

Strong Colorings of Graphs

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radio-networks [14], which can be solved using strong coloring. Another application we mention concerns the segmentation problem for files in a network [2]. Here the colors represent different (disjoint) segments of a file F , the graphs are regular with degree d and a strong coloring is desired with exactly $d + 1$ colors. (This implies that in the neighbourhood of every node a full copy of F can be assembled from the available segments.)

In this paper we present some facts about the strong coloring problem for graphs. We prove a number of basic facts and present some results when the problem is restricted to special classes of graphs. We present a method to strongly color planar graphs with at most $3(\Delta + 3)$ colors, where Δ is the maximum degree in the graph. We note that the naïve method for strongly coloring a planar graph would use up to $5\Delta + 1$ colors. It is shown that outerplanar graphs can be strongly colored with at most $\Delta + 4$ colors.

The paper is organized as follows. In section 2 we give some definitions concerning graphs and (strong) graph colorings. In section 3 we give some preliminary results for the strong coloring problem for certain classes of planar and non-planar graphs. In section 4 we give some facts about strongly coloring $(r-1)$ -regular graphs with r colors, which we refer to as “perfect coloring”. In section 5 the main results for strongly coloring planar graphs are given. In section 6 we show that every circuit with non-intersecting chords can be strongly colored with at most $\Delta + 4$ colors, which is the main technical result that underlies the constructions in section 5. Section 7 contains some remarks and open questions. In an appendix we present a new proof of the NP-completeness of the strong coloring problem for graphs. We will assume some familiarity with graph theory (cf. Harary [10]).

2 Definitions

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The distance between two nodes x and y is defined as the number of edges on the shortest path between x and y . Let $\Delta = \max\{\deg(v) | v \in V\}$, with $\deg(v)$ the degree of vertex v . The square graph of G is the graph G^2 with $V(G^2) = V(G)$ and $E(G^2) = \{(u, v) | (u, v) \in E \text{ or } (u, x) \in E \text{ and } (x, v) \in E \text{ for some } x\}$. Observe that $\Delta(G^2) \leq (\Delta(G))^2$. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the least $K \leq n$ such that G can be K -vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \dots, K\}$ with $f(u) \neq f(v)$ whenever $\{u, v\} \in E$ [10]. The *chromatic index* of a graph G , denoted by $\chi'(G)$, is the least $K \leq m$ such that G can be K -edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \dots, K\}$ with $f(\{u, v\}) \neq f(\{u, w\})$ for all $u, v, w \in V$ and $\{u, v\}, \{u, w\} \in E$ [10]. With vertex (edge) coloring, every pair of vertices (edges) that have distance one must have different colors. Whenever colorings are considered, $c(v)$ denotes the color given to a vertex v .

The generalization to distance- k coloring is now straightforward. The *k -chromatic*

number of a graph G , denoted by $\chi_k(G)$, is the least $K \leq n$ such that G can be distance- k K -vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \dots, K\}$ with $f(u) \neq f(v)$ whenever u and v lie within distance k in $G = (V, E)$. The k -chromatic index of a graph G , denoted by $\chi'_k(G)$, is the least $K \leq m$ such that G can be distance- k K -edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \dots, K\}$ with $f(\{u, v\}) \neq f(\{w, x\})$ whenever $\{u, v\}$ and $\{w, x\} \in E$ and $\{u, v\}$ and $\{w, x\}$ lie within distance k from each other. For $k = 2$, we speak of the *strong chromatic number* and the *strong chromatic index* respectively. If a $(r-1)$ -regular graph can be strongly colored with exactly r colors, then this coloring is called *perfect* [11].

McCormick [16] has proved that, given a graph and an integer K , the problem of deciding whether a graph can be distance- k vertex colored with K colors is NP-complete, for every $k \geq 1$. Another proof for the strong chromatic number problem can be found in the Appendix. Checking whether a graph can be strongly colored with $K \leq 3$ colors is trivial. If a graph has $\Delta \geq 4$ or contains a C_5 or a $K_{2,3}$, then the graph is not strongly 4-colorable. Recently Bakker [3] has shown that deciding whether a $(r-1)$ -regular graph can be perfectly colored with r colors is NP-complete, even for the case $r = 4$. It is open whether the problem of deciding $\chi'_k(G) \leq K$ is NP-complete. In this paper we will focus entirely on the strong coloring problem ($k = 2$). We will be referring to some special classes of graphs including planar graphs, outerplanar graphs, Halin graphs, chordal graphs and partial k -trees. We assume that the first three are known but include a recursive definition of partial k -trees.

Definition 2.1 ([1]) *The class of k -trees is the smallest class of graphs that can be defined as follows:*

1. *the complete graph K_k on k vertices is a k -tree.*
2. *if $G = (V, E)$ is a k -tree and v_1, \dots, v_k form a complete subgraph of G , then the graph $G' = (V \cup \{w\}, E \cup \{(v_i, w) | 1 \leq i \leq k\})$ with $w \notin V$ is also a k -tree.*

A graph is a partial k -tree if and only if it is the subgraph of a k -tree.

3 Preliminaries

To obtain some first bounds for $\chi_2(G)$, consider the smallest-last (SL) ordering of a graph, determined by the following algorithm of [15] in $O(m)$ time.

ALGORITHM SL

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j = n; H = G;
for j = n downto 1 do

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begin

Let v_j be a vertex with minimum degree in H .

Remove v_j and all edges incident to v_j from H .

end

SL = v_1, v_2, \dots, v_n .

END OF ALGORITHM

Let SL = v_1, v_2, \dots, v_n be a smallest-last ordering of G . Let p be the maximum of the degrees of the vertices as they appear in the for-loop of algorithm SL. It is easily shown that $p = \max_H \min_{v \in V(H)} \{deg_H(v) | H \text{ a subgraph of } G\}$, and that G can be colored with $p + 1$ colors.

Theorem 3.1 *With p be defined as above: $\Delta + 1 \leq \chi_2(G) \leq p \cdot \Delta + 1$.*

Proof: The lowerbound is trivial. For the upperbound, we use induction to show that the vertices can be strongly colored in SL-order. Let C be a set of $p \cdot \Delta + 1$ colors. Vertex v_1 can be assigned an arbitrary color from C . Assume we have strongly colored the vertices v_1, \dots, v_{i-1} (following the smallest-last ordering) using colors from C ($i \geq 2$). v_i is connected to at most p colored neighbours, hence v_i has at most $p \cdot \Delta$ colored vertices within distance 2 and at most $p \cdot \Delta$ colors from C are blocked for it. Hence v_i can be colored with a color from C . This completes the induction. \square

Corollary 3.2 *Partial k -trees can be strongly colored with at most $k \cdot \Delta + 1$ colors.*

Proof: From definition 2.1 it follows that there exists a smallest-last ordering of the partial k -tree by removing vertices with maximum degree $\leq k$ in each step of algorithm SL. Using theorem 3.1, the corollary follows. \square

Lemma 3.3 *Every outerplanar graph can be strongly colored with at most $2\Delta + 1$ and every planar graph with at most $5\Delta + 1$ colors.*

Proof: Every outerplanar graph has a vertex with degree at most 2. Deleting this vertex with all its incident edges gives another outerplanar graph with the same property. In this way one can easily make a smallest-last ordering in which the maximum degree in each step is at most 2, and the lemma follows from theorem 3.1 for outerplanar graphs. Noting that every planar graph has a vertex with degree at most 5, the lemma easily follows in a similar way. \square

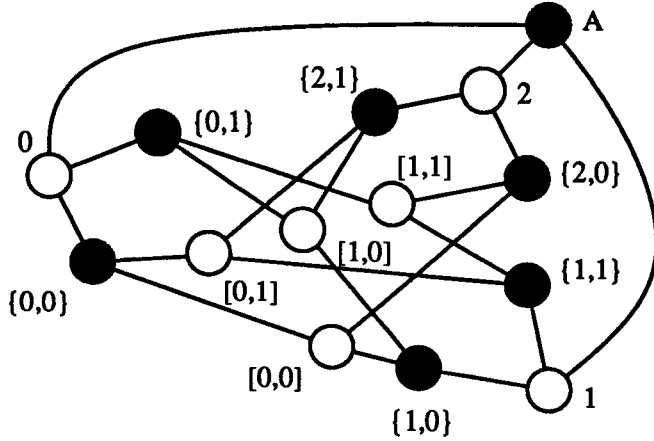


Figure 1: Example of a G_p for $p = 3$.

Similar bounds can be given for distance- k vertex and edge colorings. Observe that theorem 3.1 also implies that $\chi'(G) \leq \chi_2(G)$, as $\chi'(G) \leq \Delta + 1$ by Vizing's theorem. The question whether $\chi_k(G) \leq \chi'_k(G)$ remains as an interesting open problem. For trees it is clear that $\chi_2(G) = \Delta + 1$ and $\chi'_2(G) = \max\{\deg(u) + \deg(\text{father}(u)) \mid u \in V\}$. Every Halin graph can be strongly colored with at most $\Delta + 6$ colors. (For the latter result one uses that every tree can be strongly colored using at most $\Delta + 1$ colors and every circuit with at most 5 colors.)

Observe that from a strong vertex coloring with $\chi_2(G)$ colors one can obtain a strong edge coloring with $(\chi_2(G))^2$ colors, by assigning to every edge $\{u, v\}$ the color $[c(u), c(v)]$, where the colors of the vertices are taken from the strong vertex coloring. Notice that this large difference of the strong chromatic number and the strong chromatic index actually occurs in the case of the graph $G = K_{n,n}$, where $\chi_2(G) = 2n$ and $\chi'_2(G) = n^2$.

There appears to be no simple connection between $\chi(G)$ and $\chi_2(G)$. A reasonable conjecture like $\chi_2(G) \leq (\Delta + 1)\chi(G) + 1$ fails, by observing the construction of the following bipartite graph $G_p = (\langle V_1, V_2 \rangle, E)$. V_1 consists of p vertices $0, \dots, p-1$. V_2 consists of $p(p-1)$ vertices $\{i, j\}, 0 \leq i < p, 0 \leq j < p-1$, and a vertex A . Let $i \in V_1$ be connected to the nodes $\{i, x\}$ for $0 \leq x < p-1$ and to vertex A . Add $p(p-1)$ vertices $[k, l]$ to V_1 , with $0 \leq k, l < p-1$, and the edges $([i, j], \{0, i\})$ and $([i, j], \{k, ((k-1) \cdot i + j) \bmod (p-1)\})$, with $1 \leq k < p$ and $0 \leq i, j < p-1$. Note that every two vertices in V_2 have distance 2 to each other. Hence this bipartite p -regular graph has $\Delta = p, \chi(G) = 2$ and $\chi_2(G) = \Delta(\Delta - 1) + 1$. It shows that $\chi(G)$ and $\chi_2(G)$ can differ dramatically. In figure 1, an example is given for $p = 3$.

From the observation that $\chi_k(G) = \chi(G^k)$ and $\chi'_k(G) = \chi'(G^k)$ (using the definition of the k^{th} power graph G^k of [10]) for every $k \geq 2$, we conclude that we can use the available algorithms for ordinary graph coloring for obtaining strong colorings,

after calculating G^2 in $O(\Delta \cdot m)$ time. Also all lowerbounds for the chromatic number trivially hold for the strong chromatic number. This also leads to the following observation:

Theorem 3.4 *Chordal graphs can be strongly colored in a smallest possible number of colors in polynomial time. For every chordal graph G one has $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$.*

Proof: Note that the square graph of a chordal graph is a chordal graph too, and can be colored in an optimal number of colors in polynomial time (cf. Golumbic [9]).

For proving $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$ we induct on n , the size of G . For $n \leq 3$ the result trivially holds. Consider an arbitrary chordal graph G of n nodes, $n > 3$. Without loss of generality we may assume that G is connected. If G is a clique, then it can be strongly colored with $n = \Delta + 1$ colors and $\Delta + 1 \leq (\frac{1}{2}\Delta + 1)^2$. Thus let G not be a clique, S a minimal node separator of G and A_1, A_2, \dots, A_l the connected components of $G - S$. Let H_1 be the induced subgraph spanned by S and A_1 , and let H_2 be the induced subgraph spanned by S and A_2, \dots, A_l . By well-known facts for chordal graphs [9] S is a clique, and H_1 and H_2 are connected chordal graphs. Let $|S| = s$. By induction H_1 and H_2 are strongly colorable with the colors of some set C of $(\frac{1}{2}\Delta + 1)^2$ colors. A strong coloring of G can now be obtained as follows.

