $z = x_{r(x)}$ for $x \in V_{inex}$ and now $x \in X_i \cap X_k \subseteq X_j$, and $z \in \{x_1, x_{r(x)}\} \subseteq X'_j$. So $X'_j \supseteq X'_i \cap X'_k$. Next consider the case that $i \in I' - I$, $k \in I$. It follows that there is an $x \in V_{inex}$ with $X'_i \cap X'_k \subseteq \{x, x_1, x_{r(x)}\}$. Now note that j must also be on the path in T' between $i(x)$ and k, and also $\{x, x_1, x_{r(x)}\} \in X'_{i(x)}$. It follows that $X'_i \cap X'_i \cap X'_k \subseteq X'_{i(x)} \cap X'_k \subseteq X'_j$. The other cases are similar or easy.

Finally note that $i \in I \Rightarrow |X'_i| \leq 3 \cdot |X_i| \leq 3^k$, and $i \in I' - I \Rightarrow |X'_i| = 4 \leq 3^k$. Hence the tree-width of the decomposition is at most $3^k - 1$. □

One may observe that for every interior face of H, that contains at least one vertex of V_{ex}, (and hence has exactly 3 sides), there must be at least one $i \in I'$, with X'_i containing each of the (three) vertices on this face. (There are basically 2 cases: one has a face with vertices $\{x, x_j, x_{j+1}\}$, or a face $\{x, y, x_1 = y_{r(y)}\}$, with $x, y \in V_{inex}$, $1 \leq j \leq r(x) - 1$. For the former case, the observation is straightforward. In the latter case, observe that we have $\exists i \in I : x \in X_i$ and $y \in X_i$, by definition. Now $\{x, y, x_1 = y_{r(y)}\} \subseteq X'_i$). So we have the following, slightly stronger result.

Lemma 3.11.4
Suppose H_{ex} is a cycle and suppose H_{in} is biconnected. Then there exists a tree-decomposition $(\{X_i \mid i \in I\}, T)$ of H with tree-width at most $3^k - 1$, and for each interior face of H that contains at least one vertex of V_{ex}, there must be at least one $i \in I$ with X_i containing each of the vertices on this face.

With induction to m we now prove the following lemma:

Lemma 3.11.5
Suppose H_{ex} is a cycle and H_{in} consists of m biconnected components. Then there exists a tree-decomposition $(\{X_i \mid i \in I\}, T)$ of H with tree-width $\leq 3^k - 1$, and for each interior face of H that contains at least one vertex of V_{ex}, there must be at least one $i \in I$ with X_i containing each of the vertices on this face.

Proof.
Use induction to m. For $m = 1$, the lemma follows from lemma 3.11.4.

Let $m > 1$, and let the lemma be true for all $m' < m$. There must be at least one biconnected component of H_{in}, that shares exactly one vertex v with one or more of the other biconnected components. So we can write $V_{in} = V^1 \cup V^2$, $V^1 \cap V^2 = \{v\}$ and the subgraph of H, induced by V^1, denoted by G^1, is biconnected , and the subgraph of H, induced by V^2 has $m - 1$ biconnected components and is connected.

Let $W^1 \subseteq V_{ex}$ be the set of vertices in V_{ex}, that are adjacent to vertices in V^1, and $W^2 \subseteq V_{ex}$ be the set of vertices in V_{ex}, that are adjacent to vertices in V^2. Note that $W^1 \cup W^2 = V_{ex}$. W^1 and W^2 both induce connected subgraphs of the cycle H_{ex} (see fig. 6.) (If not, then one derives a contradiction with the fact that H is biconnected.) Now, the "left most" vertex of W^1 equals the "right most" vertex of W^2, and vice versa. Let these two vertices be x and y.

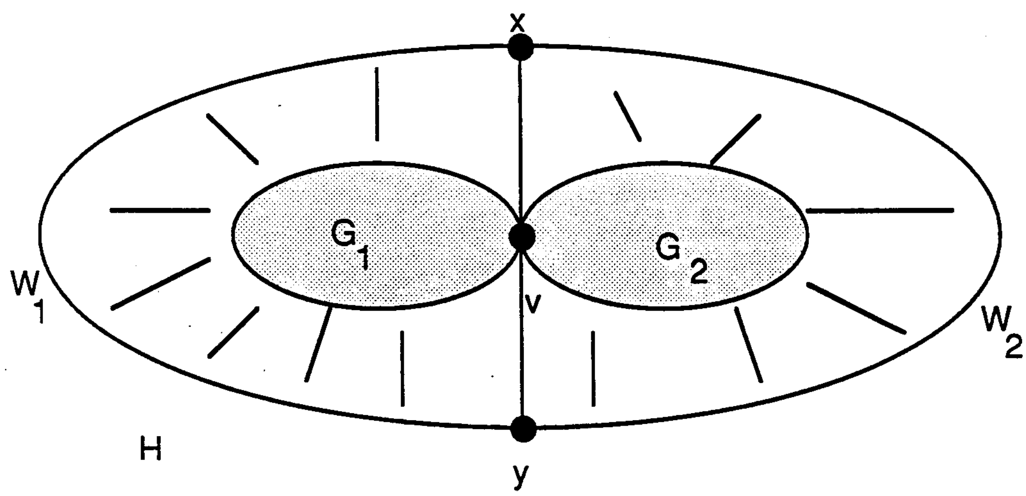


Figure 6

Now consider the graphs $H_1 = (V^1 \cup W^1, F_1)$ and $H_2 = (V^2 \cup W^2, F^2)$, with $F_i = \{(w_1, w_2) \in F \mid w_1, w_2 \in V^i \cup W^i\} \cup \{(x, y)\}$. (See fig. 7.).

Observe that $F_1 \cup F_2 = F \cup \{(x, y)\}$. If we define $(H_1)_{in}$, and $(H_2)_{in}$ similar to H_{in}, then $(H_1)_{in} = G_1$ and $(H_2)_{in} = G_2$. Now $(H_1)_{in}$ has one biconnected component and $(H_2)_{in}$ has $m-1$ biconnected components. By using the induction hypothesis, one can obtain now tree-decompositions $(\{X_i \mid i \in I_1\}, T_1 = (I_1, E_1))$ of H_1 and $(\{X_i \mid i \in I_2\}, T_2 = (I_2, E_2))$ of H_2, with the following properties:

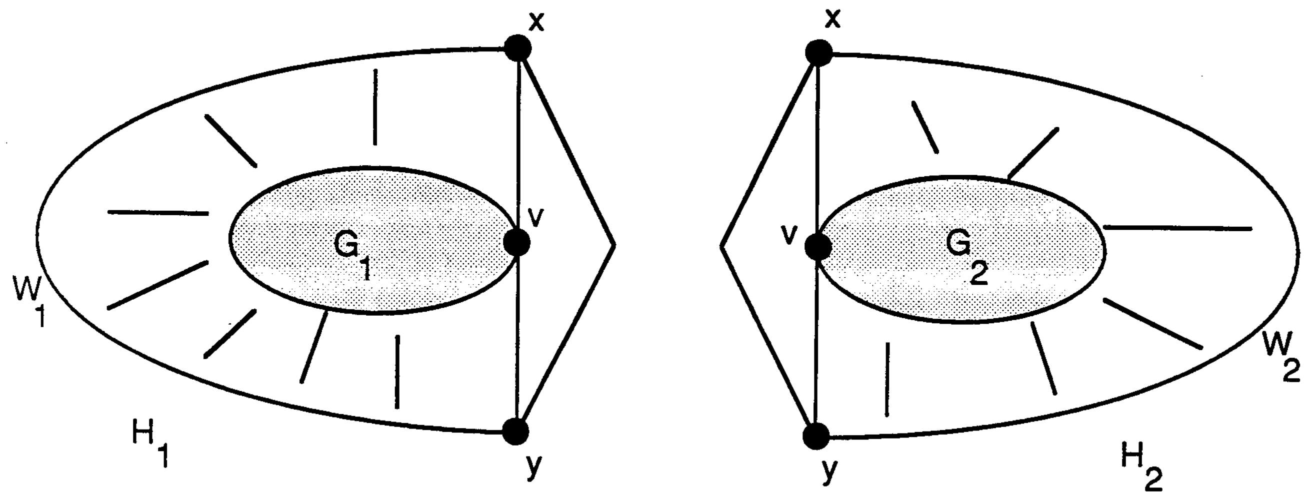


Figure 7

- $i \in I_1 \Rightarrow |X_i| \leq 3^k$.
- $i \in I_2 \Rightarrow |X_i| \leq 3^k$.
- $\exists i_1 \in I_1 : \{v,\, x,\, y\} \in X_{i_1}$. (Use that x, y are on the exterior face of H_1.)
- $\exists i_2 \in I_2 : \{v,\, x,\, y\} \in X_{i_2}$. (Use that x, y are on the exterior face of H_2.)

We now claim that $(\{X_i \mid i \in I_1 \cup I_2\}, T_2 = (I_1 \cup I_2, E_1 \cup E_2 \cup \{(i_1, i_2)\})$ is a tree-decomposition of H, with tree-width at most $3^k - 1$. (See fig. 8.)

It easily follows that $\bigcup_{i \in I_1 \cup I_2} X_i = V$. If $(v, w) \in F$ then $(v, w) \in F_1 \vee (v, w) \in F_2$, thus $(\exists i \in I_1 : v \in X_i \wedge w \in X_i) \vee (\exists i \in I_2 : v \in X_i \wedge w \in X_i)$.

Next let i, j, $k \in I_1 \cup I_2$, and suppose j is on the path from i to k in T. If i, $k \in I_1$ or i, $k \in I_2$, then it directly follows that $X_j \subseteq X_i \cap X_k$. Now suppose $i \in I_1$, $j \in I_1$, $k \in I_2$. (The other cases are similar.) Then $X_i \cap X_k \subseteq (V^1 \cup W^1) \cap (V^2 \cup W^2) = \{v,\, x,\, y\}$. Note that j is on the path in T from i to i_1, and $\{v,\, x,\, y\} \subseteq X_{i_1}$. Hence $X_i \cap X_k \subseteq X_i \cap X_{i_1} \subseteq X_j$.

Further it follows directly that $i \in I_1 \cup I_2 \Rightarrow |X_1| \leq 3^k$.

Finally notice that every interior face of H, containing at least one vertex of V_{ex}, either is an interior face of H_1, containing at least one vertex of W^1, or is an interior face of H_2, containing at least one vertex of W^2. Hence, for

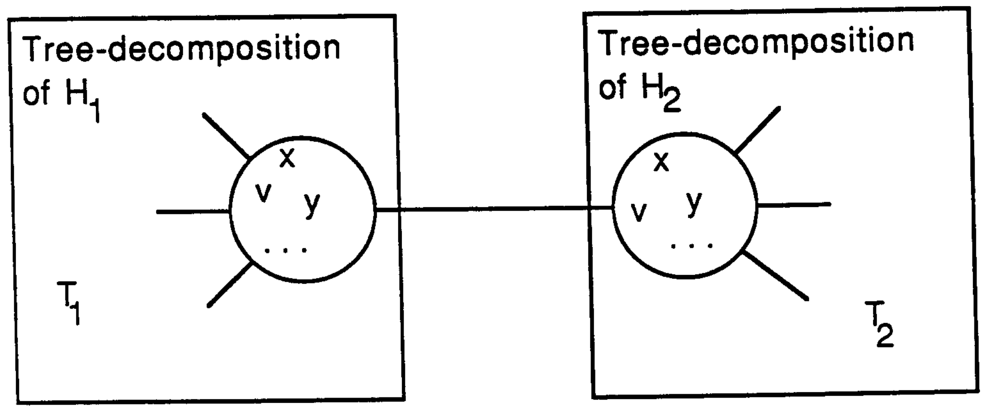


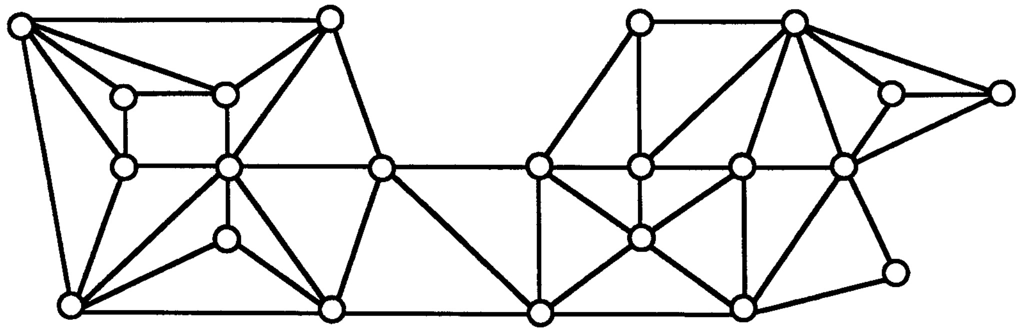
Figure 8

such a face, there will be a $i \in I_1 \cup I_2$, with X_i containing all vertices on the face. This completes the induction argument of lemma 3.11.5. □

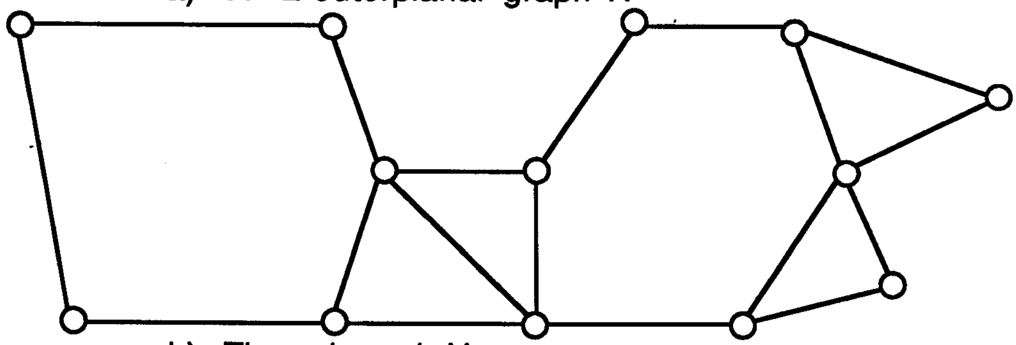
We will finally come to the proof of our main theorem 3.11. First we remark that H_{ex} always is an outerplanar graph. Because we supposed that H is biconnected, we may suppose that H has at least one face. (If not, then H is a graph with at most 2 vertices, and trivially treewidth(H) ≤ 1.)