Permute the colors such that in the strong colorings of H_1 and H_2 , the nodes of S get the same colors. (This can be done because the colors assigned to the nodes of S must all be different, by the strong coloring requirement, in both the coloring of H_1 and the coloring of H_2 .) Let N_1 be the set of nodes in A_1 that are reached by an edge from S , and N_2 the set on nodes in $A_2 \cup \dots \cup A_l$ defined similarly. Let $|N_1| = n_1$ and $|N_2| = n_2$, and observe that $n_1 + n_2 \leq s \cdot (\Delta - s + 1) \leq (\frac{1}{2}\Delta + 1)^2 - s$. Thus we have sufficiently many colors in C to arrange that s colors are fixed for the nodes in S , and the remaining colors can be permuted such that in the strong colorings of H_1 and H_2 the nodes in N_1 and N_2 are colored by disjoint sets of colors. The resulting strong colorings of H_1 and H_2 can now be combined (merged) to a correct strong coloring of G which employs no more than $(\frac{1}{2}\Delta + 1)^2$ colors. \square

We conjecture that $\chi(G^2) \leq Q + 1$, with Q the number of vertices of the largest clique in the graph G^2 . If G^2 is a linegraph, then this conjecture is true by noting that if the linegraph G^2 has a largest clique of size Q , then the linegraph of this linegraph (which is a normal graph) has maximal degree Q and can be edge-colored with $Q + 1$ colors. Hence the linegraph G^2 can be vertex-colored with $Q + 1$ colors.

4 Perfect Colorings

Recall that a $(r - 1)$ -regular graph is perfectly colorable if it can be strongly colored with r colors. This coloring is useful for the following file distribution problem [2]: “Given a connected regular network $G = (V, E)$ and a file F , assign to each

node $x \in V$ a segment $F_x \subseteq F$ such that for all $x \in V, \cup_{\{x,y\} \in E} F_y \cup F_x = F$ and in every neighbourhood the distributed fragments are free of overlaps, i.e., $\forall (x, y) \in E : F_x \cap F_y = \emptyset$." When there are r different disjoint segments of F , this problem is only meaningful for $(r-1)$ -regular networks. A perfect coloring describes the assignment of the segments for a valid solution of the file distribution problem. It has been shown by Bakker [3] that this problem is NP-complete, even for the case $r = 4$.

In this section we give some characteristics of perfectly colorable graphs. Some relationships between strong colorings, perfect colorings and edge colorings are the following:

Theorem 4.1 ([2]) *If a $(r-1)$ -regular graph with $|V| = n$ nodes can be perfectly colored, then $r|n$ and every group of equally colored nodes has $\frac{n}{r}$ nodes.*

Proof: Let $N(x)$ denote the set of vertices having distance ≤ 1 to node x . Consider any perfect coloring of the graph, and let c be one of the colors. Let x_1 and x_2 be two nodes colored c . There can be no node y in $N(x_1) \cap N(x_2)$ because, if there was, y would have two neighbours of the same color (which contradicts the strong coloring property). Thus $\forall x_1, x_2$ with $c(x_1) = c(x_2) = c : N(x_1) \cap N(x_2) = \emptyset$. Furthermore $\forall y \exists x, c(x) = c : y \in N(x)$. Hence the neighbourhoods $N(x)$ of nodes x with $c(x) = c$ form a partitioning of G . But $\forall x \in V : |N(x)| = r$. Hence $r|n$ and every group of equally colored nodes has size $\frac{n}{r}$. \square

Theorem 4.2 *Every strongly r -colorable graph is the induced subgraph of a perfectly colorable $(r-1)$ -regular graph.*

Proof: We induct on r . For $r = 1, 2$ and 3 , the theorem is trivial. Thus let $r \geq 4$ and G be a strongly r -colorable graph. Consider a strong coloring of G with the colors c_1, \dots, c_r and let H be the induced subgraph of G consisting of all nodes with a color $\in \{c_1, \dots, c_{r-1}\}$. By induction H is an induced subgraph of some $(r-2)$ -regular graph R_H that is perfectly colorable, and w.l.o.g. we can assume that it is perfectly colored with c_1, \dots, c_{r-1} . Arrange the nodes of R_H into $(r-1)$ disjoint blocks B_1, \dots, B_{r-1} , with B_i ($1 \leq i \leq r-1$) containing the nodes of color c_i , and let every block contain b nodes. (By theorem 4.1 we know that the blocks must be of equal size.) Tag the nodes of R_H that correspond to the nodes of H . Let the nodes x_1, \dots, x_s (some $s \geq 1$) of $G - H$ together form the "beginning" of the r^{th} block B_r . The nodes $\{x_1, \dots, x_s\}$ form an independent set.

Now form the graph R_G as follows. Make $\lceil \frac{s}{b} \rceil$ copies of R_H and extend B_r by another $\lceil \frac{s}{b} \rceil b - s$ nodes y . We now "connect" the x - and y -nodes to the nodes in the R_H copies in two steps, as follows:

1. for i from 1 to s do
begin

connect x_i to a node from a B_j -block for every $j, 1 \leq j \leq r - 1$,
always favoring the tagged node $z \in$ block B_j if x_i is
directly connected to z in G , and an untagged node otherwise.
end;

Observe that we have $\lceil \frac{s}{b} \rceil b \geq s$ nodes of every color, so step 1 always works. But also observe that we have exactly $\lceil \frac{s}{b} \rceil b - s$ nodes left of every color after this step.

2. **for** j **from** 1 **to** $\lceil \frac{s}{b} \rceil \cdot b - s$ **do**
begin
pick a new y -node and connect y to a node from a B_i -block that
was not used before, for every $i, 1 \leq i \leq r - 1$.
(Note that these nodes were not tagged.)
end;

Note that step 2 makes the graph G_H $(r - 1)$ -regular. The result is a graph R_G that is $(r - 1)$ -regular, perfectly colorable with r colors and clearly, by design, we have that G is an induced subgraph of R_G . \square

Another property of perfectly colorable graphs is the following. Recall that by Vizing's theorem every graph is edge-colorable with Δ or $\Delta + 1$ colors.

Lemma 4.3 *If a graph is strongly colorable with r colors and r is even, then it is $(r - 1)$ -edge colorable.*

Proof: Let G be strongly colorable with r colors. If r is even, then K_r is edge-colorable with $(r - 1)$ colors. Let G be strongly colored with the r names of nodes of K_r . Now G can be edge-colored as follows: color an edge from the node colored X to the node colored Y with z if the edge between X and Y in K_r is colored z . This gives a correct $(r - 1)$ -edge coloring of G . \square

The converse is not true, see for example figure 2. Also this theorem does not hold for r odd in general, as an edge-coloring on a K_r (which is perfectly colorable with r colors) requires r colors when r is odd.

Also the spectra of perfectly colorable graphs have some interesting properties. Because a perfectly r -colorable graph G is $(r - 1)$ -regular, its largest eigenvalue is equal to $r - 1$ and has multiplicity 1 (cf. Biggs [6]). The following more specific observation can be made as well.

Theorem 4.4 *Let G be perfectly r -colorable. Then G has an eigenvalue -1 , with multiplicity $\geq (r - 1)$.*

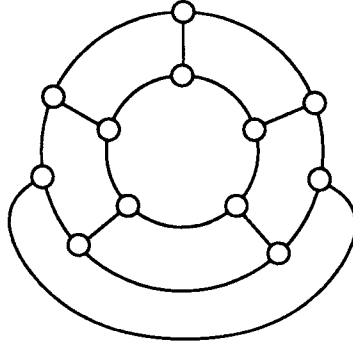


Figure 2: A 3-edge colorable 3-regular graph that is not perfectly colorable (from [12]).

Proof: Let G be perfectly r -colored, and consider the vertices of G arranged in blocks of equally colored vertices (of size $\frac{n}{r}$ each). Let $A = A(G)$ be the adjacency matrix of G corresponding to this vertex-ordering. The symmetric matrix A can be viewed as a block matrix, with the blocks along the main diagonal consisting of all zeroes and the off-diagonal blocks being $\frac{n}{r} \times \frac{n}{r}$ permutation matrices. (As an aside we note that, conversely, if the vertices of a graph G can be arranged so the adjacency matrix is of this form, then G is perfectly r -colorable.) Now consider the $r \times r$ matrix A' obtained from A by replacing every block on the main diagonal by a "0" and every off-diagonal block by a "1". A' is the adjacency matrix of the K_r , whose spectrum consists of one eigenvalue $(r - 1)$ and $(r - 1)$ eigenvalues -1 (see e.g. [6]). Also, when (x_1, \dots, x_r) is an eigenvector of A' , then the vector obtained by repeating each coordinate $\frac{n}{r}$ -fold is an eigenvector of A and independency of eigenvectors is preserved in the process. It follows in particular that A (and hence, G) has an eigenvalue -1 with multiplicity $r - 1$. \square

From the same argument some more information can be derived. Let $n > r$ and let $\lambda_1, \dots, \lambda_k$ and $-\mu_1, \dots, -\mu_l$ be the remaining positive and negative eigenvalues in the spectrum of G in decreasing order different from the r eigenvalues $(r - 1)$ and -1 that we have, with $k + l = n - r$. As the trace of A is zero, we have $\lambda_1 + \dots + \lambda_k = \mu_1 + \dots + \mu_l$. Observe also that A^2 is a symmetric matrix with all entries along the main diagonal equal to $r - 1$. It follows that $\lambda_1^2 + \dots + \lambda_k^2 + \mu_1^2 + \dots + \mu_l^2 = \text{tr}(A^2) - (r - 1)^2 - (r - 1) = (n - r)(r - 1)$. Now let $\lambda = \lambda_1 = \lambda_{\max}$, $\mu = \mu_l = \mu_{\max}$ and $\delta = \max\{\lambda, \mu\}$. One easily verifies that $\delta \geq \sqrt{r - 1}$ and $\min\{\lambda, \mu\} \geq \frac{1}{n - r} \sqrt{r - 1}$.

Some further characteristics of perfectly colorable graphs are the following:

Theorem 4.5 *Let G be regular of degree ≥ 3 and perfectly colorable. Then one can partition V as $V_1 \cup V_2$ such that*

1. the induced subgraph G_1 on V_1 is a set of chordless cycles of length divisible by 3.
2. the induced subgraph G_2 on V_2 is regular of degree $\Delta - 3$ and perfectly colorable.

Proof: Let a, b, c be three colors of the perfect coloring of G . Let V_1 be the set of nodes colored a, b or c and $V_2 = V - V_1$.

1. Consider any node in V_1 , say with color a . It has one neighbour colored b , this neighbour has one neighbour colored c , etc. This necessarily closes itself as a cycle at the point of departure. By the strong coloring property, this cycle must be chordless. This proves the statement, and the cycles are not connected to each other.

2. Consider any node in V_2 . It has *exactly* three neighbours in V_1 . Thus G_2 inherits the strong (perfect) coloring of G , with the remaining $\Delta - 3$ colors. \square

This shows that perfectly colorable graphs decompose entirely into (disjoint) chordless cycles. Note that $\frac{|V_2|}{|V_1|} = \frac{\Delta-2}{3}$, for $\Delta \geq 3$.

For the file distribution problem perfect colorings are interesting mostly for regular networks, which includes many current processor networks. In [11] a detailed study is given of the perfectly colorable processor networks. For completeness we summarize the results of [11] in the following theorem.

Theorem 4.6 ([11]) *The following processor networks are perfectly colorable:*

- The hypercube C_n , if and only if $n = 2^i - 1, i > 0$.
- The d -dimensional torus of size $l_1 \times \dots \times l_d$ if $l_i \bmod q = 0$, with q such that $\sqrt{2d+1}|q$ for some integer $r > 0$.
- The Cube-connected Cycles CCC_d , if and only if $d > 2, d \neq 5$.
- The directed shuffle-exchange network and the directed 4-pin shuffle network.
- The chordal ring network with chord length $4p - 1$ ($p > 0$) and $4kp - 4t$ ($0 \leq t < p$) nodes if and only if :
 1. k and t are even and (if $t > 0$) $\frac{t}{\gcd(t,p)}$ is even, or
 2. $k, \frac{t}{\gcd(t,p)}$ and $\frac{p}{\gcd(t,p)}$ are odd and $t + p$ is even.
- The hexagonal network of size $m \times n$ if and only if $m, n \bmod 7 = 0$.

5 Main Results for Planar Graphs

In this section we consider the strong coloring problem for planar graphs. By the results from section 3 we know that every planar graph G can be strongly colored using at most $5\Delta + 1$ colors. Our aim will be to improve this to a bound of $c\Delta + O(1)$ colors with a significantly lower constant $c < 5$ (uniformly, for all planar graphs). We will show that one can take $c = 1$ for outerplanar graphs (which is optimal) and that one can take $c = 3$ for planar graphs in general. We begin by giving a worst-case lowerbound for c (for general planar graphs).