Consider the dual graph $(H_{ex})^*$ of H_{ex}. The vertices of a dual graph of a planar graph correspond to interior faces of the graph and there is an edge between v and w in the dual graph if the corresponding faces share an edge. It is well known (see e.g. [16]), that the dual graph of an outerplanar graph G is a tree. Now consider a face of H_{ex}, corresponding to a leaf in $(H_{ex})^*$. This face must share exactly one edge (v, w) with another face of H_{ex}. Let $(V_{ex})^1$ be the set of vertices on this face, and let $(H_{ex})^1$ be the subgraph of H_{ex}, induced by $(V_{ex})^1$. Note that $(H_{ex})^1$ is a cycle.

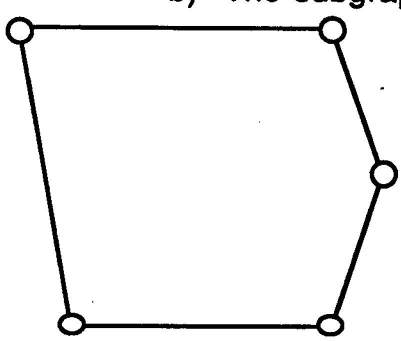
Now let $(V_{in})^1$ be the set of vertices in V_{in}, that are embedded in the area in the plane that is enclosed by H^1_{ex}. Let $V^1 = (V_{in})^1 \cup (V_{ex})^1$. Let H_1 be the subgraph of H, induced by V^1. Let $V^2 = (V - V^1) \cup \{v, w\}$. Let H_2 be the subgraph of H_1 induced by V^2. For an example see fig. 9.



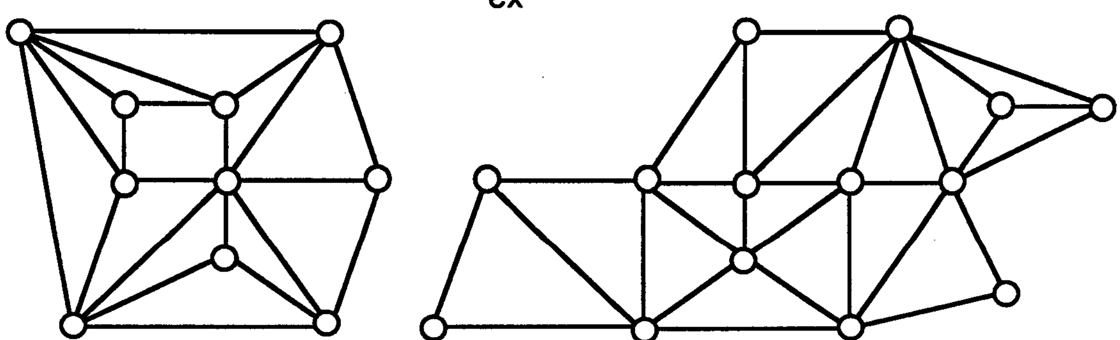
a) A 2-outerplanar graph H



b) The subgraph H_{ex}



c) The subgraph $(H_{ex})^1$



d) The subgraphs H^1 and H^2

Figure 9

Now H^1 and H^2 are also k-outerplanar graphs, but have a smaller size than H. By using induction we can obtain tree-decompositions $(\{X_i \mid i \in I_1\}, T_1 = (I_1, E_1))$ of H_1, and $(\{X_i \mid i \in I_2\}, T_2 = (I_2, E_2))$ of H_2, both with tree-width at most $3^k - 1$. There must be $i \in I_1$ with $v \in X_{i_1} \wedge w \in X_{i_2}$, and $i_2 \in I_2$ with $v \in X_{i_2} \wedge w \in X_{i_2}$. Now $(\{X_i \mid i \in I_1 \cup I_2\}, T = (I_1 \cup I_2, E_1 \cup E_2 \cup \{(i_1, i_2)\})$ is a tree-decomposition of H with tree-width at most $3^k - 1$. The proof of this fact is similar to the argument in lemma 3.11.5. This completes the proof of theorem 3.11. □

The last class of graphs that is considered in this section on classes of planar graphs is the class of the Halin graphs.

Definition.
A graph $G = (V, E)$ is a Halin graph if it can be obtained by embedding a tree without vertices with degree 2 in the plane and connecting its leaves by a cycle that crosses none of its edges.

It directly follows that the Halin graphs are contained in the 2-outerplanar graphs, and thus, by theorem 3.11 have treewidth 8 or less. However, by closer inspection one can obtain the following result.

Theorem 3.12
Let $G = (V, E)$ be a Halin graph. Then treewidth$(G) \leq 5$.

Proof.
The proof is similar to the proof of theorem 3.11. One uses that treewidth $(H_{in}) = 1$. □

3.5 Intersection graphs.

In this section we consider classes of intersection graphs. Each vertex in an intersection graph has associated with it an object in some space; there is an edge between two vertices if the corresponding objects intersect. For each of the considered classes of intersection graphs, we need on additional bound on the maximum cliquesizes of the graphs, to let the classes be subclasses of the graphs with bounded tree-width. We remark that each of the considered classes of graphs is a subclass of the class of the perfect graphs, i.e. for each of these graphs, its chromatic number equals the size of the largest clique that it contains.

Definition.
$G = (V, E)$ is a chordal graph if and only if every cycle with length exceeding three has an edge joining two non-consecutive vertices in the cycle.

Theorem 3.13 [7]
Let a graph $G = (V, E)$ be given. Then G is a chordal graph if and only if there is a tree $T = (W, F)$ such that one can associate with each vertex $v \in V$ a subtree $T_v = (W_v, F_v)$ of T, such that $(v, w) \in E$ if and only if $W_v \cap W_w \neq \emptyset$.

The following result was already noted by Robertson and Seymour [13].

Theorem 3.14 [13]
For every graph $G = (V, E)$ and integer $k \in N^+$, treewidth$(G) \leq k - 1$ if and only if G is subgraph of a chordal graph H that has maximum cliquesize at most k.

Proof.
Use the characterization of chordal graphs of theorem 3.13.
($\Leftarrow$) Use the tree-decomposition $(\{X_i \mid i \in W\}, T)$ with $X_i = \{v \in V \mid i \in W_v\}$. Note that $i \in W_v \wedge i \in W_w \Rightarrow (v, w) \in E$, hence X_i forms a clique in H, hence $|X_i| \leq k$. ($\Rightarrow$) Let a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of G be given. From the definition of tree-decomposition it follows that for all v, the subgraph of T, induced by $I_v = \{i \in I \mid v \in X_i\}$ is connected, i.e. is a subtree T_v of v. Now let H be the intersection graph given by these subtrees T_v, i.e. $H = (V, E')$ with $(v, w) \in E' \Leftrightarrow I_v \cap I_w \neq \emptyset$. One can argue that for all $v_1, \ldots, v_r \in V$, if for all i, $j \leq r$: $I_{v_i} \cap I_{v_j} \neq \emptyset$, then $\bigcup_{1 \leq i \leq r} I_{v_i} \neq \emptyset$. (Use that T is a tree, i.e. is cyclefree.) Hence if we have a clique with r vertices in H, then there must be an $i \in I$ with $|X_i| \geq r$. □

It follows that the treewidth of a chordal graph is its maximum cliquesize -1. Similar results can be obtained for subclasses of the chordal graphs.

Definition.

- The undirected path graphs are graphs with vertices corresponding to paths in a tree, and edges between vertices, if the corresponding paths have a vertex in common.
- The directed path graphs are graphs with vertices corresponding to paths in a tree with one vertex in the tree marked as root, and each

path is a subpath of a path from the root to a leaf, and there is an edge between two vertices if the corresponding paths have a vertex in common.

- The interval graphs are graphs with vertices, corresponding to connected subgraphs of a path (that is: a tree with each vertex degree ≤ 2), and edges between vertices, if the corresponding subgraphs have a vertex in common.
- The proper interval graphs are graphs with vertices, corresponding to connected subgraphs of a path, such that no subgraph is entirely contained in another, and edges between vertices, if the corresponding subgraphs have a vertex in common.

It follows that each interval graph is a directed path graph, each directed path graph is an undirected path graph, and each undirected path graph is a chordal graph. Similar as for the chordal graphs one can obtain the following results without difficulty.

Theorem 3.15

- $(G = V, E)$ has a tree-decomposition $\{(X_i \mid i \in I\}, T = (I, F))$ with tree-width $\leq k - 1$, and for each $v \in V$, $\{i \in I \mid v \in X_i\}$ induces a path in $T) \Leftrightarrow G$ is a subgraph of a undirected pathgraph $H = (V, E')$ with maximum cliquesize $\leq k$.
- $(G = (V, E)$ has a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ with tree-width $\leq k - 1$, and we can choose a root-vertex $r \in I$, such that for each $v \in V$, $\{i \in I \mid v \in X_i\}$ induces a subgraph of a path from r to a leaf in $T) \Leftrightarrow G$ is a subgraph of a directed pathgraph $H = (V, E')$ with maximum cliquesize $\leq k$.
- $(G = (V, E)$ has a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$, with tree-width $\leq k-1$, and T is a path) $\Leftrightarrow G$ is isomorphic to a subgraph of an interval graph H with maximum cliquesize $\leq k$.

The tree-width of graphs with the additional restriction that T is a path, (called path-width) was studied by Robertson and Seymour in [12]. We now consider the circular are graphs and the proper circular arc graphs.

Definition.

- $G = (V, E)$ is a circular arc graph, if we can associate with each $v \in V$ a path on a cycle, and $(v, w) \in E \Leftrightarrow$ the paths associated with v and w intersect.

- $G = (V, E)$ is a proper circular arc graph, if we can associate with each $v \in V$ a path on a cycle, and no path is entirely contained in another one, and $(v, w) \in E \Leftrightarrow$ the paths associated with v and w intersect.

Lemma 3.16
Let $G = (V_G, E_G)$ be an intersection graph, each vertex $v \in V_G$ corresponding to a connected subgraph of a graph $H = (V_H, E_H)$, and $(v, w) \in E_G$, if and only if the corresponding subgraphs have a vertex in common. Let k be the maximum cliquesize in G. Then treewidth$(G) \leq$ (treewidth $(H) + 1) * k - 1$.

Proof.
Let $(\{X_i \mid i \in I), T = (I, F))$ be a tree-decomposition of $H = (V_H, E_H)$ with tree-width $c - 1$. Associate with each vertex $v \in V_H$ a set $I_v \subseteq V_G$ with $I_v = \{w \in V_G \mid v$ is a vertex in the subgraph associated with $w\}$. By noting that $w_1 \in I_v \wedge w_2 \in I_v \Rightarrow (w_1, w_2) \in E_G$ one sees that $|I_v| \leq k$ for all $v \in V_H$. Now let for all $i \in I$, $Y_i = \bigcup_{v \in X_i} I_v$. One can verify that $(\{Y_i \mid i \in I\}, T = (I, F))$ is a tree-decomposition of G, and the tree-width of this decomposition is at most $c \cdot k - 1$. □

Corollary 3.17
Let $G = (V, E)$ be a circular arc graph (or a proper circular arc graph), with maximum clique size k. Then treewidth$(G) \leq 2k - 1$.