Lemma 5.1 *For every $\Delta \geq 1$ there exists a planar graph G with $\chi_2(G) \geq \lceil \frac{3}{2}\Delta \rceil$.*

Proof: We can assume w.l.o.g. that $\Delta > 1$. (For $\Delta = 1$ the lemma trivially holds by taking a graph that consists of a single edge). Choose $r, s \geq 0$ with $s \leq r$ such that $\Delta = r + s + 2$. It will be useful to take $r = s = \frac{1}{2}\Delta - 1$ when Δ is even and $s = r - 1 = \frac{1}{2}\Delta - \frac{3}{2}$ when Δ is odd. Construct the graph G_Δ consisting of a “triangle” of three nodes (A, B and C), r nodes that are each connected to A and to B , s nodes that are each connected to B and to C , and s more nodes that are each connected to A and to C . For Δ odd (implying $\Delta \geq 3$), a separate node D is “inserted” on the triangle-edge (A, B). This node is also connected to C . One easily verifies that G_Δ is planar, has maximum degree Δ and diameter 2. Because of the latter any strong coloring of G_Δ needs as many colors as there are nodes, which is precisely $\lceil \frac{3}{2}\Delta \rceil$. (By a result of Seyffart [17] this is about the largest possible number of nodes in any planar graph of diameter 2 and maximum degree Δ .) \square

The lemma shows that $c \geq \frac{3}{2}$ for general planar graphs. For $\Delta \leq 5$ one can construct planar graphs that need $\geq 2\Delta$ colors in any strong coloring (which does not imply that $c \geq 2$, in the given formulation).

We take an indirect approach to the strong coloring problem for planar graphs. First we show that certain subclasses of planar graphs admit strong colorings with a “very small” number of colors. It is used to obtain strong colorings of general planar graphs G that use at most $3\Delta + O(1)$ colors. The following technical lemma is instrumental, but for clarity reasons its proof is deferred to section 6.

Lemma 5.2 *Every (planar) circuit G with non-intersecting chords can be strongly colored using at most $\Delta + 4$ colors.*

We use the lemma to derive a bound on the number of colors needed to strongly color an arbitrary outerplanar graph.

Theorem 5.3 *Every outerplanar graph G can be strongly colored using at most $\Delta + 4$ colors.*

Proof: Let G be outerplanar. (Without loss of generality we can confine ourselves to connected graphs.) We proceed by induction. When G has ≤ 5 nodes, the theorem trivially holds. Thus assume that the theorem holds for all connected outerplanar graphs of $\leq n - 1$ nodes, and let G have n nodes (some $n > 5$). If G is a planar circuit with non-intersecting chords, then the result follows immediately by Lemma 5.2. If G is not, then G must contain a cutvertex v . In this case G consists of connected outerplanar graphs H_1 and H_2 such that each contain a “copy” of the node v and are joined at v , but which are otherwise disjoint. (Without loss of generality we may assume that both H_1 and H_2 have $\leq n - 1$ nodes.)

Let v have degree Δ_1 in H_1 and degree Δ_2 in H_2 , where we can assume w.l.o.g. that $\Delta_1 \leq \Delta_2$ and clearly $\Delta_1 + \Delta_2 \leq \Delta$. We can assume inductively that H_1 and H_2 can be strongly colored using at most $\Delta + 4$ colors. Shift color-names such that H_1 and H_2 use colors from the same set of $\Delta + 4$ colors and v gets the same color “ α ” in H_1 and H_2 . Joining H_1 and H_2 at v (while retaining the colorings of H_1 and H_2 respectively) results in a strong coloring of G with $\Delta + 4$ colors, except in the one case that some neighbours of v in H_1 have the same color as some neighbours of v in H_2 . We now argue how such a conflict can be removed by a mere permutation of the colors, if it arises.

Thus assume that the latter case arises. Note that v and its neighbours in H_2 use $\Delta_2 + 1$ colors. Let k neighbours of v in H_1 use colors different from these but l neighbours use colors c_1, \dots, c_l that are among the colors used by the Δ_2 neighbours in H_2 , for certain k and l with $k + l = \Delta_1$. It means that $\Delta_2 + 1 + k$ different colors are used in the neighbourhood of v . Choose l different colors d_1, \dots, d_l from among the remaining colors. (This can be done because $\Delta + 4 - (\Delta_2 + 1 + k) \geq \Delta_1 + \Delta_2 + 4 - (\Delta_2 + 1 + k) = l + 3$.) Exchanging c_i and d_i (for i from 1 to l) in the coloring of H_1 throughout leaves a strong coloring in H_1 and removes the color conflicts at v , thus leading to a correct strong coloring of G using at most $\Delta + 4$ colors.

This completes the inductive argument. □

The theorem enables us to prove the main result of this section on strong colorings of planar graphs.

Theorem 5.4 *Every planar graph G can be strongly colored using at most $3\Delta + 9$ colors.*

Proof: Let G be an arbitrary planar graph, v a node of G (e.g. chosen to lie on the exterior face of G). Define L_i to be the set of nodes that lie at distance i from v , for any $i \geq 1$. (This leads to a decomposition of V into finitely many disjoint sets which are easily determined algorithmically by breadth-first-search.) Consider the subgraphs of G induced by the sets L_i , and let Δ_i be the maximum degree of any node in the L_i -induced subgraph. Now observe that each L_i -induced subgraph is outerplanar, and that nodes in L_i can only be adjacent to nodes in L_{i-1} , L_i and

L_{i+1} . Also every node in L_i must be adjacent to a node in L_{i-1} (for $i \geq 1$), which means that $\Delta_i \leq \Delta - 1$ (for all $i \geq 0$).

We can now obtain a strong coloring of G as follows. By theorem 5.3 each L_i -induced subgraph can be strongly colored using $\leq \Delta_i + 4 \leq \Delta + 3$ colors. Take three sets of $\Delta + 3$ different colors T_0, T_1, T_2 and color the nodes of every L_i -induced subgraph strongly using the colors from the set $T_{i \bmod 3}$, by the method implicit in theorem 5.3. This necessarily results in a strong coloring of G using (at most) $3(\Delta + 3) = 3\Delta + 9$ colors. \square

The same technique as used in theorem 5.4. can be used to prove some further results. For example, a very similar argument can be used to show that every k -outerplanar graph can be strongly colored using at most $3\Delta + 9$ colors.

6 Strongly Coloring Circuits with Non-Intersecting Chords

This section is devoted entirely to the proof of Lemma 5.2, which asserts that every planar circuit G with non-intersecting chords can be strongly colored using at most $\Delta + 4$ colors. First we formalize a useful technique that will be applied repeatedly in the proof. Let G be an arbitrary graph, with colors assigned to the nodes.

Definition 6.1 *A node v is said to “miss” color α if neither v nor any of its neighbours is colored α .*

Definition 6.2 *Let H be any connected (but not necessarily induced) subgraph of G , and let α, β be two different colors in the coloring. The operation $\text{SWITCH}_H(\alpha, \beta)$ acts on the given coloring of G by interchanging the colors α and β in the color assignment for the nodes of H .*

(We will assume that the operation SWITCH_H is defined *only* for subgraphs H as stated.)

Lemma 6.1 *Let v miss color α and $c(v) \neq \beta$, and H be any connected component of $G - \{v\}$. If G is strongly colored, then so it is after performing $\text{SWITCH}_H(\alpha, \beta)$.*

Proof: Let $H_0 = H, H_1, \dots$ be the connected components of $G - \{v\}$. In G , v is connected by an edge to selected nodes in H_0, H_1, \dots but the subgraphs H_0, H_1, \dots themselves are mutually disjoint. $\text{SWITCH}_H(\alpha, \beta)$ preserves the strong coloring property in the nodes of H_0, H_1, \dots trivially, hence we only need to verify that it does at v . If v misses β , then the operation has no effect on the neighbourhood of v at all. If v has a neighbour w with $c(w) = \beta$, then two cases can arise. If $w \in H_0$, then $c(w) = \alpha$ after performing the SWITCH_H operation. As v missed α , this gives

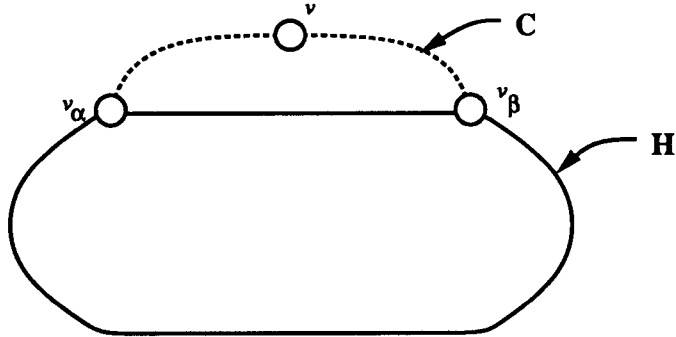


Figure 3: The inductive construction of a circuit with non-intersecting chords.

no conflict at v . If $w \in H_i$ for some $i > 0$, then the neighbourhood of v again is not affected by the operation. \square

We now analyze the strong colorings of planar circuits G with non-intersecting chords. We assume inductively that such graphs can always be strongly colored with $\leq \Delta + c$ colors, for some constant $c \geq 1$ that will be fixed later. (Lemma 5.2 indicates that we will later choose $c = 4$.) This certainly holds for all circuits G with non-intersecting chords that have ≤ 5 nodes, provided we take $c \geq 3$. Assume that the hypothesis holds for all G with $\leq n - 1$ nodes (some $n > 5$) and consider an arbitrary planar circuit G with non-intersecting chords that has n nodes. We will prove the induction hypothesis for G by case analysis.

If G has no chords, then it is a simple C_n (which has $\Delta = 2$). One easily verifies that C_n can be strongly colored with 3 colors when $n \equiv 0 \pmod{3}$, and with 4 colors otherwise (and $n > 5$, like we assumed). This satisfies the induction hypothesis.

Now assume that G has non-intersecting chords. As G is outerplanar, it must have a node v of degree 2. Orient the circuit, and let v_α and v_β be the first nodes to the left and to the right of v respectively that are incident to chords. Note that there must be a chord between v_α and v_β , which also is the first chord “seen” from v .

It follows that G can be decomposed into a chain C , which contains v and the part of the circuit to its left and to its right up to (but not including) v_α and v_β , and a remaining graph H . See figure 3.

Note that H is a planar circuit with non-intersecting chords of $\leq n - 1$ nodes. By induction H can be strongly colored using $\Delta_H + c$ colors, where $\Delta_H = \max \{ \deg(v) \mid v \in H \}$ (with degrees as counted in H). Clearly $\Delta_H \leq \Delta$, and w.l.o.g. we can assume that $c(v_\alpha) = \alpha$ and $c(v_\beta) = \beta$. It remains to color the nodes of C such that a strong coloring of the graph G results.

Assume first that C is a chain of k nodes for some $k \geq 2$. Let C consist of the nodes x_1, \dots, x_k “from left to right”, with x_1 adjacent to v_α and x_k to v_β . As we

have $\Delta_H + c$ colors at our disposal and both v_α and v_β only used $\Delta_H + 1$ of them (at most), there must be two different colors γ and δ such that v_α misses γ and v_β misses δ (in the strong coloring of H). Note that necessarily $\gamma, \delta \neq \alpha, \beta$. Now color C as follows. Assign γ to x_1 , δ to x_k and assign the colors β, α and γ alternately (in this order) to x_2, x_3, \dots when $k > 2$. If x_{k-1} is assigned α or γ , we are done. If x_{k-1} is assigned β , then a conflict arises with the strong coloring requirement at x_k . To resolve it we choose any color $\mu \notin \{\alpha, \beta, \gamma, \delta\}$ and assign it to x_{k-1} instead. (Such a μ exists because we have at least $\Delta_H + c \geq 6$ colors available.) This gives a valid, strong coloring of the entire graph with $\Delta_H + c \leq \Delta + c$ colors.

Next assume that C consists of only 1 node, namely the node v . This (final) case gives more difficulties, as we will see. Let M_α and M_β be the sets of colors, missed by v_α and v_β respectively. If $M_\alpha \cap M_\beta \neq \emptyset$ and (say) $\mu \in M_\alpha \cap M_\beta$, then we can “complete” the strong coloring of G by simply assigning the color of μ to v . Let us therefore assume that M_α and M_β are fully disjoint, for the remaining analysis. The idea of the remainder of the proof is to try and make suitable “color flips” such that a new strong coloring arises in which the corresponding M_α and M_β are no longer disjoint (which would be sufficient for our proof by the previous case).

If $\Delta_H < \Delta$, then necessarily $\Delta_H = \Delta - 1$. But we used $\Delta_H + c$ colors for H and have $\Delta + c$ colors available for G . It follows that there must be a “free” color which must thus belong to M_α and M_β , contradicting their disjointness. Thus we have $\Delta_H = \Delta$, and necessarily $\deg_H(v_\alpha) \leq \Delta - 1$ and $\deg_H(v_\beta) \leq \Delta - 1$. We can also observe that both v_α and v_β must have chords “inside” H . For suppose e.g. that v_β had no chord inside H and (thus) $\deg_H(v_\beta) = 2$. Then $|M_\beta| = (\Delta + c) - 3 = \Delta + (c - 3)$. At the same time, $|M_\alpha| \geq (\Delta + c) - \Delta = c$. As $|M_\alpha| + |M_\beta| \geq \Delta + (2c - 3) > \Delta + c$ for $c > 3$, it would follow that M_α and M_β necessarily intersect, a contradiction.

Let u be the leftmost node to which v_β is connected by a chord. Let M_u be the set of colors missed by u . Suppose that M_α and M_u were not disjoint and (say) that $\gamma \in M_u \cap M_\alpha$ for some color γ . As $\gamma \notin M_\beta$, v_β must have a neighbour colored γ (which cannot be a neighbour of u as $\gamma \in M_u$). See figure 4. Choose a color $\varphi \in M_\beta$ with $\varphi \neq c(u)$ and do a $\text{SWITCH}_{H_r}(\gamma, \varphi)$ on the given coloring. It is easily argued that this preserves the strong coloring of H_r , but an argument similar to Lemma 6.1 shows that it actually preserves the strong coloring of H . This leads to a coloring in which both v_α and v_β miss γ , which suffices for our claim. A similar argument applies in case M_u and M_β were not disjoint. Thus we proceed on the assumption now that M_u is disjoint from M_α and M_β .

Next assume that there is a left neighbour of u colored with some color $\gamma \in M_\beta$ and there is another left neighbour of u colored with some color $\delta \in M_\alpha$. Now change the coloring of H_l by executing a $\text{SWITCH}_{H_l}(\gamma, \delta)$. Note that this changing of colors preserves the strong coloring property and has no effect on the right neighbours of v_u and (hence) has no effect on the nodes of H_r . It follows that after this operation both v_α and v_β miss color δ , which suffices for our claim. A similar argument applies in the case there is a right neighbour of u colored with some color $\gamma \in M_\beta$ and there is another right neighbour of u colored with some color $\delta \in M_\alpha$. So we proceed

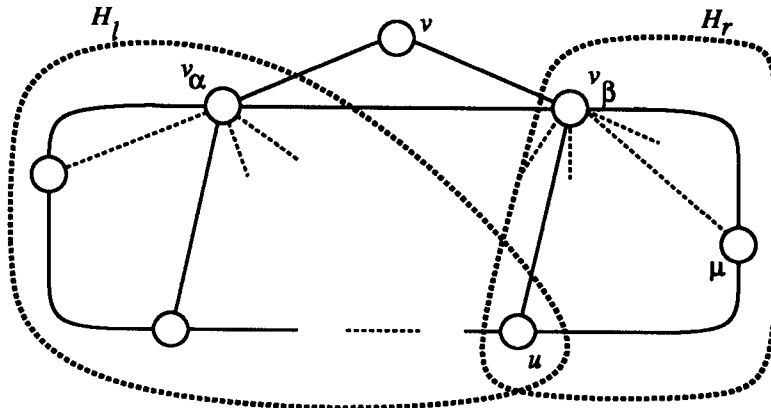


Figure 4: Both M_u and M_α miss a color γ .

on the assumption that this is not the case, i.e., all neighbours of u colored with elements of M_α are *left* neighbours of u and all neighbours of u colored with elements of M_β are *right* neighbours of u , or the other way around.

If the M_α -colored neighbours of u are right neighbours of u and the M_β -colored neighbours of u are left neighbours of u then we do the following. Let $\gamma \in M_u$, $\delta \in M_\alpha$ and $\mu \in M_\beta$. Change the coloring in H by doing a $\text{SWITCH}_{H_l}(\gamma, \delta)$ and a $\text{SWITCH}_{H_r}(\gamma, \mu)$ operation. It is easily argued that $\text{SWITCH}_{H_l}(\gamma, \delta)$ preserves the strong coloring of H_l and by Lemma 6.1 has no effect on the strong coloring of H_r . Similarly a $\text{SWITCH}_{H_r}(\gamma, \mu)$ has no effect on the strong coloring of H_l . These two recolorings have the effect that now both v_α and v_β miss color γ , which suffices for our claim.

Thus we proceed on the assumption that all M_α -colored neighbours of u are *left* neighbours of u and all M_β -colored neighbours of u are *right* neighbours of u . Let J_u be the set of colors used at u that are *not* in $M_\alpha \cup M_\beta$. Let L_u be the subset of colors of J_u , used by left neighbours of u , and let R_u be the subset of colors of J_u used by right neighbours of u . Note that J_u is disjoint from M_u, M_α and M_β , by definition. We have come some way in reconstructing the neighbourhood of v_α and v_β . We now know that v_α has (distinct) neighbours that are colored with the colors in $J_u \cup M_u \cup M_\beta$ and that v_β has (distinct) neighbours that are colored with the colors in $J_u \cup M_u \cup M_\beta$.

Lemma 6.2 *The colors can be flipped such the strong coloring requirement is preserved and some M_β -colored neighbour of v_α precedes all $R_u \cup M_u$ -colored neighbours (along the arc from v_α to u) and some M_α -colored neighbour of v_β precedes all $L_u \cup M_u$ -colored neighbours (along the arc from v_β to u).*

Proof: We only prove this for the neighbours of v_α , as the argument is similar for v_β and respective color-flips that are needed do not interfere. Suppose the

property is not satisfied yet, i.e., there exists a $\gamma \in R_u \cup M_u$ that precedes all colors μ with $\mu \in M_\beta$ along the arc from v_α to u for correspondingly colored neighbours of v_α (along the arc towards u). We let γ be the first color from $R_u \cup M_u$ that occurs along the arc and has this property. Let $\mu \in M_\beta$ be some color. Consider first the case $\gamma \in M_u$. It is easily seen that $\text{SWITCH}_{H_l}(\gamma, \mu)$ preserves the strong coloring of H_l and of H , and swaps μ into the desired “leading” position. Next, we consider the case that $\gamma \in R_u$, thus there is a right neighbour of u , colored with γ . Note that all M_β -colored neighbours of u are *right* neighbours of u , thus there is right neighbour of u colored with μ . Note that now a $\text{SWITCH}_{H_l}(\gamma, \mu)$ will again do the trick, for any $\mu \in M_\beta$ and swaps μ into the desired “leading” position. \square

(The proof of Lemma 6.2 is easily extended to show that all M_β -colored neighbours of v_α precede the $R_u \cup M_u$ -colored neighbours, but this is not needed for our argument here.) Let β_1 be the color of the first M_β -colored neighbour of v_α along the arc, and likewise α_1 the color of the first M_α -colored neighbour of v_β along the arc.

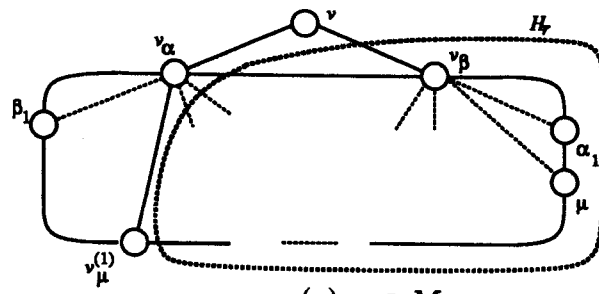
Observe that $|J_u| \geq (\deg(u) + 1) - |M_\alpha| - |M_\beta|$, $|M_u| = \Delta + c - (\deg(u) + 1)$ and $|M_\alpha| + |M_\beta| \leq \Delta + c$. This means that $|J_u| + |M_u| + |M_\alpha| + |M_\beta| \geq \Delta + c$ and (hence) $= \Delta + c$ because they are all disjoint sets, and in particular $|J_u| = (\deg(u) + 1) - |M_\alpha| - |M_\beta|$. It follows that $J_u \cup M_u \cup M_\alpha \cup M_\beta$ is the full set of colors. We can now estimate the number of neighbours of v_α and v_β as follows:

$$\begin{aligned}
|L_u| + |R_u| + 2|M_u| + |M_\alpha| + |M_\beta| &= |J_u| + 2|M_u| + |M_\alpha| + |M_\beta| \\
&= (\deg(u) + 1) - |M_\alpha| - |M_\beta| + \\
&\quad 2(\Delta + c - (\deg(u) + 1)) + |M_\alpha| + |M_\beta| \\
&= 2\Delta + 2c - (\deg(u) + 1) \\
&\geq \Delta + 2c - 1.
\end{aligned}$$

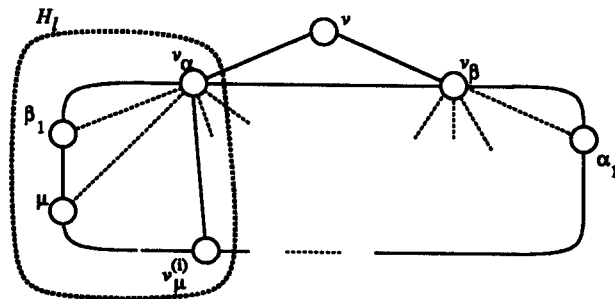
As each of these nodes must miss at least $c - 1 \geq 3$ colors out of the full set of $\Delta + c$ colors and $3(\Delta + 2c - 1) > 3(\Delta + c)$, there must be some color μ that is missed in at least 4 of the nodes under consideration (i.e., neighbours between the β_1 -colored one of v_α and the α_1 -colored one of the v_β). In fact, because $3(\Delta + 2c - 3) > 3(\Delta + c)$ we can even claim that these 4 nodes occur *strictly in between* the β_1 -colored neighbour of v_α and the α_1 -colored neighbour of v_β . Let the nodes be $v_\mu^{(1)}, v_\mu^{(2)}, v_\mu^{(3)}$ and $v_\mu^{(4)}$, in this order. Assume w.l.o.g. that $v_\mu^{(i)}$ is a neighbour of v_α .

We will now complete the proof by a final case-analysis. The $v_\mu^{(1)}$ -nodes act as “separating nodes” and we can perform any sort of $\text{SWITCH}_*(\mu, *)$ operation on the arcs left or right of $v_\mu^{(i)}$ that preserves the strong coloring requirement, and try to “free” a color that can be assigned to v . We always use the same argument for it as in Lemma 6.1.

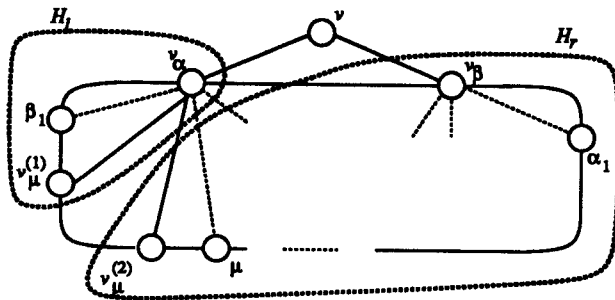
First assume that $\mu \in M_\alpha$ (see figure 5(a)). Observe that in this case v_β has a neighbour colored μ . Performing a $\text{SWITCH}_{H_r}(\mu, \beta_1)$ preserves the strong coloring



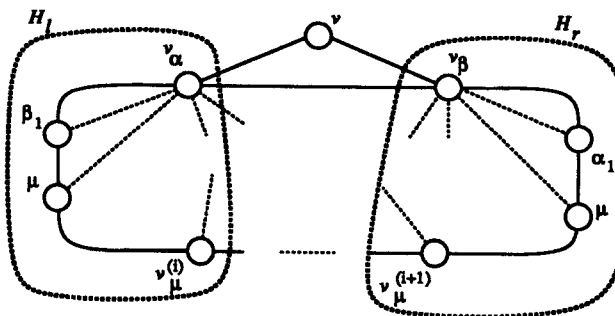
(a) $\mu \in M_\alpha$



(b) $v_\mu^{(i)}$ to the right of the μ -colored neighbour of v_α



(c) All $v_\mu^{(i)}$ nodes in between the μ -colored neighbours of v_α and v_β



(d) Two $v_\mu^{(i)}$ nodes in between the μ -colored neighbours of v_α and v_β

Figure 5: Final case analysis.

requirement in H_r and H , but it has the additional effect that in the new coloring v_β also misses μ (and v_α continues to miss it as well). Thus μ can be assigned to v and we are done.