Proof.
Use the definitions, lemma 3.16, and the fact that the tree-width of a cycle is 2. □

Also we have the following properties of the proper interval graphs and the proper circular graphs.

Theorem 3.18
Let $G = (V, E)$ be a proper interval graph with maximum cliquesize k. Then: bandwidth$(G) \leq k - 1$.

Proof.
Suppose we have associated with each $v \in V$ a set of vertices $S_v = \{l_v, l_v + 1, \ldots, r_v - 1, r_v\} \subseteq \{1, 2, \ldots, N\}$, such that $(v, w) \in E \Leftrightarrow S_v \cap S_w \neq \emptyset$; and $S_v \subseteq S_w \Rightarrow v = w$.

Let $|V| = n$. We pose a total ordering $\prec$ on V, such that $v \prec w \Leftrightarrow l_v < l_w$. Note that $v \neq w$ implies that $l_v \neq l_w$, else $S_v \subseteq S_w$ or $S_w \subseteq S_v$. One can now obtain a unique linear ordering f of G, with $f(v) < f(w) \Leftrightarrow v \prec w$, i.e. $f(v) < f(w) \Leftrightarrow l_v < l_w$. We claim that bandwidth$(f) \leq k-1$.

Consider an edge $(v, w) \in E$ and without loss of generality suppose $f^{-1}(v) + i$, for all i, $0 \leq i \leq l$. (Thus $v = v_o$, $w = v_l$). We have that $(v, w) \in E \Rightarrow S_v \cap S_w \neq \emptyset$, thus $l_w < r_v$, i.e. $l_w \in S_v$. For all i, $1 \leq i \leq l-1$, we have $\neg(S_{v_i} \subseteq S_v)$ and $l_v < l_{v_i} < l_w < r_w$, thus $r_{v_i} > l_w$, hence $l_w \in S_{v_i}$. So $l_w \in \bigcap_{0 \leq i \leq l} S_{v_i}$. So the vertices $v_0, v_1, v_2, \ldots v_{l-1}, v_l$ form a clique in G with $l+1$ vertices. It follows that $f(w) - f(v) = l \leq k-1$. We now have : $\forall (v, w) \in E : |f(v) - f(w)| \leq k-1$, hence bandwidth$(f) \leq k-1$. □

Theorem 3.19
Let $G = (V, E)$ be a proper circular arc graph with maximum cliquesize k. Then: cyclic bandwidth$(G) \leq k-1$.

Proof.
Similar to the proof of theorem 3.18. □

Corollary 3.20
For all proper circular arc graphs $G = (V, E)$: treewidth$(G) \leq 2k-2$.

Proof.
Use corollary 3.9 and theorem 3.19. □

3.6 Tree-partite graphs.

Seese [15] introduced the notion of ($n-$bounded) tree-partite graphs, and showed that a number of important graph-problems can be solved in polynomial time for graphs, represented as $n-$bounded tree-partite graphs, with n bounded by some fixed constant, and the degree of the graph also bounded by some fixed constant.

Definition. [15]
Let $n \geq 1$. A graph $G = (V, E)$ is said to $n-$bounded tree-partite, if there is a tree $T = (I, F)$ and a collection $\{A_t \mid t \in I\}$, such that

- $V = \bigcup_{t \in I} A_t$.
- $A_t \cap A_{t'} = \emptyset$, for all $t, t' \in V, t \neq t'$.

- For all $e = (v, w) \in E$, either there is a $t \in I$ such that $v \in A_t \wedge w \in A_t$ or there are $t, t' \in I$ with t, t' adjacent in the tree T, and $v \in A_t \wedge w \in A_{t'}$.

- For all $t \in I$, $|A_t| \leq n$.

Theorem 3.21 [15]
If G is $k-$bounded tree-partite, then treewidth $(G) \leq 2k - 1$.

We have also a connection of this notion with the notion of emulation ([4]).

Definition.

- Let $G = (V_G, E_G), H = (V_H, E_H)$ be graphs. A mapping $f : V_G \rightarrow V_H$ is called an emulation, if for all $(v, w) \in E_G : f(v) = f(w) \vee (f(v), f(w)) \in E_H$.

- The cost of an emulation $f; V_G \rightarrow V_H$ is cost $(f) = \max_{v \in V_H} |f^{-1}(v)|$.

One easily obtains the following relation between the two notions.

Theorem 3.22
Let $G = (V, E)$ be a graph. G is $k-$bounded tree-partite, if and only if there is a tree $T = (W, F)$ and an emulation $f : V \rightarrow W$ of G on T with $\text{cost}(f) \leq k$.

One can show the following relation between the cost of emulations of G on a path and the bandwidth of G, very similar to a result in [4].

Theorem 3.23
Let $G = (V, E)$ be a graph, $k \in N^+$. Then: bandwidth $(G) \leq k \Rightarrow$
$\Rightarrow$ There is an emulation f of G on a path P with $\text{cost}(f) \leq k$
$\Rightarrow$ bandwidth$(G) \leq 2^k - 1$.

Corollary 3.24
For all graphs $G = (V, E)$, if bandwidth$(G) \leq k$, then G is a $k-$bounded tree-partite graph.

4 Conclusions

This paper shows some inclusion relations between the class of the graphs with treewidth bounded by some fixed number, and a number of subclasses. This also resolves some open problems of [10]. For instance, it follows that the problems to determine, whether an almost tree with parameter k (k fixed), or a graph with bandwidth k (k fixed) has a Hamiltonian circuit; and the Chromatic Number problem, restricted to almost trees with parameter k (k fixed), all can be solved in polynomial time.

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H.L. Bodlaender

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Utrecht University
Department of Computer Science
Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

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H.L. Bodlaender

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Department of Computer Science
Utrecht University
P.O.Box 80.089
3508 TB Utrecht
The Netherlands

CLASSES OF GRAPHS WITH BOUNDED TREE-WIDTH*

H.L. Bodlaender

December 1986

Abstract

In this paper we show a number of classes of graphs to be subclasses of the graphs with tree-width, bounded by some constant integer k, (also called the partial k-trees). These classes include all trees, forests, almost trees with parameter k_1 (k_1 a constant), graphs with bandwidth or cutwidth bounded by some constant, outerplanar graphs, series-parallel graphs, Halin graphs, k_2-outerplanar graphs (k_2 a constant), k_3-bounded tree-partite graphs (k_3 a constant), chordal graphs with maximum clique size k_4 (k_4 a constant) and circular arc graphs with maximum clique size k_5 (k_5 a constant). Some of these results were well-known, others are new. Also some other relations between the considered classes of graphs are shown. For many of the classes, it has been shown that many NP-complete problems can be solved in polynomial time, when restricted to graphs in the specific class. The results in this paper illustrate why this similarity occurs.

1 Introduction

NP-complete problems are generally believed not to be solvable in polynomial time. Hence there is much effort spent on finding subproblems of NP-complete problems for which polynomial time algorithms can be designed. For a number of classes of graphs, it has been shown that many NP-complete graph problems become solvable in polynomial time, when restricted to graphs in the specific class. An overview of some important

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NP-complete graph problems, and their (known) complexity when restricted to a number of important classes of graphs is given in [10].

In this paper we show that many of the classes, that yield polynomial time solutions for many problems that are NP-complete for general graphs, are contained in the class of graphs with tree-width bounded by some constant integer k, also called the partial k-graphs. Arnborg and Proskurowski [2] show that for many NP-complete graph problems linear time algorithms can be obtained when one restricts the instances to graphs with tree-width bounded by some constant k. These results illustrate the similarity in the complexity results that are known for the various discussed classes of graphs.

We consider the following classes of graphs, and show them to have tree-width $\leq k$, for some constant k:

- Trees and forests
- Almost trees with parameter k_1
- Graphs with bandwidth at most k_2
- Graphs with cutwidth at most k_3
- Outerplanar graphs
- Series-parallel graphs
- Halin graphs
- k_4-outerplanar graphs
- Chordal graphs with maximum cliquesize k_5
- Undirected path graphs with maximum clique size k_6
- Directed path graphs with maximum clique size k_7
- Interval graphs with maximum cliquesize k_8
- Proper interval graphs with maximum cliquesize k_9
- Circular arc graphs with maximum cliquesize k_{10}
- Proper circular arc graphs with maximum cliquesize k_{11}
- k_{12}-bounded tree-partite graphs

where k_1, k_2, $\cdots$, k_{12} are fixed constants. Some of the inclusion-relations are already well known, but are included in this paper for completeness sake. A schematic overview of the results is given in fig. 1 and fig. 2.

Throughout this paper we will assume all graphs to be undirected and free from self-loops and parallel edges, unless mentioned otherwise.

Class of graphs	Upperbound for Maximum Tree-width	Reference
Trees, forests	1	
Almost trees with parameter k	$k+1$	
Graphs with bandwidth $\leq k$	k	
Graphs with cyclic bandwidth $\leq k$	$2k$	
Graphs with cutwidth $\leq k$	k	
Series-parallel graphs	2	[17]
Outerplanar graphs	2	[17]
Halin graphs	5	
k-outerplanar graphs	$3^k - 1$	
Chordal graphs with max. cliquesize k	$k-1$	[7,13]
Undirected pathgraphs with max. cliquesize k	$k-1$	[13]
Directed pathgraphs with max. cliquesize k	$k-1$	[13]
Interval graphs with max. cliquesize k	$k-1$	[13]
Proper interval graphs with max. cliquesize k	$k-1$	[13]
Circular arc graphs with max. cliquesize k	$2k-1$	
Proper circular arc graphs with max. cliquesize k	$2k-2$	
k-bounded tree-partite graphs	$2k-1$	[15]

Figure 1: Classes of graphs and upperbounds for the maximum tree-width of graphs in the classes.

2 Partial k-trees and the tree-width of a graph

Let C_k be the complete graph on k vertices. The (partial) k-trees are defined as follows:

Definition.

- C_k is a k-tree.

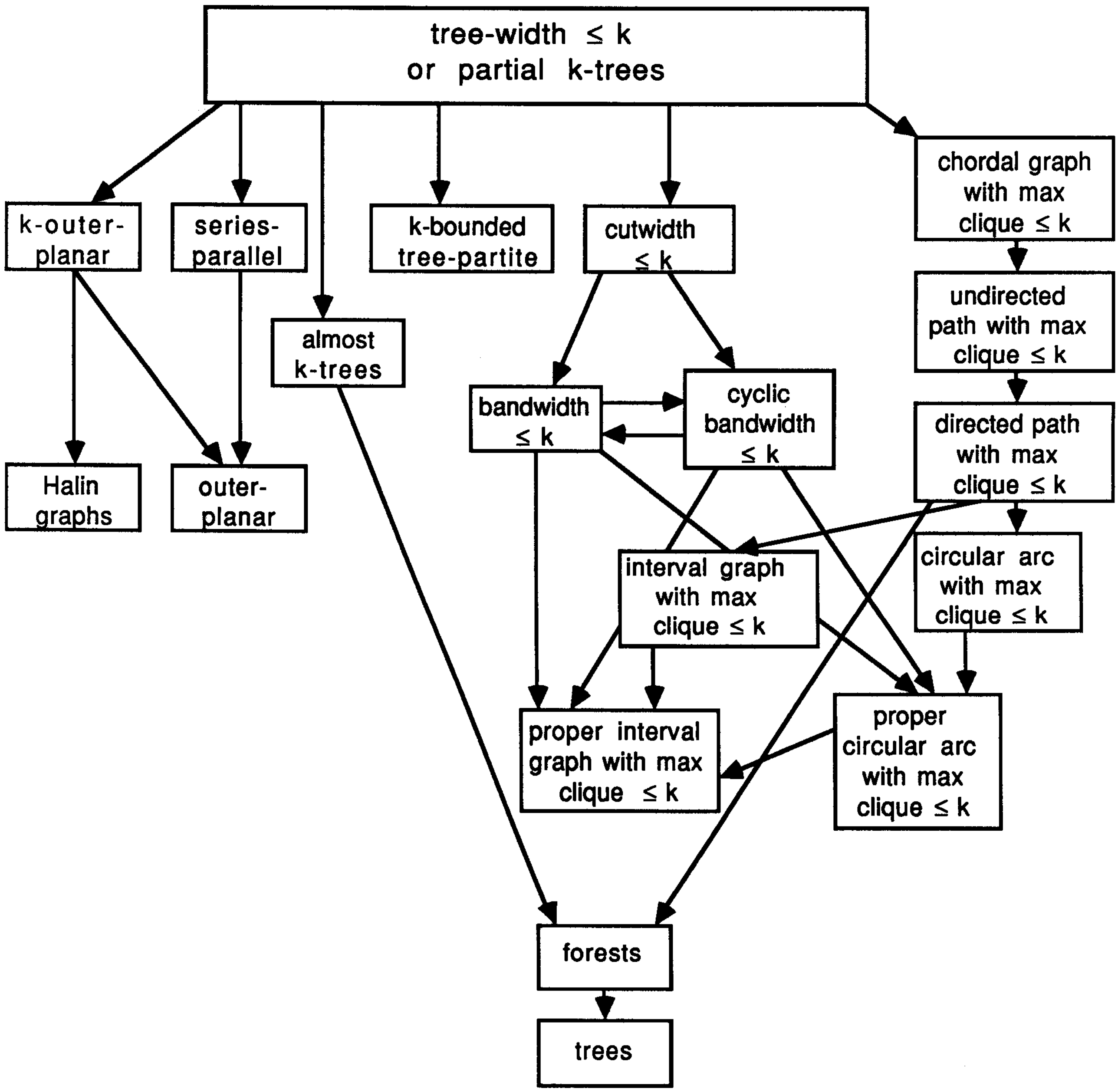


Figure 2. Schematic overview of containment relations between discussed classes of graphs. Note that the constants k in different boxes may denote different constants. E.g. not every graph with bandwidth ≤ k has cutwidth ≤ k, but for every k, there is an m, such that every graph with bandwidth ≤ k has cutwidth ≤ m,

- If $G = (V, E)$ is a k-tree, and $V' \leq V$ is a set of k vertices, inducing a complete subgraph in G, then $G' = (V \cup \{v\}, E \cup \{(w, v) \mid w \in V'\})$, with v a new vertex, i.e. $v \notin V$, is a k-tree.

I.e. a new k-tree G' can be obtained by taking a k-tree G and adding a new vertex v with edges to each vertex in a clique with k vertices in G.

Definition.
H is a partial k-tree, if H is a subgraph of a k-tree G.

Lemma 2.1
Let G be a partial k-tree. Then G does not contain a clique with $k + 2$ vertices.

Proof.
It is sufficient to prove the lemma for G a k-tree. Use induction. If a k-tree $G = (V, E)$ does not contain a clique with $k + 2$ vertices, then $G' = (V \cup \{v\}, E \cup \{(w, v) \mid w \in V'\})$, with $V' \subseteq V$, $|V'| \leq k$, $v \notin V$, will also not contain a clique with $k + 2$ vertices. □

Another way of characterizing partial k-trees is with help of the notion of tree-width, introduced by Robertson and Seymour [13].

Definition.
Let $G = (V, E)$ be a graph. A tree-decomposition of G is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, with $\{X_i \mid i \in I\}$ a family of subsets of V, and T a tree, with the following properties:

- $\bigcup_{i \in I} X_i = V$
- For every edge $e = (v, w) \in E$, there is a subset X_i, $i \in I$ with $v \in X_i$ and $w \in X_i$.
- For all $i, j, k \in I$, if j lies on the path in T from i to k, then $X_i \cap X_k \subseteq X_j$.

The tree-width of a tree-decomposition $(\{X_i \mid i \in I\}, T)$ is $\max_{i \in I} |X_i| - 1$. The tree-width of G, denoted by tree-width(G) is the minimum tree-width of a tree-decomposition of G, taken over all possible tree-decomposition of G.

Theorem 2.2 [1,13]
G is a partial k-tree if and only if G has tree-width k or less.

Independently, Arnborg, Corneil and Proskurowski [1], and Robertson and Seymour [13] have shown that there exist polynomial algorithms to test whether a graph has tree-width $\leq k$, for any given, fixed k. Arnborg, Corneil and Proskurowski [1] have also shown that the problem to determine the tree-width of a graph is NP-complete. Arnborg and Proskurowski [2] have shown that several NP-complete graphs problems are solvable in linear time when restricted to graphs with tree-width bounded by some fixed k (or equivalently, to partial k-graphs, for fixed k). Similar results can be found in [5]. The following lemma's will be used in section 3.

Lemma 2.3
Let $G = (V, E)$ be a graph and let $k \in N^+$. Then treewidth$(G) \leq k$, if and only if for each biconnected component $G_i = (V_i, E_i)$ of G, treewidth$(G_i) \leq k$.

Proof.
$\Rightarrow$ Trivial.
$\Leftarrow$ Suppose we have tree-decompositions $(\{X_i^1 \mid i \in I^1\}, T_1), \cdots, (\{X_i^c \mid i \in I^c\}, T_c)$ of the biconnected components $G_1, \ldots, G_c$ of G with tree-width $\leq k$ each. Now one can obtain tree-decompositions of connected subgraphs of G, consisting of more and more biconnected components, each tree-decomposition having tree-width k or less, in the following manner. Suppose we have connected subgraphs $G_\alpha = (V_\alpha, E_\alpha)$, $G_\beta = (V_\beta, E_\beta)$, each consisting of one or more biconnected components of G. Let $(\{X_i^\alpha \mid i \in I^\alpha$, $T_\alpha = (I^\alpha, F^\alpha))$, $(\{X_i^\beta \mid i \in I^\beta\}$, $T_\beta = (I^\beta, F^\beta))$, be tree-decompositions of G_α and G_β, respectively, with tree-width $\leq k$ each. Further suppose $\{v\} = V_\alpha \cap V_\beta$. (I.e. G_α and G_β share exactly one vertex v). There are $i_0 \in I^\alpha$, $i_1 \in I^\beta$ with $v \in X_{i_0}^\alpha$, $v \in X_{i_1}^\beta$. Now let $T_\gamma = (I^\alpha \cup I^\beta, F^\alpha \cup F^\beta \cup \{(i_0, i_1)\})$. T_γ is a tree. Now it is easy to check that $(\{X_i^\alpha \mid i \in I^\alpha\} \cup \{X_i^\beta \mid i \in I^\beta\}$, $T_\gamma)$ is a tree-decomposition of $G_\gamma = (V_\alpha \cup V_\beta, E_\alpha \cup E_\beta)$ with tree-width $\leq k$.

We can repeat this construction, obtaining tree-decompositions of connected subgraphs of G, containing more and more biconnected components, each with tree-width $\leq k$. (If G is not connected, then a similar, but still easier construction can be used). Finally one obtains a tree-decomposition of G with width $\leq k$. □

Lemma 2.4
Let G be a subgraph of H. Then treewidth$(G) \leq$ treewidth (H).

Proof.
Trivial. □

3 Classes of graphs with bounded tree-width

In this section we will discuss a number of classes of graphs with the property that the maximum tree-width of all graphs in the class is bounded by some fixed number.

3.1 Trees and forests.

The following well-known propositions follow directly from the definitions of "k-tree" and " partial k-tree ".

Proposition 3.1
$G = (V, E)$ is a tree, if and only if G is a 1-tree.

Proposition 3.2
$G = (V, E)$ is a forest, if and only if treewidth(G) ≤ 1.

3.2 Almost trees with parameter k.

Definition.
$G = (V, E)$ is an almost tree with parameter $\leq k$ iff for some spanning tree T of G, in each biconnected component of G there are at most k edges of G that are not in T.

With other words, $G = (V, E)$ is an almost tree with parameter $\leq k$ if and only if for each biconnected component $G_i = (V_i, E_i)$ of G one has $|E_i| - |V_i| + 1 \leq k$.

Theorem 3.3
Let $G = (V, E)$ be an almost tree with parameter k. Then treewidth(G) $\leq$ $k + 1$.

Proof.
From lemma 2.3 it follows that is sufficient to prove the theorem for biconnected graphs $G = (V, E)$. Let $G = (V, E)$ be a biconnected almost tree with parameter k. Let $T_0 = (V, F)$ be a spanning tree of G. Note that $|E - F| \leq k$. Now let $(\{X_i \mid i \in I\}, T)$ be a tree-decomposition of T_0 with tree-width 1, i.e. for all $i \in I: |X_i| \leq 2$. We now write $E - F = \{(v_1, w_1), (v_2, w_2), \ldots, (v_l, w_l)\}$ $(l \leq k)$.

We can now obtain a tree-decomposition of G with tree-width $\leq k+1$, by adding the vertices $v_1, \ldots, v_e$ to each set X_i, i.e. we have the tree-decomposition $(\{X_i \cup \{v_1, \ldots, v_e\} \mid i \in I\}, T$. One easily verifies that this is a correct tree-decomposition. For instance, for every edge $(v_j, w_j) \in E - F$, there is a $i \in I$, with $w_j \in X_i$. Hence $w_j \in X_i \cup \{v_1, \ldots, v_e\}$, and by definition $v_j \in X_i \cup \{v_1, \ldots, v_e\}$. The tree-width of this tree-decomposition is $\max_{i \in I} \mid X_i \cup \{v_1, \ldots, v_e\}| - 1 \leq 2 + l - 1 \leq k + 1$. □

Coppersmith and Viskin [6] and Gurevich, Stockmeyer and Viskin [9] have shown that a number of important NP-complete graph problems can be solved in linear time for graphs, that are almost trees with parameter k, for fixed k. The time, needed for these algorithms is exponential in k. Theorem 3.4 shows the relation of these results with the results of Arnborg and Proskurowski [2].

3.3 Graphs with bounded bandwidth or cutwidth.

In this section we consider graphs with bandwidth or cutwidth bounded by some fixed number. These graphs can be recognized in polynomial time (see [14],[8]), (but the time is exponential in the bandwidth or cutwidth of the graph). In [11] it is shown that several NP-complete problems can be solved in polynomial time for graphs $G = (V, E)$ with bandwidth bounded by $c \cdot \log(|V|)$ for some constant c. In [5] similar results are obtained for the larger class of graphs with treewidth, bounded by $c \log(|V|)$ for some constant c. (One must assume that the graphs are given together with the corresponding linear orderings or tree-decompositions.)

Definition.
Let $G = (V, E)$ be a graph, with $n = |V|$.

- A linear ordering of G is a bijective mapping $f\colon V \to \{1, \ldots, n\}$.
- A linear ordering f of G is said to have bandwidth k if $k = \max_{(u,v) \in E} |f(u) - f(v)|$.
- The bandwidth of G, denoted by bandwidth(G), is the minimum bandwidth of a linear ordering f, over all possible linear orderings of G.
- A linear ordering f of G is said to have cutwidth k if $k = \max_{1 \leq i \leq n} |\{(u, v) \in E \mid f(u) \leq i < f(v)\}|$.

- the cutwidth of G, denoted by cutwidth(G), is the minimum cutwidth of a linear ordering f over all possible linear orderings of G.

The following variant of the notion of bandwidth, called "cyclic bandwidth" was introduced in [11].

Definition.
Let $G = (V, E)$ be a graph, with $n = |V|$.

- A linear ordering f of G is said to have cyclic bandwidth k if $k = \max_{(u,v)\in E}(\min(|f(u)-f(v)|, n-|f(u)-f(v)|))$ (= the maximum distance of $f(u)$ and $f(v)$ in a ring with n vertices $R_n = (\{1, \ldots, n\}, \{(1,2), (2,3), \ldots, (n-1, n), (n, 1)\})$, taken over all $(u, v) \in E$).
- The cyclic bandwidth of G, denoted by cyclic bandwidth(G), is the minimum cyclic bandwidth of a linear ordering f over all possible linear orderings of G.

Lemma 3.4
Let $G = (V, E)$ be a graph, with bandwidth(G) = k. Then cutwidth (G) $\leq \frac{k(k+1)}{2}$.

Proof.
Consider a linear ordering f of G with bandwidth k. Let $n = |V|$. For all i, $1 \leq i \leq n$, $|\{(u,v) \in E \mid f(u) \leq i < f(v)\}| \leq |\{(j_1, j_2) \in \{1, \ldots, n\} * \{1, \ldots, n\} \mid j_1 \leq i < j_2 \wedge |j_1 - j_2| \leq k\}| \leq k(k+1)/2$. Hence cutwidth($G$) $\leq k(k+1)/2$. □

Showing that every graph with bandwidth $\leq k$ is a partial k-tree, i.e. has tree-width $\leq k$, is, in particular, very simple.

Definition.
The maximal graph on n vertices with bandwidth k is the graph $G_{k,n} = (V_n, E_{k,n})$, with $V_n = \{1, 2, \ldots, n\}$ and $E_{k,n} = \{(i,j) \mid i,j \in V_n \wedge |i-j| \leq k\}$. The following observation was made by Saxe [14].

Lemma 3.5 [14]
Let $G = (V, E)$ with $|V| = n$. Then bandwidth(G) $\leq k$, if and only if G is isomorphic to a subgraph of $G_{k,n}$.

Lemma 3.6

1. For all k, n, $n \geq k \geq 1$, $G_{k,n}$ is a k-tree.
2. For all k, $n \geq 1$, $G_{k,n}$ is a partial k-tree.

Proof.