Next assume that $\mu \in M_\beta$. In this case v_α has a neighbour colored μ . If v_β has one of the $v_\mu^{(i)}$ nodes as neighbour, then we can proceed as in the previous case. Thus let all $v_\mu^{(i)}$ be neighbours of v_α . First assume that there is a $v_\mu^{(i)}$ right of the μ -colored neighbour of v_α , as shown in figure 5(b). Now perform a $\text{SWITCH}_{H_l}(\mu, \alpha_1)$ to achieve the same effect as in the previous case. It leads to a strong coloring of H in which both v_α and v_β miss μ . Next assume that all $v_\mu^{(i)}$ are in between the β_1 - and μ -colored neighbours of v_α , see figure 5(c). (Note that H_l and H_r do not include the part of H “between” $v_\mu^{(1)}$ and $v_\mu^{(2)}$.) Now perform a $\text{SWITCH}_{H_r}(\mu, \alpha_1)$ and also a $\text{SWITCH}_{H_l}(\mu, \beta_1)$. A straightforward argument shows that this preserves the strong coloring requirement, but it has the additional effect of removing β_1 from the neighbourhood of v_α . Thus β_1 is “freed” and can be assigned to v .

Finally assume that $\mu \notin M_\alpha \cup M_\beta$. Now both v_α and v_β have μ -colored neighbours, say x and y respectively. The nodes x and y divide the arc from the β_1 -colored neighbour of v_α to the α_1 -colored neighbour of v_β into three intervals. Thus at least one of these intervals must contain two $v_\mu^{(i)}$ nodes. If either the first or the last interval contains two $v_\mu^{(i)}$ nodes, then we can reason exactly as in the previous case and are done. (Consider e.g. figure 5(b) and add a μ -colored neighbour of v_β . The argument remains unchanged.) Thus the only case left is the case in which two $v_\mu^{(i)}$ nodes occur between the μ -colored neighbours of v_α and v_β , see figure 5(d). Now perform a $\text{SWITCH}_{H_l}(\mu, \alpha_1)$ and also a $\text{SWITCH}_{H_r}(\mu, \beta_1)$. This preserves the strong coloring requirement and frees μ at both v_α and v_β . Thus the strong coloring of G can be completed by assigning μ to v . This completes the proof of Lemma 5.2.

The proof shows that we can indeed take $c = 4$.

7 Conclusions and further remarks

In this paper we have presented some basic facts about the strong coloring problem for graphs. We gave a summary of results for perfect colorings of regular graphs and for strongly coloring special classes of graphs. We proved a new upperbound, namely $3\Delta + 9$, for strongly coloring arbitrary planar graphs. For outerplanar graphs a much tighter bound for the strong coloring problem is proved, namely $\Delta + 4$. The bound for general planar graphs, while nontrivial, is not likely to be the best possible. At present the only known worst-case lowerbound is about $\frac{3}{2}\Delta + O(1)$.

Throughout the paper we have reported some results for the strong coloring problem for various classes of non-planar graphs as well. Here many interesting problems are left. For example, given a coloring algorithm A which gives a good approximate bound on the chromatic number of a graph G , does this algorithm give a good approximate bound for the strong chromatic number of G , when it is applied to the square graph G^2 ? What if G belong to a special class of graphs?

Another open question is the following. Is there an analog for strong chromatic numbers of the following theorem of Garey and Johnson [8] : "If for some constant $r < 2$ and constant d there exists a polynomial-time algorithm A which guarantees $A(G) \leq r\chi(G) + d$, then there exist a polynomial-time algorithm A which guarantees $A(G) = \chi(G)$."? The best performance ratio known for approximation algorithms for the chromatic number problem is $\frac{n \log \log n}{(\log n)^2}$ [5]. What is the corresponding best performance ratio for the strong chromatic number by applying this to the square graph G^2 ?

It would be interesting to investigate other relationships between the strong coloring problem and the well-studied coloring problem (see e.g. [12]), as well as relationships between the strong vertex coloring problem and the strong edge coloring problem.

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Appendix

Theorem *Given a graph G and an integer K , the problem of determining whether G can be strongly colored with $\leq K$ colors is NP-complete (STRONG CHROMATIC NUMBER).*

Proof: The problem trivially belongs to NP. (One can assign $\leq K$ colors to the nodes of G and verify in polynomial time whether it is a strong coloring.) For proving the NP-completeness, we reduce 3-SAT to STRONG CHROMATIC NUMBER. Let F be a CNF formula having r clauses, with at most three literals per clause. Let x_i ($1 \leq i \leq n$) be the variables in F . We may assume $n \geq 4$. We shall construct, in polynomial time, a graph G that is strongly colorable with $rn + 2n + 2$ colors iff F is satisfiable. The graph $G = (V, E)$ is defined by:

$$V = \{x_1, x_2, \dots, x_n\} \cup \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\} \cup \{y_1, y_2, \dots, y_{n+1}\} \cup \{p_{1,1}, \dots, p_{n,r}\} \\ \cup \{p_{n+1,r}\} \cup \{z_1, z_2, \dots, z_n\} \cup \{C_1, C_2, \dots, C_r\}$$

and

$$E = \{(y_i, y_j) | i \neq j\} \cup \{(z_i, z_j) | i \neq j\} \cup \{(z_i, x_i), 1 \leq i \leq n\} \cup \\ \{(p_{i,j}, p_{k,l}) | i \neq k \text{ or } j \neq l\} \cup \{(z_i, \bar{x}_i), 1 \leq i \leq n\} \cup \{(p_{n+1,r}, y_{n+1})\} \cup \\ \{(y_i, z_j) | 1 \leq i \leq n, i \neq j\} \cup \{(p_{i,j}, C_j), 1 \leq i \leq n, 1 \leq j \leq r\} \cup \\ \{(p_{i,j}, z_k), 1 \leq i, k \leq n, 1 \leq j \leq r\} \cup \{(x_i, p_{i,k}) | x_i \notin C_k\} \cup \{(\bar{x}_i, p_{i,k}) | \bar{x}_i \notin C_k\}$$

To see that G is $rn + 2n + 2$ colorable iff F is satisfiable, we first observe that the y_i 's form a complete subgraph on $n + 1$ vertices. Hence, each y_i must be assigned a distinct color. Without loss of generality we may assume that in any coloring of G y_i is given the color i for $1 \leq i \leq n + 1$. Then we observe that the z_i 's together form a complete subgraph on n vertices. Every z_i is at most at distance two from every y_i , hence the z_i must be colored differently from the y_i . Assume w.l.o.g. that z_i is given the color $n + i + 1$ for $1 \leq i \leq n$. We also observe that the $p_{i,j}$'s together form a complete subgraph on $rn + 1$ vertices. Every $p_{i,j}$ is at most at distance two from every y_k , and every $p_{i,j}$ is at most distance two from every z_k , so the colors of the $p_{i,j}$ must be different from the colors of the y_k and different from the colors of the z_l . Thus we can assume that $p_{i,j}$ is given the color $2n + in + j + 1$ and $p_{n+1,r}$ is given the color $rn + 2n + 2$. Since y_i lies within distance two from all the x_j 's and the \bar{x}_j 's, except x_i and \bar{x}_i , the color i can only be assigned to x_i or \bar{x}_i . x_i lies within distance two from \bar{x}_i , so one of these two vertices must have a different color. x_i and \bar{x}_i lie within distance two from every z_k and $p_{k,l}$ and every other $y_j, j \leq i, j \neq i$, so only color $n + 1$ is available for one of these two vertices, for every $i, 1 \leq i \leq n$, because no x_i or \bar{x}_i lies within distance two from any other x_j or \bar{x}_j . The vertex that is assigned to color $n + 1$ will be called the *false* vertex. The other is the *true* vertex. The only way to color G using $rn + 2n + 2$ colors, is to assign color $n + 1$ to one of $\{x_i, \bar{x}_i\}$ for each $i, 1 \leq i \leq n$.

Under what conditions can the remaining vertices be colored using no further colors? Since $n \geq 4$ and each clause has at most three literals, each C_i lies within distance two from a pair x_j, \bar{x}_j , for at least one j . Consequently no C_i may be assigned the color $n + 1$. Also every C_i lies within distance two from every $p_{k,l}$ and every z_j , so C_i must be assigned a color less than $n + 1$.

Also no C_i can be assigned a color corresponding to an x_j or an \bar{x}_j that does not occur in clause C_i . These observations imply that the only colors that can be assigned to C_i correspond to vertices x_j or \bar{x}_j that are in clause C_i and are *true* vertices.

Hence G is strongly $rn + 2n + 2$ colorable iff there is a true vertex corresponding to each C_i , and thus iff F is satisfiable. \square

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Abstract

We consider the generalization of graph coloring to distance- k coloring, also termed *strong coloring* in the case $k = 2$. Some basic facts about strong coloring of graphs are given, and several auxiliary results are presented for strong colorings of special classes of graphs. We show that every planar graph can be strongly colored with at most $3\Delta + 9$ colors, where Δ denotes the maximum degree of the graph. (A straightforward algorithm would use as many as $5\Delta + 1$ colors.) It is shown that every outerplanar graph can be strongly colored with at most $\Delta + 4$ colors.

1 Introduction

The coloring problem for graphs has a longstanding mathematical interest. In this paper we consider the generalization to distance- k coloring for any $k \geq 1$, that is, we consider the problem in which it is required that all vertices with distance $\leq k$ have a different color. The distance- k coloring problem for graphs is NP-complete for every $k \geq 1$ [16]. For $k = 1$ one has the old definition of graph coloring, and for $k = 2$ the concept is also referred to as *strong graph coloring* [2, 4, 7]. Alternatively, a strong coloring can be defined as a coloring with the property that adjacent nodes have different colors (the usual “coloring condition”) and, moreover, all neighbours of any node are colored differently (the “strong coloring condition”).

The strong coloring problem for graphs has several applications. For example, in computing approximations to sparse Hessian matrices [16] the following typical problem arises: Given an $n \times n$ matrix M of 0's and 1's, one wishes to partition the columns of M into a number of sets such that no two columns in the same set have a 1 in the same row. This is equal to the strong coloring problem when we let M be the adjacency matrix of a graph with 1's along the main diagonal. Another application concerns the design of collision-free multi-hop channel access protocols in

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radio-networks [14], which can be solved using strong coloring. Another application we mention concerns the segmentation problem for files in a network [2]. Here the colors represent different (disjoint) segments of a file F , the graphs are regular with degree d and a strong coloring is desired with exactly $d + 1$ colors. (This implies that in the neighbourhood of every node a full copy of F can be assembled from the available segments.)

In this paper we present some facts about the strong coloring problem for graphs. We prove a number of basic facts and present some results when the problem is restricted to special classes of graphs. We present a method to strongly color planar graphs with at most $3(\Delta + 3)$ colors, where Δ is the maximum degree in the graph. We note that the naïve method for strongly coloring a planar graph would use up to $5\Delta + 1$ colors. It is shown that outerplanar graphs can be strongly colored with at most $\Delta + 4$ colors.

The paper is organized as follows. In section 2 we give some definitions concerning graphs and (strong) graph colorings. In section 3 we give some preliminary results for the strong coloring problem for certain classes of planar and non-planar graphs. In section 4 we give some facts about strongly coloring $(r - 1)$ -regular graphs with r colors, which we refer to as “perfect coloring”. In section 5 the main results for strongly coloring planar graphs are given. In section 6 we show that every circuit with non-intersecting chords can be strongly colored with at most $\Delta + 4$ colors, which is the main technical result that underlies the constructions in section 5. Section 7 contains some remarks and open questions. In an appendix we present a new proof of the NP-completeness of the strong coloring problem for graphs. We will assume some familiarity with graph theory (cf. Harary [10]).

2 Definitions

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The distance between two nodes x and y is defined as the number of edges on the shortest path between x and y . Let $\Delta = \max\{deg(v) | v \in V\}$, with $deg(v)$ the degree of vertex v . The square graph of G is the graph G^2 with $V(G^2) = V(G)$ and $E(G^2) = \{(u, v) | (u, v) \in E \text{ or } (u, x) \in E \text{ and } (x, v) \in E \text{ for some } x\}$. Observe that $\Delta(G^2) \leq (\Delta(G))^2$. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the least $K \leq n$ such that G can be K -vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \dots, K\}$ with $f(u) \neq f(v)$ whenever $\{u, v\} \in E$ [10]. The *chromatic index* of a graph G , denoted by $\chi'(G)$, is the least $K \leq m$ such that G can be K -edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \dots, K\}$ with $f(\{u, v\}) \neq f(\{u, w\})$ for all $u, v, w \in V$ and $\{u, v\}, \{u, w\} \in E$ [10]. With vertex (edge) coloring, every pair of vertices (edges) that have distance one must have different colors. Whenever colorings are considered, $c(v)$ denotes the color given to a vertex v .