1. Use induction to n.
2. Use (1) and the fact that every graph on $n \leq k$ vertices is a partial k-tree.

□

Corollary 3.7
For every graph $G = (V, E)$, bandwidth(G) $\geq$ treewidth(G).

Lemma 3.8
For every graph $G = (V, E)$:
cyclic bandwidth(G) $\leq$ bandwidth(G) $\leq 2\cdot$ cyclic bandwidth(G).

Proof. First we remark that cyclic bandwidth(G)$\leq$ bandwidth(G) follows directly from the definitions.

Now let $|V| = n$. Let $f : V \rightarrow \{1, \ldots, n\}$ be a linear ordering of G with cyclic bandwidth(f) $\leq k$. We suppose n is even. If n is odd, then a similar construction can be made. Let $g : V \rightarrow \{1, \ldots, n\}$ be defined by

$$g(v) = \begin{cases} 2 \cdot f(v), & \text{if } f(v) \leq n/2 \\ 2n + 1 - 2 \cdot f(v) & \text{if } f(v) > n/2 \end{cases}$$

It is easy to verify that f is a linear ordering of G with bandwidth(f) $\leq 2k$. Hence bandwidth(G)$\leq 2\cdot$ cyclic bandwidth(G). □

Corollary 3.9
For every graph $G = (V, E)$, cyclic bandwidth(G) $\geq \frac{1}{2}\cdot$ tree-width(G).

Theorem 3.10
For every graph $G = (V, E)$, cutwidth(G) $\geq$ treewidth (G).

Proof.
Suppose we have a linear ordering f of $G = (V, E)$ with cutwidth k. Let $n = |V|$. We now let $I = \{1, 2, \ldots, n\}$, for all $i \in I$; $X_i = \{w \mid f(w) > i \wedge \exists v \in V\colon (v,w) \in E \wedge f(v) \leq i\} \cup \{f^{-1}(i)\}$, and P is the pathgraph on n vertices, i.e. $P = (I, \{(i, i+1) \mid 1 \leq i \leq n-1\})$. Now we claim that $(\{X_i \mid i \in I\}, P)$ is a tree-decomposition of G with tree-width $\leq k$.

First note that for all $v \in V$, $v \in X_{f^{-1}(v)}$.

Secondly consider an edge $(v, w) \in E$. Either $f(v) < f(w)$ or $f(w) < f(v)$. Without loss of generality assume the former. Then $v \in X_{f^{-1}(v)}$, and $w \in X_{f^{-1}(v)}$, by definition.

Next suppose $i < j < k$ and $w \in X_i \cap X_k$. From $w \in X_k$ it follows that $f(w) > k \vee f(w) = k$, hence $f(w) \geq k > j > i$. Thus there must be a $v \in V$, with $(v, w) \in E \wedge f(v) \leq i$. So we have $f(w) > j \wedge \exists v \in V : (v, w) \in E \wedge f(v) \leq j$. Hence $w \in X_j$. It follows that $X_i \cap X_k \subseteq X_j$.

Finally note that for all i, $1 \leq i \leq n$, $|X_i| \leq |\{(v,w) \in E \mid f(v) \leq i < f(w)\}| + 1 \leq k + 1$. So the tree-width of the tree-decomposition is at most k. □

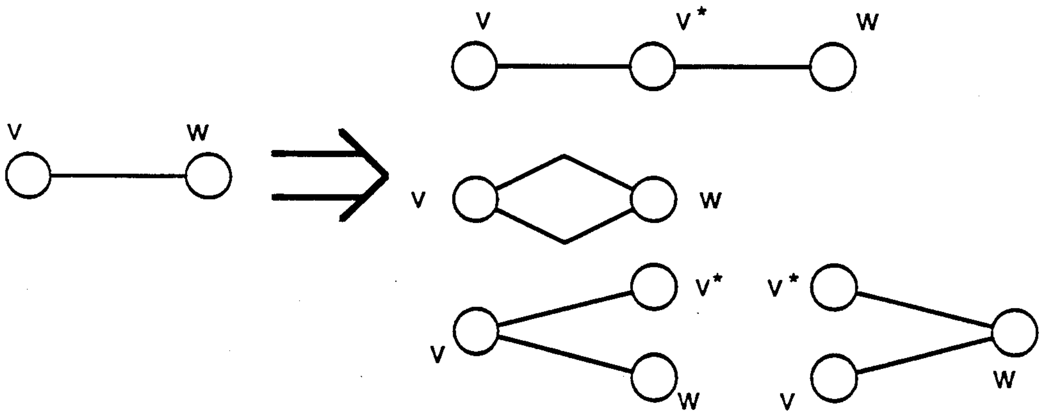
3.4 Classes of planar graphs.

In this section we consider a number of classes of planar graphs. Arbitrary planar graphs can have arbitrary large tree-width. For instance, an $n * n$ grid network has tree-width n [13]. For a number of classes it was already known that the tree-width of graphs in the class is bounded by some fixed number, e.g. the series-parallel graphs and the outerplanar graphs. We will review some of these results and will also show that every Halin graph has tree-width ≤ 5 and every k-outerplanar graph has tree-width $\leq 3^k - 1$.

First we consider series-parallel graphs. A series-parallel graph can have parallel edges, i.e. it is a multigraph. One can define a series-parallel graph recursively as follows:

- The graph with 2 vertices and one edge is a series-parallel graph:

- Let $G = (V, E)$ be a series parallel graph. One obtains a new series parallel graph G', by replacing any edge $(v, w) \in E$ in one of the following 3 manners:



where $v^* \notin V$, i.e. v^* is a new vertex.

From this definition and the definition of the tree-width of a graph, one easily can proof, with induction, that every series-parallel graph has tree-width ≤ 2. To be precise, the class of graphs with tree-width ≤ 2 equals the class of series-parallel graphs [17].

Next we consider the outerplanar graphs. A graph G is outerplanar, if it is planar and it can be drawn in such a way in the plane that all vertices lie on the exterior face. It can be shown that every outerplanar graph is a series-parallel graph [10]. Thus outerplanar graphs have tree-width 2. This can also shown in the following way.

For every outerplanar graph $G = (V, E)$, there must be a vertex v with $degree(v) = 1 \vee degree(v) = 2$. Suppose $degree(v) = 2$. Let $(v, w) \in E$, $(v, x) \in E$, $w \neq x$. Now $G' = (V - \{v\}, (E - \{(v, w), (v, x)\}) \cup \{(w, x)\})$ is an outerplanar graph, and we may assume, with induction, that we have a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of G with tree-width ≤ 2. There must be an $i \in I$, with $w \in X_i \wedge x \in X_i$. Now let $i^* \notin I$, $I^* = I \cup \{i^*\}$, $X_{i^*} = \{v, w, x\}$ and $T^* = (I^*, F \cup \{(i, i^*)\})$. One easily verifies that $(\{X_i \mid i \in I^*\}, T^*)$ is a tree-decomposition of G with tree-width at most 2.

A generalization of the outerplanar graphs are the k-outerplanar graphs.

Definition.

- A graph $G = (V, E)$ is 1-outerplanar if and only if it is outerplanar.
- For $k \geq 2$, a graph $G = (V, E)$ is k-outerplanar if and only if it is planar and it has a planar embedding such that if all vertices on the exterior face (and all adjacent edges) are deleted, then the connected components of the remaining graph are all $(k-1)$-outerplanar.

The notion of k-outerplanar graphs was introduced by Baker [3], who also showed that several NP-complete graph problems can be solved in polynomial time, when restricted to k-outerplanar graphs.

Theorem 3.11
Let $G = (V, E)$ be a k-outerplanar graph. Then treewidth$(G) \leq 3^k - 1$.

Proof.
We use induction to k. For $k = 1$, the theorem holds because every outerplanar graph has tree-width ≤ 2, as discussed above.

Now let $k \geq 2$ and assume that the theorem holds for all $k' \leq k - 1$ and let $G = (V, E)$ be a k-outerplanar graph, embedded in the plane, such that if all vertices on the exterior face are removed, then each connected component of the remaining plane is $k - 1$-outerplaner. We first prove the following lemma.

Lemma 3.11.1
There is a k-outerplanar graph $H = (V, F)$, with $E \subseteq F$ (i.e. G is a subgraph of H), and there is an embedding of H in the plane such that

- if we remove all vertices on the exterior face, then each of the remaining connected components is $(k-1)$-outerplanar.
- every interior face which contains at least one vertex on the exterior face of H has exactly 3 sides.

Proof.
For every interior face of G, containing at least one vertex v on the exterior face, we add edges from v to every other vertex in the face (if not already present). The resulting graph H fulfills the stated conditions. □

Let $H = (V, F)$ be given, as indicated by the previous lemma. From lemma 2.4 it follows that it is sufficient to prove that treewidth$(H) \leq 3^k - 1$. By lemma 2.3 we may suppose that H is biconnected.

Now we define $V_{ex} = \{v \in V \mid v$ is on the exterior face of $H\}$; $V_{in} = V - V_{ex}$; H_{ex} is the subgraph of H, induced by V_{ex} and H_{in} is the subgraph of H induced by V_{in}. It follows that H_{in} is $(k-1)$-outerplanar, and thus treewidth$(H_{in}) \leq 3^{k-2} - 1$.

Lemma 3.11.2
Suppose H_{ex} is a cycle. Then H_{in} is connected.

Proof.
Suppose not. Then there must be vertices $v, w \in V_{ex}$ with v and w adjacent to vertices in different components of H_{in}. We can choose v and w such that $(v, w) \in F$. Consider the vertices in V_{in} adjacent to v. We can order these $x_1, \ldots, x_r$, such that $\forall i \leq r$: $(x_i, x_{i+1}) \in F$. (Use the property that each face containing v has exactly 3 sides.) Likewise, let $y_1, \ldots, y_s$ be the vertices in V_{in}, adjacent to w, with $\forall i \leq s$, $(y_i, y_{i+1}) \in F$. (See fig. 3.) Now,

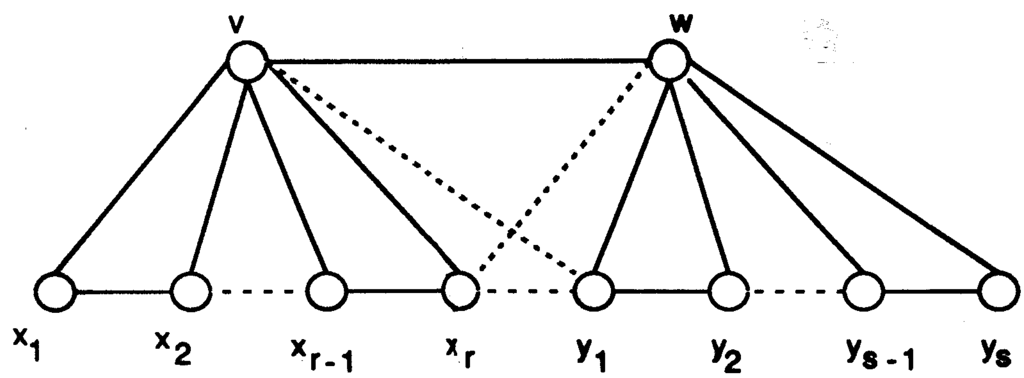


Figure 3:

because of the property that each face containing v and/or w, must have 3 sides, either $\{v, w, y_1\}$ and $\{v, y_1, x_r\}$ are faces of H, or $\{v, w, x_r\}$ and $\{w, y_1, x_r\}$ are faces of H. In both cases $(x_r, y_1) \in F$ and $x_1, x_2, \ldots, x_r, y_1, \ldots, y_s$ are in the same connected component in H_{in}. Contradiction. □

Now suppose H_{in} is connected. Let V_{inex} denote the set of vertices on the exterior face of H_{in}. It follows that each vertex $v \in V_{ex}$ only is adjacent to vertices in $V_{ex} \cup V_{inex}$.

Lemma 3.11.3
Suppose H_{ex} is a cycle and suppose H_{in} is biconnected. Then treewidth(H) $\leq 3^k - 1$.