The generalization to distance- k coloring is now straightforward. The *k -chromatic*

number of a graph G , denoted by $\chi_k(G)$, is the least $K \leq n$ such that G can be distance- k K -vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \dots, K\}$ with $f(u) \neq f(v)$ whenever u and v lie within distance k in $G = (V, E)$. The k -chromatic index of a graph G , denoted by $\chi'_k(G)$, is the least $K \leq m$ such that G can be distance- k K -edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \dots, K\}$ with $f(\{u, v\}) \neq f(\{w, x\})$ whenever $\{u, v\}$ and $\{w, x\} \in E$ and $\{u, v\}$ and $\{w, x\}$ lie within distance k from each other. For $k = 2$, we speak of the *strong chromatic number* and the *strong chromatic index* respectively. If a $(r-1)$ -regular graph can be strongly colored with exactly r colors, then this coloring is called *perfect* [11].

McCormick [16] has proved that, given a graph and an integer K , the problem of deciding whether a graph can be distance- k vertex colored with K colors is NP-complete, for every $k \geq 1$. Another proof for the strong chromatic number problem can be found in the Appendix. Checking whether a graph can be strongly colored with $K \leq 3$ colors is trivial. If a graph has $\Delta \geq 4$ or contains a C_5 or a $K_{2,3}$, then the graph is not strongly 4-colorable. Recently Bakker [3] has shown that deciding whether a $(r-1)$ -regular graph can be perfectly colored with r colors is NP-complete, even for the case $r = 4$. It is open whether the problem of deciding $\chi'_k(G) \leq K$ is NP-complete. In this paper we will focus entirely on the strong coloring problem ($k = 2$). We will be referring to some special classes of graphs including planar graphs, outerplanar graphs, Halin graphs, chordal graphs and partial k -trees. We assume that the first three are known but include a recursive definition of partial k -trees.

Definition 2.1 ([1]) *The class of k -trees is the smallest class of graphs that can be defined as follows:*

1. *the complete graph K_k on k vertices is a k -tree.*
2. *if $G = (V, E)$ is a k -tree and v_1, \dots, v_k form a complete subgraph of G , then the graph $G' = (V \cup \{w\}, E \cup \{(v_i, w) | 1 \leq i \leq k\})$ with $w \notin V$ is also a k -tree.*

A graph is a partial k -tree if and only if it is the subgraph of a k -tree.

3 Preliminaries

To obtain some first bounds for $\chi_2(G)$, consider the smallest-last (SL) ordering of a graph, determined by the following algorithm of [15] in $O(m)$ time.

ALGORITHM SL

```

j = n; H = G;
for j = n downto 1 do

```

begin

Let v_j be a vertex with minimum degree in H .

Remove v_j and all edges incident to v_j from H .

end

SL = v_1, v_2, \dots, v_n .

END OF ALGORITHM

Let SL = v_1, v_2, \dots, v_n be a smallest-last ordering of G . Let p be the maximum of the degrees of the vertices as they appear in the for-loop of algorithm SL. It is easily shown that $p = \max_H \min_{v \in V(H)} \{deg_H(v) | H \text{ a subgraph of } G\}$, and that G can be colored with $p + 1$ colors.

Theorem 3.1 *With p be defined as above: $\Delta + 1 \leq \chi_2(G) \leq p \cdot \Delta + 1$.*

Proof: The lowerbound is trivial. For the upperbound, we use induction to show that the vertices can be strongly colored in SL-order. Let C be a set of $p \cdot \Delta + 1$ colors. Vertex v_1 can be assigned an arbitrary color from C . Assume we have strongly colored the vertices v_1, \dots, v_{i-1} (following the smallest-last ordering) using colors from C ($i \geq 2$). v_i is connected to at most p colored neighbours, hence v_i has at most $p \cdot \Delta$ colored vertices within distance 2 and at most $p \cdot \Delta$ colors from C are blocked for it. Hence v_i can be colored with a color from C . This completes the induction. \square

Corollary 3.2 *Partial k -trees can be strongly colored with at most $k \cdot \Delta + 1$ colors.*

Proof: From definition 2.1 it follows that there exists a smallest-last ordering of the partial k -tree by removing vertices with maximum degree $\leq k$ in each step of algorithm SL. Using theorem 3.1, the corollary follows. \square

Lemma 3.3 *Every outerplanar graph can be strongly colored with at most $2\Delta + 1$ and every planar graph with at most $5\Delta + 1$ colors.*

Proof: Every outerplanar graph has a vertex with degree at most 2. Deleting this vertex with all its incident edges gives another outerplanar graph with the same property. In this way one can easily make a smallest-last ordering in which the maximum degree in each step is at most 2, and the lemma follows from theorem 3.1 for outerplanar graphs. Noting that every planar graph has a vertex with degree at most 5, the lemma easily follows in a similar way. \square

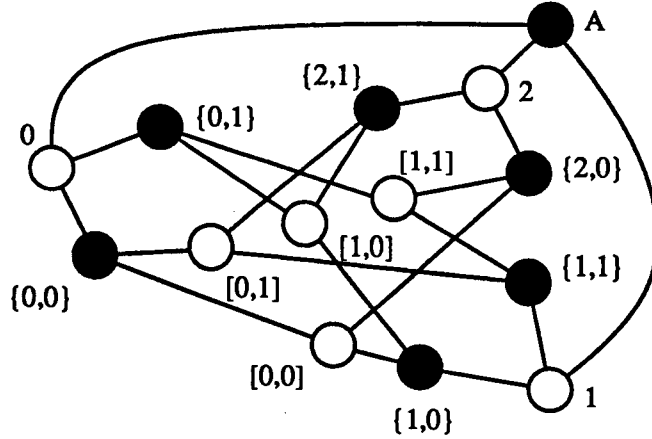


Figure 1: Example of a G_p for $p = 3$.

Similar bounds can be given for distance- k vertex and edge colorings. Observe that theorem 3.1 also implies that $\chi'(G) \leq \chi_2(G)$, as $\chi'(G) \leq \Delta + 1$ by Vizing's theorem. The question whether $\chi_k(G) \leq \chi'_k(G)$ remains as an interesting open problem. For trees it is clear that $\chi_2(G) = \Delta + 1$ and $\chi'_2(G) = \max\{deg(u) + deg(father(u)) | u \in V\}$. Every Halin graph can be strongly colored with at most $\Delta + 6$ colors. (For the latter result one uses that every tree can be strongly colored using at most $\Delta + 1$ colors and every circuit with at most 5 colors.)

Observe that from a strong vertex coloring with $\chi_2(G)$ colors one can obtain a strong edge coloring with $(\chi_2(G))^2$ colors, by assigning to every edge $\{u, v\}$ the color $[c(u), c(v)]$, where the colors of the vertices are taken from the strong vertex coloring. Notice that this large difference of the strong chromatic number and the strong chromatic index actually occurs in the case of the graph $G = K_{n,n}$, where $\chi_2(G) = 2n$ and $\chi'_2(G) = n^2$.

There appears to be no simple connection between $\chi(G)$ and $\chi_2(G)$. A reasonable conjecture like $\chi_2(G) \leq (\Delta + 1)\chi(G) + 1$ fails, by observing the construction of the following bipartite graph $G_p = (\langle V_1, V_2 \rangle, E)$. V_1 consists of p vertices $0, \dots, p-1$. V_2 consists of $p(p-1)$ vertices $\{i, j\}, 0 \leq i < p, 0 \leq j < p-1$, and a vertex A . Let $i \in V_1$ be connected to the nodes $\{i, x\}$ for $0 \leq x < p-1$ and to vertex A . Add $p(p-1)$ vertices $[k, l]$ to V_1 , with $0 \leq k, l < p-1$, and the edges $([i, j], \{0, i\})$ and $([i, j], \{k, ((k-1) \cdot i + j) \bmod (p-1)\})$, with $1 \leq k < p$ and $0 \leq i, j < p-1$. Note that every two vertices in V_2 have distance 2 to each other. Hence this bipartite p -regular graph has $\Delta = p, \chi(G) = 2$ and $\chi_2(G) = \Delta(\Delta - 1) + 1$. It shows that $\chi(G)$ and $\chi_2(G)$ can differ dramatically. In figure 1, an example is given for $p = 3$.

From the observation that $\chi_k(G) = \chi(G^k)$ and $\chi'_k(G) = \chi'(G^k)$ (using the definition of the k^{th} power graph G^k of [10]) for every $k \geq 2$, we conclude that we can use the available algorithms for ordinary graph coloring for obtaining strong colorings,

after calculating G^2 in $O(\Delta \cdot m)$ time. Also all lowerbounds for the chromatic number trivially hold for the strong chromatic number. This also leads to the following observation:

Theorem 3.4 *Chordal graphs can be strongly colored in a smallest possible number of colors in polynomial time. For every chordal graph G one has $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$.*

Proof: Note that the square graph of a chordal graph is a chordal graph too, and can be colored in an optimal number of colors in polynomial time (cf. Golumbic [9]).

For proving $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$ we induct on n , the size of G . For $n \leq 3$ the result trivially holds. Consider an arbitrary chordal graph G of n nodes, $n > 3$. Without loss of generality we may assume that G is connected. If G is a clique, then it can be strongly colored with $n = \Delta + 1$ colors and $\Delta + 1 \leq (\frac{1}{2}\Delta + 1)^2$. Thus let G not be a clique, S a minimal node separator of G and A_1, A_2, \dots, A_l the connected components of $G - S$. Let H_1 be the induced subgraph spanned by S and A_1 , and let H_2 be the induced subgraph spanned by S and A_2, \dots, A_l . By well-known facts for chordal graphs [9] S is a clique, and H_1 and H_2 are connected chordal graphs. Let $|S| = s$. By induction H_1 and H_2 are strongly colorable with the colors of some set C of $(\frac{1}{2}\Delta + 1)^2$ colors. A strong coloring of G can now be obtained as follows.

Permute the colors such that in the strong colorings of H_1 and H_2 , the nodes of S get the same colors. (This can be done because the colors assigned to the nodes of S must all be different, by the strong coloring requirement, in both the coloring of H_1 and the coloring of H_2 .) Let N_1 be the set of nodes in A_1 that are reached by an edge from S , and N_2 the set on nodes in $A_2 \cup \dots \cup A_l$ defined similarly. Let $|N_1| = n_1$ and $|N_2| = n_2$, and observe that $n_1 + n_2 \leq s \cdot (\Delta - s + 1) \leq (\frac{1}{2}\Delta + 1)^2 - s$. Thus we have sufficiently many colors in C to arrange that s colors are fixed for the nodes in S , and the remaining colors can be permuted such that in the strong colorings of H_1 and H_2 the nodes in N_1 and N_2 are colored by disjoint sets of colors. The resulting strong colorings of H_1 and H_2 can now be combined (merged) to a correct strong coloring of G which employs no more than $(\frac{1}{2}\Delta + 1)^2$ colors. \square

We conjecture that $\chi(G^2) \leq Q + 1$, with Q the number of vertices of the largest clique in the graph G^2 . If G^2 is a linegraph, then this conjecture is true by noting that if the linegraph G^2 has a largest clique of size Q , then the linegraph of this linegraph (which is a normal graph) has maximal degree Q and can be edge-colored with $Q + 1$ colors. Hence the linegraph G^2 can be vertex-colored with $Q + 1$ colors.

4 Perfect Colorings

Recall that a $(r - 1)$ -regular graph is perfectly colorable if it can be strongly colored with r colors. This coloring is useful for the following file distribution problem [2] : "Given a connected regular network $G = (V, E)$ and a file F , assign to each

node $x \in V$ a segment $F_x \subseteq F$ such that for all $x \in V, \bigcup_{\{x,y\} \in E} F_y \cup F_x = F$ and in every neighbourhood the distributed fragments are free of overlaps, i.e., $\forall (x,y) \in E : F_x \cap F_y = \emptyset$." When there are r different disjoint segments of F , this problem is only meaningful for $(r-1)$ -regular networks. A perfect coloring describes the assignment of the segments for a valid solution of the file distribution problem. It has been shown by Bakker [3] that this problem is NP-complete, even for the case $r = 4$.

In this section we give some characteristics of perfectly colorable graphs. Some relationships between strong colorings, perfect colorings and edge colorings are the following:

Theorem 4.1 ([2]) *If a $(r-1)$ -regular graph with $|V| = n$ nodes can be perfectly colored, then $r|n$ and every group of equally colored nodes has $\frac{n}{r}$ nodes.*

Proof: Let $N(x)$ denote the set of vertices having distance ≤ 1 to node x . Consider any perfect coloring of the graph, and let c be one of the colors. Let x_1 and x_2 be two nodes colored c . There can be no node y in $N(x_1) \cap N(x_2)$ because, if there was, y would have two neighbours of the same color (which contradicts the strong coloring property). Thus $\forall x_1, x_2$ with $c(x_1) = c(x_2) = c : N(x_1) \cap N(x_2) = \emptyset$. Furthermore $\forall y \exists x, c(x) = c : y \in N(x)$. Hence the neighbourhoods $N(x)$ of nodes x with $c(x) = c$ form a partitioning of G . But $\forall x \in V : |N(x)| = r$. Hence $r|n$ and every group of equally colored nodes has size $\frac{n}{r}$. \square

Theorem 4.2 *Every strongly r -colorable graph is the induced subgraph of a perfectly colorable $(r-1)$ -regular graph.*

Proof: We induct on r . For $r = 1, 2$ and 3 , the theorem is trivial. Thus let $r \geq 4$ and G be a strongly r -colorable graph. Consider a strong coloring of G with the colors c_1, \dots, c_r and let H be the induced subgraph of G consisting of all nodes with a color $\in \{c_1, \dots, c_{r-1}\}$. By induction H is an induced subgraph of some $(r-2)$ -regular graph R_H that is perfectly colorable, and w.l.o.g. we can assume that it is perfectly colored with c_1, \dots, c_{r-1} . Arrange the nodes of R_H into $(r-1)$ disjoint blocks B_1, \dots, B_{r-1} , with B_i ($1 \leq i \leq r-1$) containing the nodes of color c_i , and let every block contain b nodes. (By theorem 4.1 we know that the blocks must be of equal size.) Tag the nodes of R_H that correspond to the nodes of H . Let the nodes x_1, \dots, x_s (some $s \geq 1$) of $G - H$ together form the "beginning" of the r^{th} block B_r . The nodes $\{x_1, \dots, x_s\}$ form an independent set.

Now form the graph R_G as follows. Make $\lceil \frac{r}{b} \rceil$ copies of R_H and extend B_r by another $\lceil \frac{r}{b} \rceil b - s$ nodes y . We now "connect" the x - and y -nodes to the nodes in the R_H copies in two steps, as follows:

1. for i from 1 to s do
begin

connect x_i to a node from a B_j -block for every $j, 1 \leq j \leq r - 1$,
always favoring the tagged node $z \in$ block B_j if x_i is
directly connected to z in G , and an untagged node otherwise.
end;

Observe that we have $\lceil \frac{s}{b} \rceil b \geq s$ nodes of every color, so step 1 always works. But also observe that we have exactly $\lceil \frac{s}{b} \rceil b - s$ nodes left of every color after this step.

2. **for** j **from** 1 **to** $\lceil \frac{s}{b} \rceil \cdot b - s$ **do**
begin
pick a new y -node and connect y to a node from a B_i -block that
was not used before, for every $i, 1 \leq i \leq r - 1$.
(Note that these nodes were not tagged.)
end;

Note that step 2 makes the graph G_H $(r - 1)$ -regular. The result is a graph R_G that is $(r - 1)$ -regular, perfectly colorable with r colors and clearly, by design, we have that G is an induced subgraph of R_G . \square

Another property of perfectly colorable graphs is the following. Recall that by Vizing's theorem every graph is edge-colorable with Δ or $\Delta + 1$ colors.

Lemma 4.3 *If a graph is strongly colorable with r colors and r is even, then it is $(r - 1)$ -edge colorable.*

Proof: Let G be strongly colorable with r colors. If r is even, then K_r is edge-colorable with $(r - 1)$ colors. Let G be strongly colored with the r names of nodes of K_r . Now G can be edge-colored as follows: color an edge from the node colored X to the node colored Y with z if the edge between X and Y in K_r is colored z . This gives a correct $(r - 1)$ -edge coloring of G . \square

The converse is not true, see for example figure 2. Also this theorem does not hold for r odd in general, as an edge-coloring on a K_r (which is perfectly colorable with r colors) requires r colors when r is odd.

Also the spectra of perfectly colorable graphs have some interesting properties. Because a perfectly r -colorable graph G is $(r - 1)$ -regular, its largest eigenvalue is equal to $r - 1$ and has multiplicity 1 (cf. Biggs [6]). The following more specific observation can be made as well.

Theorem 4.4 *Let G be perfectly r -colorable. Then G has an eigenvalue -1 , with multiplicity $\geq (r - 1)$.*

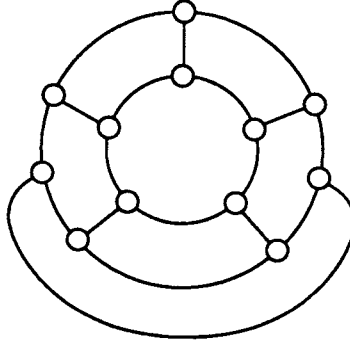


Figure 2: A 3-edge colorable 3-regular graph that is not perfectly colorable (from [12]).

Proof: Let G be perfectly r -colored, and consider the vertices of G arranged in blocks of equally colored vertices (of size $\frac{n}{r}$ each). Let $A = A(G)$ be the adjacency matrix of G corresponding to this vertex-ordering. The symmetric matrix A can be viewed as a block matrix, with the blocks along the main diagonal consisting of all zeroes and the off-diagonal blocks being $\frac{n}{r} \times \frac{n}{r}$ permutation matrices. (As an aside we note that, conversely, if the vertices of a graph G can be arranged so the adjacency matrix is of this form, then G is perfectly r -colorable.) Now consider the $r \times r$ matrix A' obtained from A by replacing every block on the main diagonal by a "0" and every off-diagonal block by a "1". A' is the adjacency matrix of the K_r , whose spectrum consists of one eigenvalue $(r - 1)$ and $(r - 1)$ eigenvalues -1 (see e.g. [6]). Also, when (x_1, \dots, x_r) is an eigenvector of A' , then the vector obtained by repeating each coordinate $\frac{n}{r}$ -fold is an eigenvector of A and independency of eigenvectors is preserved in the process. It follows in particular that A (and hence, G) has an eigenvalue -1 with multiplicity $r - 1$. \square

From the same argument some more information can be derived. Let $n > r$ and let $\lambda_1, \dots, \lambda_k$ and $-\mu_1, \dots, -\mu_l$ be the remaining positive and negative eigenvalues in the spectrum of G in decreasing order different from the r eigenvalues $(r - 1)$ and -1 that we have, with $k + l = n - r$. As the trace of A is zero, we have $\lambda_1 + \dots + \lambda_k = \mu_1 + \dots + \mu_l$. Observe also that A^2 is a symmetric matrix with all entries along the main diagonal equal to $r - 1$. It follows that $\lambda_1^2 + \dots + \lambda_k^2 + \mu_1^2 + \dots + \mu_l^2 = \text{tr}(A^2) - (r - 1)^2 - (r - 1) = (n - r)(r - 1)$. Now let $\lambda = \lambda_1 = \lambda_{\max}$, $\mu = \mu_l = \mu_{\max}$ and $\delta = \max\{\lambda, \mu\}$. One easily verifies that $\delta \geq \sqrt{r - 1}$ and $\min\{\lambda, \mu\} \geq \frac{1}{n - r} \sqrt{r - 1}$.

Some further characteristics of perfectly colorable graphs are the following:

Theorem 4.5 *Let G be regular of degree ≥ 3 and perfectly colorable. Then one can partition V as $V_1 \cup V_2$ such that*

1. the induced subgraph G_1 on V_1 is a set of chordless cycles of length divisible by 3.
2. the induced subgraph G_2 on V_2 is regular of degree $\Delta - 3$ and perfectly colorable.

Proof: Let a, b, c be three colors of the perfect coloring of G . Let V_1 be the set of nodes colored a, b or c and $V_2 = V - V_1$.

1. Consider any node in V_1 , say with color a . It has one neighbour colored b , this neighbour has one neighbour colored c , etc. This necessarily closes itself as a cycle at the point of departure. By the strong coloring property, this cycle must be chordless. This proves the statement, and the cycles are not connected to each other.

2. Consider any node in V_2 . It has *exactly* three neighbours in V_1 . Thus G_2 inherits the strong (perfect) coloring of G , with the remaining $\Delta - 3$ colors. \square

This shows that perfectly colorable graphs decompose entirely into (disjoint) chordless cycles. Note that $\frac{|V_2|}{|V_1|} = \frac{\Delta-2}{3}$, for $\Delta \geq 3$.

For the file distribution problem perfect colorings are interesting mostly for regular networks, which includes many current processor networks. In [11] a detailed study is given of the perfectly colorable processor networks. For completeness we summarize the results of [11] in the following theorem.

Theorem 4.6 ([11]) *The following processor networks are perfectly colorable:*

- The hypercube C_n , if and only if $n = 2^i - 1, i > 0$.
- The d -dimensional torus of size $l_1 \times \dots \times l_d$ if $l_i \bmod q = 0$, with q such that $\sqrt{2d+1}|q$ for some integer $r > 0$.
- The Cube-connected Cycles CCC_d , if and only if $d > 2, d \neq 5$.
- The directed shuffle-exchange network and the directed 4-pin shuffle network.
- The chordal ring network with chord length $4p - 1$ ($p > 0$) and $4kp - 4t$ ($0 \leq t < p$) nodes if and only if :
 1. k and t are even and (if $t > 0$) $\frac{t}{\gcd(t,p)}$ is even, or
 2. $k, \frac{t}{\gcd(t,p)}$ and $\frac{p}{\gcd(t,p)}$ are odd and $t + p$ is even.
- The hexagonal network of size $m \times n$ if and only if $m, n \bmod 7 = 0$.

5 Main Results for Planar Graphs

In this section we consider the strong coloring problem for planar graphs. By the results from section 3 we know that every planar graph G can be strongly colored using at most $5\Delta + 1$ colors. Our aim will be to improve this to a bound of $c\Delta + O(1)$ colors with a significantly lower constant $c < 5$ (uniformly, for all planar graphs). We will show that one can take $c = 1$ for outerplanar graphs (which is optimal) and that one can take $c = 3$ for planar graphs in general. We begin by giving a worst-case lowerbound for c (for general planar graphs).

Lemma 5.1 *For every $\Delta \geq 1$ there exists a planar graph G with $\chi_2(G) \geq \lceil \frac{3}{2}\Delta \rceil$.*

Proof: We can assume w.l.o.g. that $\Delta > 1$. (For $\Delta = 1$ the lemma trivially holds by taking a graph that consists of a single edge). Choose $r, s \geq 0$ with $s \leq r$ such that $\Delta = r + s + 2$. It will be useful to take $r = s = \frac{1}{2}\Delta - 1$ when Δ is even and $s = r - 1 = \frac{1}{2}\Delta - \frac{3}{2}$ when Δ is odd. Construct the graph G_Δ consisting of a “triangle” of three nodes (A, B and C), r nodes that are each connected to A and to B , s nodes that are each connected to B and to C , and s more nodes that are each connected to A and to C . For Δ odd (implying $\Delta \geq 3$), a separate node D is “inserted” on the triangle-edge (A, B). This node is also connected to C . One easily verifies that G_Δ is planar, has maximum degree Δ and diameter 2. Because of the latter any strong coloring of G_Δ needs as many colors as there are nodes, which is precisely $\lceil \frac{3}{2}\Delta \rceil$. (By a result of Seyffart [17] this is about the largest possible number of nodes in any planar graph of diameter 2 and maximum degree Δ .) \square

The lemma shows that $c \geq \frac{3}{2}$ for general planar graphs. For $\Delta \leq 5$ one can construct planar graphs that need $\geq 2\Delta$ colors in any strong coloring (which does not imply that $c \geq 2$, in the given formulation).

We take an indirect approach to the strong coloring problem for planar graphs. First we show that certain subclasses of planar graphs admit strong colorings with a “very small” number of colors. It is used to obtain strong colorings of general planar graphs G that use at most $3\Delta + O(1)$ colors. The following technical lemma is instrumental, but for clarity reasons its proof is deferred to section 6.

Lemma 5.2 *Every (planar) circuit G with non-intersecting chords can be strongly colored using at most $\Delta + 4$ colors.*

We use the lemma to derive a bound on the number of colors needed to strongly color an arbitrary outerplanar graph.