Proof.
Note that the vertices in V_{inex} form a cycle. For all $x \in V_{inex}$ there are one and more *consecutive* vertices $x_1, \ldots, x_{r(x)} \in V_{ex}$ that are adjacent to x. (If not, then one can find a face with 4 sides, containing at least one vertex $v \in V_{ex}$). See fig. 4. H_{in} is $(k-1)$-outerplanar so we have, with induction,

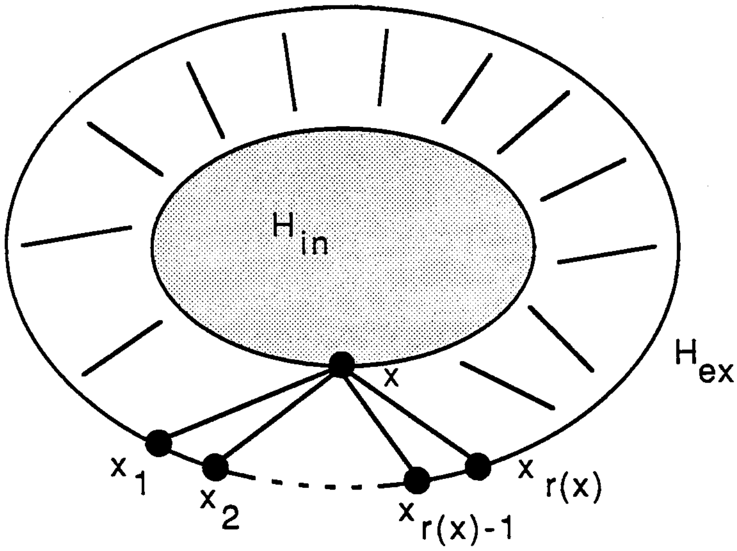


Figure 4

that there is a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, D))$ of H_{in} with tree-width at most $3^{k-1} - 1$.

For all $i \in I$, let $X_i' = X_i \cup \{x_1 \mid x \in X_i \cap V_{inex}\} \cup \{x_{r(x)} \mid x \in X_i \cap V_{inex}\}$. Recall that x_1 and $x_{r(x)}$ are the first and last vertex in V_{ex}, adjacent to x. (If $r(x) = 1$ then $x_1 = x_{r(x)}$).

Further, for all $x \in V_{inex}$ with $r(x) > 2$ we choose an $i(x) \in I$, with $x \in X_{i(x)}$. Let $I' = I \cup \{i_{x,j} \mid x \in V_{inex} \wedge r(x) > 2 \wedge 1 \leq j \leq r(x) - 2\}$, where all $i_{x,j} \notin I$, i.e. are new elements. Let $X'_{i_{x,j}} = \{x, x_j, x_{j+1}, x_{r(x)}\}$. Let $T' = (I', D')$, with $D' = D \cup \{(i(x), i_{x,1}) \mid x \in V_{inex} \wedge r(x) > 2\} \cup \{i_{x,j}, i_{x,j+1}) \mid x \in V_{inex} \wedge r(x) > 2 \wedge 1 \leq j \leq r(x) - 2\}$.

In other words, to each X_i of the old decomposition we add for each $x \in V_{inex}$ the first and last of the neighbors of x, that are in V_{ex}. For $x \in V_{inex}$ that are adjacent to more than 2 vertices, we choose a vertex $i \in I$

and add an extra branch with $r(x) - 2$ vertices to i in the tree, in order to represent $x_2, \ldots, x_{r(x)-1}$, as illustrated in fig. 5.

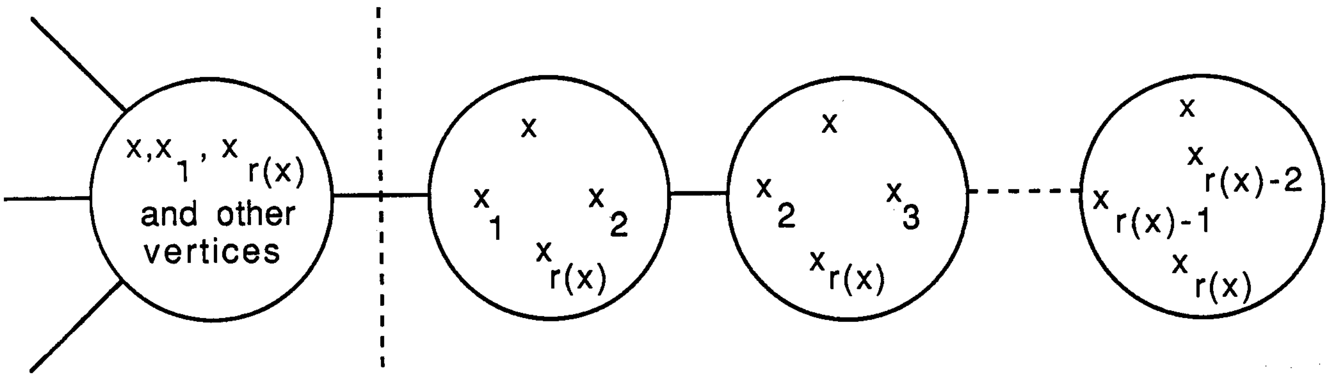


Figure 5

We now claim that $(\{X'_i \mid i \in I'\}, T' = (I', F'))$ is a tree-decomposition of H with tree-width $\leq 3^k - 1$.

It is easy to see that for all $v \in V$, there is a $i \in I'$ with $v \in X'_i$. Next we claim that $(v, w) \in F \Rightarrow \exists i \in I'$ with $v \in X'_i \wedge w \in X'_i$. We consider the following cases.

Case 1. $v \in V_{in} \wedge w \in V_{in}$. Then $\exists i \in I$ with $v \in X_i \wedge w \in X_i$. Hence also $v \in X'_i \supseteq X_i$ and $w \in X'_i$.

Case 2. $v \in V_{in} \wedge w \in V_{ex}$. Then $v \in V_{inex}$, and we can write $w = v_j$ with $1 \leq j \leq r(v)$. If $j = 1$ or $j = r(v)$, then for all $i \in I$ with $v \in X_i$ one has $v \in X'_i \wedge w = v_j \in X'_i$. If $1 < j < r(v)$, then $v \in X_{i_{v,j+1}}$ and $w = v_j \in X_{i_{v,j+1}}$.

Case 3. $v \in V_{ex} \wedge w \in V_{in}$. Similar to case 2.

Case 4. $v \in V_{ex} \wedge w \in V_{ex}$. Now there is a $x \in V_{inex}$, with $(v, x) \in F \wedge (w, x) \in F$, else we have a face in H, containing v and w and at least 4 sides. Hence, we can write $(v = x_j \wedge w = x_{j+1}, 1 \leq j \leq r(x) - 1)$ or $(v = x_{j+1} \wedge w = x_j, 1 \leq j \leq r(x) - 1)$. Without loss of generality, suppose the former. We have: if $r(x) \leq 2$ then for all $i \in I$ with $x \in X_i$:

$v = x_1 \in X'_i \wedge w = x_2 \in X'_i$, and if $r(x) > 2$, then, if $j < r(x) - 1$ then $v = x_j \in X_{i_{x,j}} \wedge w = x_j \in X_{i_{x,j}}$, and if $j = r(x) - 1$ then $v = x_{r(x)-1} \in V_{i_{x,r(x)-2}} \wedge w = x_{r(x)} \in X_{i_{x,r(x)-2}}$.
So we have that in all cases $\exists i \in I' : v \in X'_i \wedge w \in X'_i$.

Now we show that if $j \in I'$ is on the path in T' from $i \in I'$ to $k \in I'$ then $X'_j \supseteq X'_i \cap X'_k$. First consider the case that $i, k \in I$. It follows that $j \in I$ and if $z \in X'_i \cap X'_k$, then either $z \in V_{in}$, and hence $z \in X_i \cap X_k \subseteq X_j \subseteq X'_j$ or $z \in V_{ex}$, and hence z can be written $z = x_1$ or $z = x_{r(x)}$ for $x \in V_{inex}$ and now $x \in X_i \cap X_k \subseteq X_j$, and $z \in \{x_1, x_{r(x)}\} \subseteq X'_j$. So $X'_j \supseteq X'_i \cap X'_k$. Next consider the case that $i \in I' - I$, $k \in I$. It follows that there is an $x \in V_{inex}$ with $X'_i \cap X'_k \subseteq \{x, x_1, x_{r(x)}\}$. Now note that j must also be on the path in T' between $i(x)$ and k, and also $\{x, x_1, x_{r(x)}\} \in X'_{i(x)}$. It follows that $X'_i \cap X'_i \cap X'_k \subseteq X'_{i(x)} \cap X'_k \subseteq X'_j$. The other cases are similar or easy.

Finally note that $i \in I \Rightarrow |X'_i| \leq 3 \cdot |X_i| \leq 3^k$, and $i \in I' - I \Rightarrow |X'_i| = 4 \leq 3^k$. Hence the tree-width of the decomposition is at most $3^k - 1$. □

One may observe that for every interior face of H, that contains at least one vertex of V_{ex}, (and hence has exactly 3 sides), there must be at least one $i \in I'$, with X'_i containing each of the (three) vertices on this face. (There are basically 2 cases: one has a face with vertices $\{x, x_j, x_{j+1}\}$, or a face $\{x, y, x_1 = y_{r(y)}\}$, with $x, y \in V_{inex}$, $1 \leq j \leq r(x) - 1$. For the former case, the observation is straightforward. In the latter case, observe that we have $\exists i \in I : x \in X_i$ and $y \in X_i$, by definition. Now $\{x, y, x_1 = y_{r(y)}\} \subseteq X'_i$). So we have the following, slightly stronger result.

Lemma 3.11.4
Suppose H_{ex} is a cycle and suppose H_{in} is biconnected. Then there exists a tree-decomposition $(\{X_i \mid i \in I\}, T)$ of H with tree-width at most $3^k - 1$, and for each interior face of H that contains at least one vertex of V_{ex}, there must be at least one $i \in I$ with X_i containing each of the vertices on this face.

With induction to m we now prove the following lemma:

Lemma 3.11.5
Suppose H_{ex} is a cycle and H_{in} consists of m biconnected components. Then there exists a tree-decomposition $(\{X_i \mid i \in I\}, T)$ of H with tree-width $\leq 3^k - 1$, and for each interior face of H that contains at least one vertex of V_{ex}, there must be at least one $i \in I$ with X_i containing each of the vertices on this face.

Proof.
Use induction to m. For $m = 1$, the lemma follows from lemma 3.11.4.

Let $m > 1$, and let the lemma be true for all $m' < m$. There must be at least one biconnected component of H_{in}, that shares exactly one vertex v with one or more of the other biconnected components. So we can write $V_{in} = V^1 \cup V^2$, $V^1 \cap V^2 = \{v\}$ and the subgraph of H, induced by V^1, denoted by G^1, is biconnected , and the subgraph of H, induced by V^2 has $m-1$ biconnected components and is connected.

Let $W^1 \subseteq V_{ex}$ be the set of vertices in V_{ex}, that are adjacent to vertices in V^1, and $W^2 \subseteq V_{ex}$ be the set of vertices in V_{ex}, that are adjacent to vertices in V^2. Note that $W^1 \cup W^2 = V_{ex}$. W^1 and W^2 both induce connected subgraphs of the cycle H_{ex} (see fig. 6.) (If not, then one derives a contradiction with the fact that H is biconnected.) Now, the "left most" vertex of W^1 equals the "right most" vertex of W^2, and vice versa. Let these two vertices be x and y.

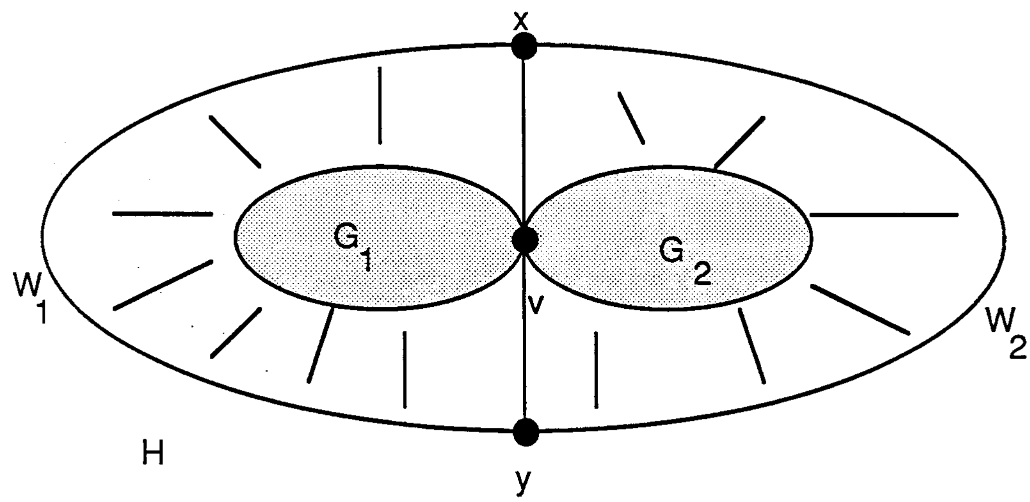


Figure 6

Now consider the graphs $H_1 = (V^1 \cup W^1, F_1)$ and $H_2 = (V^2 \cup W^2, F^2)$, with $F_i = \{(w_1, w_2) \in F \mid w_1, w_2 \in V^i \cup W^i\} \cup \{(x, y)\}$. (See fig. 7.).

Observe that $F_1 \cup F_2 = F \cup \{(x, y)\}$. If we define $(H_1)_{in}$, and $(H_2)_{in}$ similar to H_{in}, then $(H_1)_{in} = G_1$ and $(H_2)_{in} = G_2$. Now $(H_1)_{in}$ has one biconnected component and $(H_2)_{in}$ has $m-1$ biconnected components. By using the induction hypothesis, one can obtain now tree-decompositions $(\{X_i \mid i \in I_1\}, T_1 = (I_1, E_1))$ of H_1 and $(\{X_i \mid i \in I_2\}, T_2 = (I_2, E_2))$ of H_2, with the following properties:

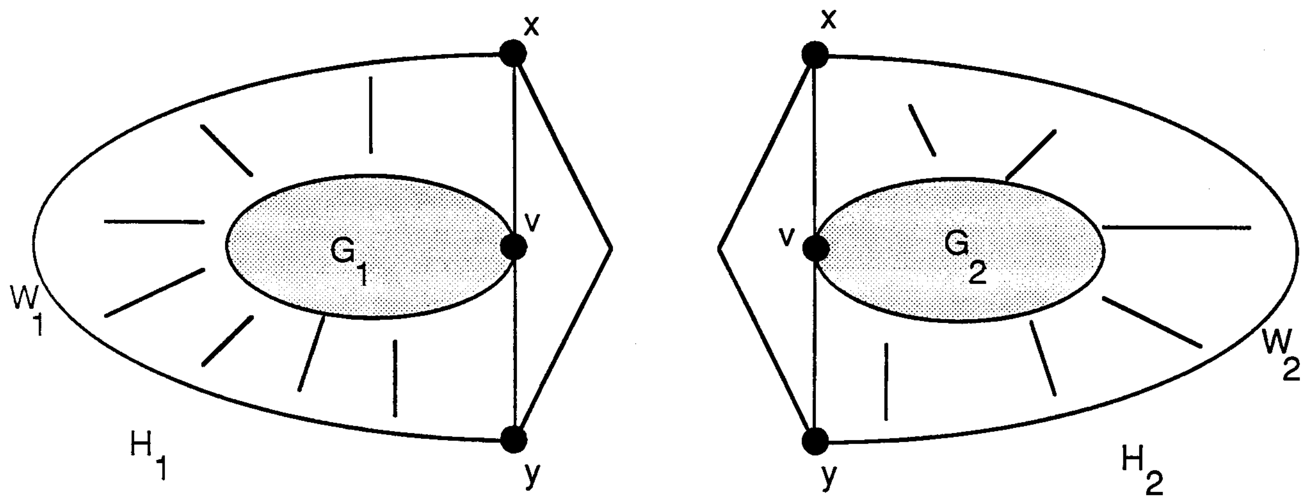


Figure 7

- $i \in I_1 \Rightarrow |X_i| \leq 3^k$.
- $i \in I_2 \Rightarrow |X_i| \leq 3^k$.
- $\exists i_1 \in I_1 : \{v, x, y\} \in X_{i_1}$. (Use that x, y are on the exterior face of H_1.)
- $\exists i_2 \in I_2 : \{v, x, y\} \in X_{i_2}$. (Use that x, y are on the exterior face of H_2.)

We now claim that $(\{X_i \mid i \in I_1 \cup I_2\}, T_2 = (I_1 \cup I_2, E_1 \cup E_2 \cup \{(i_1, i_2)\})$ is a tree-decomposition of H, with tree-width at most $3^k - 1$. (See fig. 8.)

It easily follows that $\bigcup_{i \in I_1 \cup I_2} X_i = V$. If $(v, w) \in F$ then $(v, w) \in F_1 \vee (v, w) \in F_2$, thus $(\exists i \in I_1 : v \in X_i \wedge w \in X_i) \vee (\exists i \in I_2 : v \in X_i \wedge w \in X_i)$.

Next let i, j, $k \in I_1 \cup I_2$, and suppose j is on the path from i to k in T. If i, $k \in I_1$ or i, $k \in I_2$, then it directly follows that $X_j \subseteq X_i \cap X_k$. Now suppose $i \in I_1$, $j \in I_1$, $k \in I_2$. (The other cases are similar.) Then $X_i \cap X_k \subseteq (V^1 \cup W^1) \cap (V^2 \cup W^2) = \{v, x, y\}$. Note that j is on the path in T from i to i_1, and $\{v, x, y\} \subseteq X_{i_1}$. Hence $X_i \cap X_k \subseteq X_i \cap X_{i_1} \subseteq X_j$.

Further it follows directly that $i \in I_1 \cup I_2 \Rightarrow |X_1| \leq 3^k$.

Finally notice that every interior face of H, containing at least one vertex of V_{ex}, either is an interior face of H_1, containing at least one vertex of W^1, or is an interior face of H_2, containing at least one vertex of W^2. Hence, for

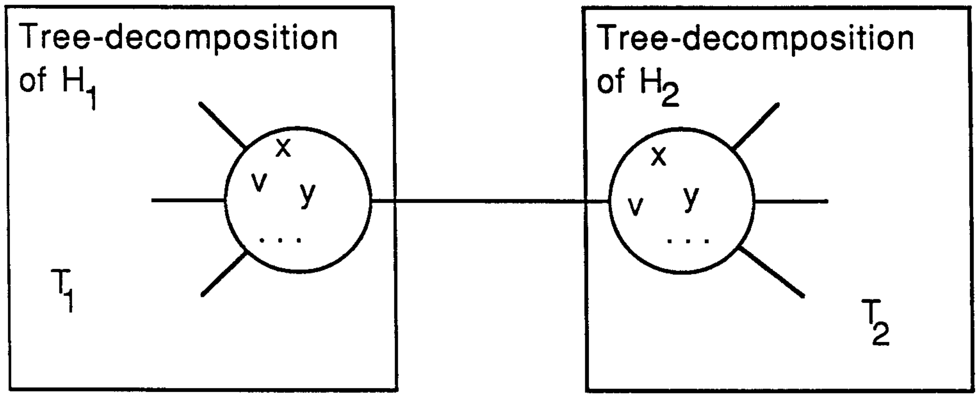


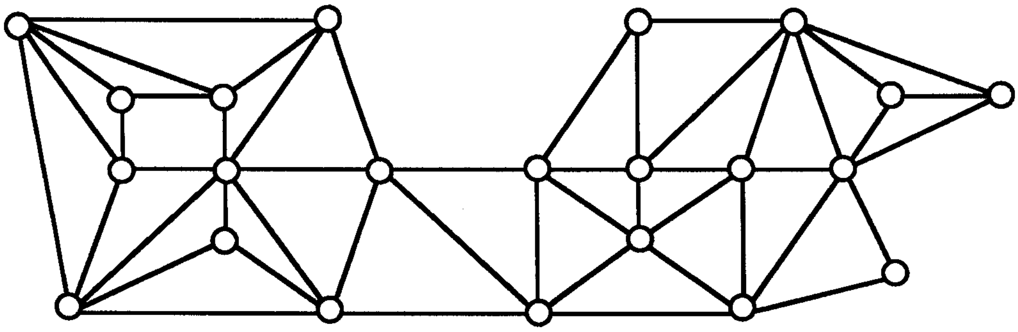
Figure 8

such a face, there will be a $i \in I_1 \cup I_2$, with X_i containing all vertices on the face. This completes the induction argument of lemma 3.11.5. □

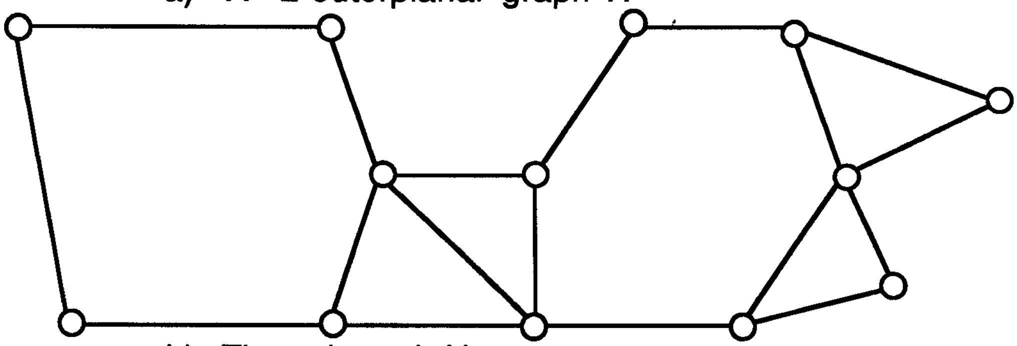
We will finally come to the proof of our main theorem 3.11. First we remark that H_{ex} always is an outerplanar graph. Because we supposed that H is biconnected, we may suppose that H has at least one face. (If not, then H is a graph with at most 2 vertices, and trivially treewidth(H) ≤ 1.)

Consider the dual graph $(H_{ex})^*$ of H_{ex}. The vertices of a dual graph of a planar graph correspond to interior faces of the graph and there is an edge between v and w in the dual graph if the corresponding faces share an edge. It is well known (see e.g. [16]), that the dual graph of an outerplanar graph G is a tree. Now consider a face of H_{ex}, corresponding to a leaf in $(H_{ex})^*$. This face must share exactly one edge (v, w) with another face of H_{ex}. Let $(V_{ex})^1$ be the set of vertices on this face, and let $(H_{ex})^1$ be the subgraph of H_{ex}, induced by $(V_{ex})^1$. Note that $(H_{ex})^1$ is a cycle.

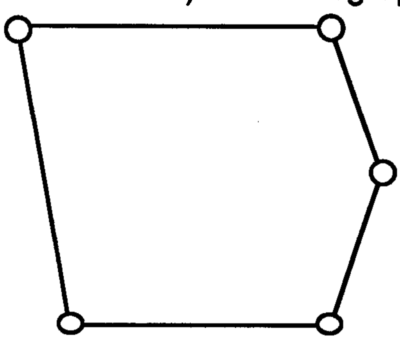
Now let $(V_{in})^1$ be the set of vertices in V_{in}, that are embedded in the area in the plane that is enclosed by H_{ex}^1. Let $V^1 = (V_{in})^1 \cup (V_{ex})^1$. Let H_1 be the subgraph of H, induced by V^1. Let $V^2 = (V - V^1) \cup \{v, w\}$. Let H_2 be the subgraph of H_1 induced by V^2. For an example see fig. 9.



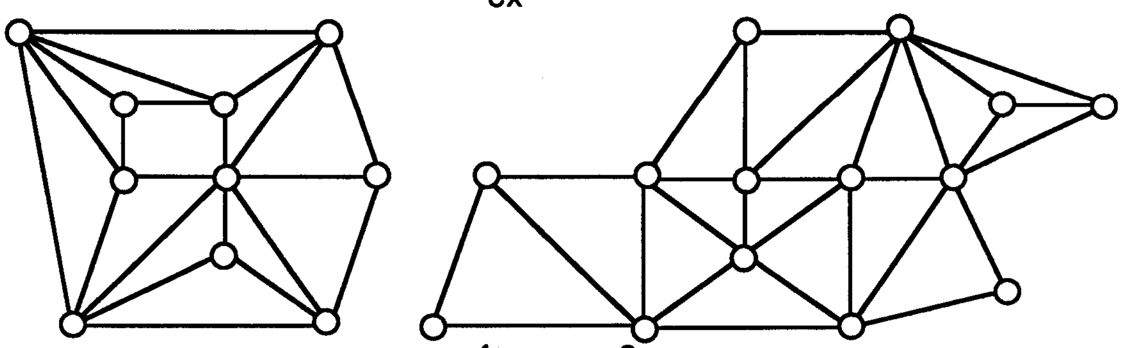
a) A 2-outerplanar graph H



b) The subgraph H_{ex}



c) The subgraph $(H_{ex})^1$



d) The subgraphs H^1 and H^2

Figure 9

Now H^1 and H^2 are also k-outerplanar graphs, but have a smaller size than H. By using induction we can obtain tree-decompositions $(\{X_i \mid i \in I_1\}, T_1 = (I_1, E_1))$ of H_1, and $(\{X_i \mid i \in I_2\}, T_2 = (I_2, E_2))$ of H_2, both with tree-width at most $3^k - 1$. There must be $i \in I_1$ with $v \in X_{i_1} \wedge w \in X_{i_2}$, and $i_2 \in I_2$ with $v \in X_{i_2} \wedge w \in X_{i_2}$. Now $(\{X_i \mid i \in I_1 \cup I_2\}, T = (I_1 \cup I_2, E_1 \cup E_2 \cup \{(i_1, i_2)\})$ is a tree-decomposition of H with tree-width at most $3^k - 1$. The proof of this fact is similar to the argument in lemma 3.11.5. This completes the proof of theorem 3.11. □

The last class of graphs that is considered in this section on classes of planar graphs is the class of the Halin graphs.

Definition.
A graph $G = (V, E)$ is a Halin graph if it can be obtained by embedding a tree without vertices with degree 2 in the plane and connecting its leaves by a cycle that crosses none of its edges.

It directly follows that the Halin graphs are contained in the 2-outerplanar graphs, and thus, by theorem 3.11 have treewidth 8 or less. However, by closer inspection one can obtain the following result.

Theorem 3.12
Let $G = (V, E)$ be a Halin graph. Then treewidth$(G) \leq 5$.

Proof.
The proof is similar to the proof of theorem 3.11. One uses that treewidth $(H_{in}) = 1$. □

3.5 Intersection graphs.

In this section we consider classes of intersection graphs. Each vertex in an intersection graph has associated with it an object in some space; there is an edge between two vertices if the corresponding objects intersect. For each of the considered classes of intersection graphs, we need on additional bound on the maximum cliquesizes of the graphs, to let the classes be subclasses of the graphs with bounded tree-width. We remark that each of the considered classes of graphs is a subclass of the class of the perfect graphs, i.e. for each of these graphs, its chromatic number equals the size of the largest clique that it contains.

Definition.
$G = (V, E)$ is a chordal graph if and only if every cycle with length exceeding three has an edge joining two non-consecutive vertices in the cycle.

Theorem 3.13 [7]
Let a graph $G = (V, E)$ be given. Then G is a chordal graph if and only if there is a tree $T = (W, F)$ such that one can associate with each vertex $v \in V$ a subtree $T_v = (W_v, F_v)$ of T, such that $(v, w) \in E$ if and only if $W_v \cap W_w \neq \emptyset$.

The following result was already noted by Robertson and Seymour [13].

Theorem 3.14 [13]
For every graph $G = (V, E)$ and integer $k \in N^+$, treewidth$(G) \leq k - 1$ if and only if G is subgraph of a chordal graph H that has maximum cliquesize at most k.

Proof.
Use the characterization of chordal graphs of theorem 3.13.
($\Leftarrow$) Use the tree-decomposition $(\{X_i \mid i \in W\}, T)$ with $X_i = \{v \in V \mid i \in W_v\}$. Note that $i \in W_v \wedge i \in W_w \Rightarrow (v, w) \in E$, hence X_i forms a clique in H, hence $|X_i| \leq k$. ($\Rightarrow$) Let a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of G be given. From the definition of tree-decomposition it follows that for all v, the subgraph of T, induced by $I_v = \{i \in I \mid v \in X_i\}$ is connected, i.e. is a subtree T_v of v. Now let H be the intersection graph given by these subtrees T_v, i.e. $H = (V, E')$ with $(v, w) \in E' \Leftrightarrow I_v \cap I_w \neq \emptyset$. One can argue that for all $v_1, \ldots, v_r \in V$, if for all $i, j \leq r$: $I_{v_i} \cap I_{v_j} \neq \emptyset$, then $\bigcup_{1 \leq i \leq r} I_{v_i} \neq \emptyset$. (Use that T is a tree, i.e. is cyclefree.) Hence if we have a clique with r vertices in H, then there must be an $i \in I$ with $|X_i| \geq r$. □

It follows that the treewidth of a chordal graph is its maximum cliquesize -1. Similar results can be obtained for subclasses of the chordal graphs.

Definition.

- The undirected path graphs are graphs with vertices corresponding to paths in a tree, and edges between vertices, if the corresponding paths have a vertex in common.
- The directed path graphs are graphs with vertices corresponding to paths in a tree with one vertex in the tree marked as root, and each

path is a subpath of a path from the root to a leaf, and there is an edge between two vertices if the corresponding paths have a vertex in common.

- The interval graphs are graphs with vertices, corresponding to connected subgraphs of a path (that is: a tree with each vertex degree ≤ 2), and edges between vertices, if the corresponding subgraphs have a vertex in common.
- The proper interval graphs are graphs with vertices, corresponding to connected subgraphs of a path, such that no subgraph is entirely contained in another, and edges between vertices, if the corresponding subgraphs have a vertex in common.

It follows that each interval graph is a directed path graph, each directed path graph is an undirected path graph, and each undirected path graph is a chordal graph. Similar as for the chordal graphs one can obtain the following results without difficulty.

Theorem 3.15

- $(G = V, E)$ has a tree-decomposition $\{(X_i \mid i \in I\}, T = (I, F))$ with tree-width $\leq k - 1$, and for each $v \in V$, $\{i \in I \mid v \in X_i\}$ induces a path in $T) \Leftrightarrow G$ is a subgraph of a undirected pathgraph $H = (V, E')$ with maximum cliquesize $\leq k$.
- $(G = (V, E)$ has a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ with tree-width $\leq k - 1$, and we can choose a root-vertex $r \in I$, such that for each $v \in V$, $\{i \in I \mid v \in X_i\}$ induces a subgraph of a path from r to a leaf in $T) \Leftrightarrow G$ is a subgraph of a directed pathgraph $H = (V, E')$ with maximum cliquesize $\leq k$.
- $(G = (V, E)$ has a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$, with tree-width $\leq k - 1$, and T is a path) $\Leftrightarrow G$ is isomorphic to a subgraph of an interval graph H with maximum cliquesize $\leq k$.

The tree-width of graphs with the additional restriction that T is a path, (called path-width) was studied by Robertson and Seymour in [12]. We now consider the circular are graphs and the proper circular arc graphs.

Definition.

- $G = (V, E)$ is a circular arc graph, if we can associate with each $v \in V$ a path on a cycle, and $(v, w) \in E \Leftrightarrow$ the paths associated with v and w intersect.

- $G = (V, E)$ is a proper circular arc graph, if we can associate with each $v \in V$ a path on a cycle, and no path is entirely contained in another one, and $(v, w) \in E \Leftrightarrow$ the paths associated with v and w intersect.

Lemma 3.16
Let $G = (V_G, E_G)$ be an intersection graph, each vertex $v \in V_G$ corresponding to a connected subgraph of a graph $H = (V_H, E_H)$, and $(v, w) \in E_G$, if and only if the corresponding subgraphs have a vertex in common. Let k be the maximum cliquesize in G. Then treewidth$(G) \leq$ (treewidth $(H) + 1) * k - 1$.

Proof.
Let $(\{X_i \mid i \in I), T = (I, F))$ be a tree-decomposition of $H = (V_H, E_H)$ with tree-width $c - 1$. Associate with each vertex $v \in V_H$ a set $I_v \subseteq V_G$ with $I_v = \{w \in V_G \mid v$ is a vertex in the subgraph associated with $w\}$. By noting that $w_1 \in I_v \wedge w_2 \in I_v \Rightarrow (w_1, w_2) \in E_G$ one sees that $|I_v| \leq k$ for all $v \in V_H$. Now let for all $i \in I$, $Y_i = \bigcup_{v \in X_i} I_v$. One can verify that $(\{Y_i \mid i \in I\}, T = (I, F))$ is a tree-decomposition of G, and the tree-width of this decomposition is at most $c \cdot k - 1$. □

Corollary 3.17
Let $G = (V, E)$ be a circular arc graph (or a proper circular arc graph), with maximum clique size k. Then treewidth$(G) \leq 2k - 1$.

Proof.
Use the definitions, lemma 3.16, and the fact that the tree-width of a cycle is 2. □

Also we have the following properties of the proper interval graphs and the proper circular graphs.

Theorem 3.18
Let $G = (V, E)$ be a proper interval graph with maximum cliquesize k. Then: bandwidth$(G) \leq k - 1$.

Proof.
Suppose we have associated with each $v \in V$ a set of vertices $S_v = \{l_v, l_v + 1, \ldots, r_v - 1, r_v\} \subseteq \{1, 2, \ldots, N\}$, such that $(v, w) \in E \Leftrightarrow S_v \cap S_w \neq \emptyset$; and $S_v \subseteq S_w \Rightarrow v = w$.

Let $|V| = n$. We pose a total ordering $\prec$ on V, such that $v \prec w \Leftrightarrow l_v < l_w$. Note that $v \neq w$ implies that $l_v \neq l_w$, else $S_v \subseteq S_w$ or $S_w \subseteq S_v$. One can now obtain a unique linear ordering f of G, with $f(v) < f(w) \Leftrightarrow v \prec w$, i.e. $f(v) < f(w) \Leftrightarrow l_v < l_w$. We claim that bandwidth$(f) \leq k - 1$.

Consider an edge $(v, w) \in E$ and without loss of generality suppose $f^{-1}(v) + i$, for all i, $0 \leq i \leq l$. (Thus $v = v_o$, $w = v_l$). We have that $(v, w) \in E \Rightarrow S_v \cap S_w \neq \emptyset$, thus $l_w < r_v$, i.e. $l_w \in S_v$. For all i, $1 \leq i \leq l-1$, we have $\neg(S_{v_i} \subseteq S_v)$ and $l_v < l_{v_i} < l_w < r_w$, thus $r_{v_i} > l_w$, hence $l_w \in S_{v_i}$. So $l_w \in \bigcap_{0 \leq i \leq l} S_{v_i}$. So the vertices $v_0, v_1, v_2, \ldots v_{l-1}, v_l$ form a clique in G with $l + 1$ vertices. It follows that $f(w) - f(v) = l \leq k - 1$. We now have : $\forall(v, w) \in E : |f(v) - f(w)| \leq k - 1$, hence bandwidth$(f) \leq k - 1$. □

Theorem 3.19
Let $G = (V, E)$ be a proper circular arc graph with maximum cliquesize k. Then: cyclic bandwidth$(G) \leq k - 1$.

Proof.
Similar to the proof of theorem 3.18. □

Corollary 3.20
For all proper circular arc graphs $G = (V, E)$: treewidth$(G) \leq 2k - 2$.

Proof.
Use corollary 3.9 and theorem 3.19. □

3.6 Tree-partite graphs.

Seese [15] introduced the notion of ($n-$bounded) tree-partite graphs, and showed that a number of important graph-problems can be solved in polynomial time for graphs, represented as $n-$bounded tree-partite graphs, with n bounded by some fixed constant, and the degree of the graph also bounded by some fixed constant.

Definition. [15]
Let $n \geq 1$. A graph $G = (V, E)$ is said to $n-$bounded tree-partite, if there is a tree $T = (I, F)$ and a collection $\{A_t \mid t \in I\}$, such that

- $V = \bigcup_{t \in I} A_t$.
- $A_t \cap A_{t'} = \emptyset$, for all $t, t' \in V, t \neq t'$.

- For all $e = (v, w) \in E$, either there is a $t \in I$ such that $v \in A_t \wedge w \in A_t$ or there are $t, t' \in I$ with t, t' adjacent in the tree T, and $v \in A_t \wedge w \in A_{t'}$.
- For all $t \in I$, $|A_t| \leq n$.

Theorem 3.21 [15]
If G is $k-$bounded tree-partite, then treewidth $(G) \leq 2k - 1$.

We have also a connection of this notion with the notion of emulation ([4]).

Definition.

- Let $G = (V_G, E_G), H = (V_H, E_H)$ be graphs. A mapping $f : V_G \rightarrow V_H$ is called an emulation, if for all $(v, w) \in E_G : f(v) = f(w) \vee (f(v), f(w)) \in E_H$.
- The cost of an emulation $f; V_G \rightarrow V_H$ is cost $(f) = \max_{v \in V_H} |f^{-1}(v)|$.

One easily obtains the following relation between the two notions.

Theorem 3.22
Let $G = (V, E)$ be a graph. G is $k-$bounded tree-partite, if and only if there is a tree $T = (W, F)$ and an emulation $f : V \rightarrow W$ of G on T with cost$(f) \leq k$.

One can show the following relation between the cost of emulations of G on a path and the bandwidth of G, very similar to a result in [4].

Theorem 3.23
Let $G = (V, E)$ be a graph, $k \in N^+$. Then: bandwidth $(G) \leq k \Rightarrow$
$\Rightarrow$ There is an emulation f of G on a path P with cost$(f) \leq k$
$\Rightarrow$ bandwidth$(G) \leq 2^k - 1$.

Corollary 3.24
For all graphs $G = (V, E)$, if bandwidth$(G) \leq k$, then G is a $k-$bounded tree-partite graph.

4 Conclusions

This paper shows some inclusion relations between the class of the graphs with treewidth bounded by some fixed number, and a number of subclasses. This also resolves some open problems of [10]. For instance, it follows that the problems to determine, whether an almost tree with parameter k (k fixed), or a graph with bandwidth k (k fixed) has a Hamiltonian circuit; and the Chromatic Number problem, restricted to almost trees with parameter k (k fixed), all can be solved in polynomial time.

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