Theorem 5.3 *Every outerplanar graph G can be strongly colored using at most $\Delta + 4$ colors.*

Proof: Let G be outerplanar. (Without loss of generality we can confine ourselves to connected graphs.) We proceed by induction. When G has ≤ 5 nodes, the theorem trivially holds. Thus assume that the theorem holds for all connected outerplanar graphs of $\leq n - 1$ nodes, and let G have n nodes (some $n > 5$). If G is a planar circuit with non-intersecting chords, then the result follows immediately by Lemma 5.2. If G is not, then G must contain a cutvertex v . In this case G consists of connected outerplanar graphs H_1 and H_2 such that each contain a “copy” of the node v and are joined at v , but which are otherwise disjoint. (Without loss of generality we may assume that both H_1 and H_2 have $\leq n - 1$ nodes.)

Let v have degree Δ_1 in H_1 and degree Δ_2 in H_2 , where we can assume w.l.o.g. that $\Delta_1 \leq \Delta_2$ and clearly $\Delta_1 + \Delta_2 \leq \Delta$. We can assume inductively that H_1 and H_2 can be strongly colored using at most $\Delta + 4$ colors. Shift color-names such that H_1 and H_2 use colors from the same set of $\Delta + 4$ colors and v gets the same color “ α ” in H_1 and H_2 . Joining H_1 and H_2 at v (while retaining the colorings of H_1 and H_2 respectively) results in a strong coloring of G with $\Delta + 4$ colors, except in the one case that some neighbours of v in H_1 have the same color as some neighbours of v in H_2 . We now argue how such a conflict can be removed by a mere permutation of the colors, if it arises.

Thus assume that the latter case arises. Note that v and its neighbours in H_2 use $\Delta_2 + 1$ colors. Let k neighbours of v in H_1 use colors different from these but l neighbours use colors c_1, \dots, c_l that are among the colors used by the Δ_2 neighbours in H_2 , for certain k and l with $k + l = \Delta_1$. It means that $\Delta_2 + 1 + k$ different colors are used in the neighbourhood of v . Choose l different colors d_1, \dots, d_l from among the remaining colors. (This can be done because $\Delta + 4 - (\Delta_2 + 1 + k) \geq \Delta_1 + \Delta_2 + 4 - (\Delta_2 + 1 + k) = l + 3$.) Exchanging c_i and d_i (for i from 1 to l) in the coloring of H_1 throughout leaves a strong coloring in H_1 and removes the color conflicts at v , thus leading to a correct strong coloring of G using at most $\Delta + 4$ colors.

This completes the inductive argument. □

The theorem enables us to prove the main result of this section on strong colorings of planar graphs.

Theorem 5.4 *Every planar graph G can be strongly colored using at most $3\Delta + 9$ colors.*

Proof: Let G be an arbitrary planar graph, v a node of G (e.g. chosen to lie on the exterior face of G). Define L_i to be the set of nodes that lie at distance i from v , for any $i \geq 1$. (This leads to a decomposition of V into finitely many disjoint sets which are easily determined algorithmically by breadth-first-search.) Consider the subgraphs of G induced by the sets L_i , and let Δ_i be the maximum degree of any node in the L_i -induced subgraph. Now observe that each L_i -induced subgraph is outerplanar, and that nodes in L_i can only be adjacent to nodes in L_{i-1} , L_i and

L_{i+1} . Also every node in L_i must be adjacent to a node in L_{i-1} (for $i \geq 1$), which means that $\Delta_i \leq \Delta - 1$ (for all $i \geq 0$).

We can now obtain a strong coloring of G as follows. By theorem 5.3 each L_i -induced subgraph can be strongly colored using $\leq \Delta_i + 4 \leq \Delta + 3$ colors. Take three sets of $\Delta + 3$ different colors T_0, T_1, T_2 and color the nodes of every L_i -induced subgraph strongly using the colors from the set $T_{i \bmod 3}$, by the method implicit in theorem 5.3. This necessarily results in a strong coloring of G using (at most) $3(\Delta + 3) = 3\Delta + 9$ colors. \square

The same technique as used in theorem 5.4. can be used to prove some further results. For example, a very similar argument can be used to show that every k -outerplanar graph can be strongly colored using at most $3\Delta + 9$ colors.

6 Strongly Coloring Circuits with Non-Intersecting Chords

This section is devoted entirely to the proof of Lemma 5.2, which asserts that every planar circuit G with non-intersecting chords can be strongly colored using at most $\Delta + 4$ colors. First we formalize a useful technique that will be applied repeatedly in the proof. Let G be an arbitrary graph, with colors assigned to the nodes.

Definition 6.1 *A node v is said to “miss” color α if neither v nor any of its neighbours is colored α .*

Definition 6.2 *Let H be any connected (but not necessarily induced) subgraph of G , and let α, β be two different colors in the coloring. The operation $\text{SWITCH}_H(\alpha, \beta)$ acts on the given coloring of G by interchanging the colors α and β in the color assignment for the nodes of H .*

(We will assume that the operation SWITCH_H is defined *only* for subgraphs H as stated.)

Lemma 6.1 *Let v miss color α and $c(v) \neq \beta$, and H be any connected component of $G - \{v\}$. If G is strongly colored, then so it is after performing $\text{SWITCH}_H(\alpha, \beta)$.*

Proof: Let $H_0 = H, H_1, \dots$ be the connected components of $G - \{v\}$. In G , v is connected by an edge to selected nodes in H_0, H_1, \dots but the subgraphs H_0, H_1, \dots themselves are mutually disjoint. $\text{SWITCH}_H(\alpha, \beta)$ preserves the strong coloring property in the nodes of H_0, H_1, \dots trivially, hence we only need to verify that it does at v . If v misses β , then the operation has no effect on the neighbourhood of v at all. If v has a neighbour w with $c(w) = \beta$, then two cases can arise. If $w \in H_0$, then $c(w) = \alpha$ after performing the SWITCH_H operation. As v missed α , this gives

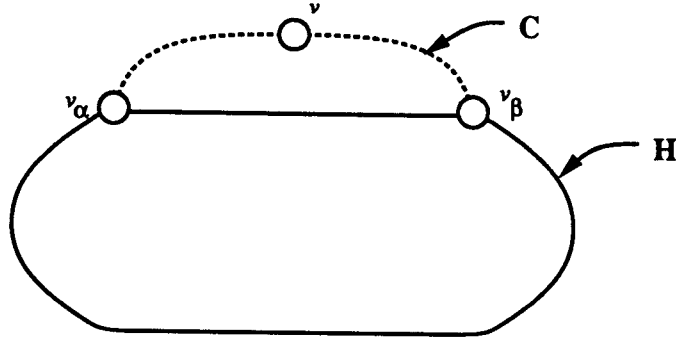


Figure 3: The inductive construction of a circuit with non-intersecting chords.

no conflict at v . If $w \in H_i$ for some $i > 0$, then the neighbourhood of v again is not affected by the operation. \square

We now analyze the strong colorings of planar circuits G with non-intersecting chords. We assume inductively that such graphs can always be strongly colored with $\leq \Delta + c$ colors, for some constant $c \geq 1$ that will be fixed later. (Lemma 5.2 indicates that we will later choose $c = 4$.) This certainly holds for all circuits G with non-intersecting chords that have ≤ 5 nodes, provided we take $c \geq 3$. Assume that the hypothesis holds for all G with $\leq n - 1$ nodes (some $n > 5$) and consider an arbitrary planar circuit G with non-intersecting chords that has n nodes. We will prove the induction hypothesis for G by case analysis.

If G has no chords, then it is a simple C_n (which has $\Delta = 2$). One easily verifies that C_n can be strongly colored with 3 colors when $n \equiv 0 \pmod{3}$, and with 4 colors otherwise (and $n > 5$, like we assumed). This satisfies the induction hypothesis.

Now assume that G has non-intersecting chords. As G is outerplanar, it must have a node v of degree 2. Orient the circuit, and let v_α and v_β be the first nodes to the left and to the right of v respectively that are incident to chords. Note that there must be a chord between v_α and v_β , which also is the first chord “seen” from v .

It follows that G can be decomposed into a chain C , which contains v and the part of the circuit to its left and to its right up to (but not including) v_α and v_β , and a remaining graph H . See figure 3.

Note that H is a planar circuit with non-intersecting chords of $\leq n - 1$ nodes. By induction H can be strongly colored using $\Delta_H + c$ colors, where $\Delta_H = \max \{deg(v) | v \in H\}$ (with degrees as counted in H). Clearly $\Delta_H \leq \Delta$, and w.l.o.g. we can assume that $c(v_\alpha) = \alpha$ and $c(v_\beta) = \beta$. It remains to color the nodes of C such that a strong coloring of the graph G results.

Assume first that C is a chain of k nodes for some $k \geq 2$. Let C consist of the nodes x_1, \dots, x_k “from left to right”, with x_1 adjacent to v_α and x_k to v_β . As we

have $\Delta_H + c$ colors at our disposal and both v_α and v_β only used $\Delta_H + 1$ of them (at most), there must be two different colors γ and δ such that v_α misses γ and v_β misses δ (in the strong coloring of H). Note that necessarily $\gamma, \delta \neq \alpha, \beta$. Now color C as follows. Assign γ to x_1 , δ to x_k and assign the colors β, α and γ alternately (in this order) to x_2, x_3, \dots when $k > 2$. If x_{k-1} is assigned α or γ , we are done. If x_{k-1} is assigned β , then a conflict arises with the strong coloring requirement at x_k . To resolve it we choose any color $\mu \notin \{\alpha, \beta, \gamma, \delta\}$ and assign it to x_{k-1} instead. (Such a μ exists because we have at least $\Delta_H + c \geq 6$ colors available.) This gives a valid, strong coloring of the entire graph with $\Delta_H + c \leq \Delta + c$ colors.

Next assume that C consists of only 1 node, namely the node v . This (final) case gives more difficulties, as we will see. Let M_α and M_β be the sets of colors, missed by v_α and v_β respectively. If $M_\alpha \cap M_\beta \neq \emptyset$ and (say) $\mu \in M_\alpha \cap M_\beta$, then we can “complete” the strong coloring of G by simply assigning the color of μ to v . Let us therefore assume that M_α and M_β are fully disjoint, for the remaining analysis. The idea of the remainder of the proof is to try and make suitable “color flips” such that a new strong coloring arises in which the corresponding M_α and M_β are no longer disjoint (which would be sufficient for our proof by the previous case).

If $\Delta_H < \Delta$, then necessarily $\Delta_H = \Delta - 1$. But we used $\Delta_H + c$ colors for H and have $\Delta + c$ colors available for G . It follows that there must be a “free” color which must thus belong to M_α and M_β , contradicting their disjointness. Thus we have $\Delta_H = \Delta$, and necessarily $\deg_H(v_\alpha) \leq \Delta - 1$ and $\deg_H(v_\beta) \leq \Delta - 1$. We can also observe that both v_α and v_β must have chords “inside” H . For suppose e.g. that v_β had no chord inside H and (thus) $\deg_H(v_\beta) = 2$. Then $|M_\beta| = (\Delta + c) - 3 = \Delta + (c - 3)$. At the same time, $|M_\alpha| \geq (\Delta + c) - \Delta = c$. As $|M_\alpha| + |M_\beta| \geq \Delta + (2c - 3) > \Delta + c$ for $c > 3$, it would follow that M_α and M_β necessarily intersect, a contradiction.

Let u be the leftmost node to which v_β is connected by a chord. Let M_u be the set of colors missed by u . Suppose that M_α and M_u were not disjoint and (say) that $\gamma \in M_u \cap M_\alpha$ for some color γ . As $\gamma \notin M_\beta$, v_β must have a neighbour colored γ (which cannot be a neighbour of u as $\gamma \in M_u$). See figure 4. Choose a color $\varphi \in M_\beta$ with $\varphi \neq c(u)$ and do a $\text{SWITCH}_{H_r}(\gamma, \varphi)$ on the given coloring. It is easily argued that this preserves the strong coloring of H_r , but an argument similar to Lemma 6.1 shows that it actually preserves the strong coloring of H . This leads to a coloring in which both v_α and v_β miss γ , which suffices for our claim. A similar argument applies in case M_u and M_β were not disjoint. Thus we proceed on the assumption now that M_u is disjoint from M_α and M_β .

Next assume that there is a left neighbour of u colored with some color $\gamma \in M_\beta$ and there is another left neighbour of u colored with some color $\delta \in M_\alpha$. Now change the coloring of H_l by executing a $\text{SWITCH}_{H_l}(\gamma, \delta)$. Note that this changing of colors preserves the strong coloring property and has no effect on the right neighbours of v_u and (hence) has no effect on the nodes of H_r . It follows that after this operation both v_α and v_β miss color δ , which suffices for our claim. A similar argument applies in the case there is a right neighbour of u colored with some color $\gamma \in M_\beta$ and there is another right neighbour of u colored with some color $\delta \in M_\alpha$. So we proceed

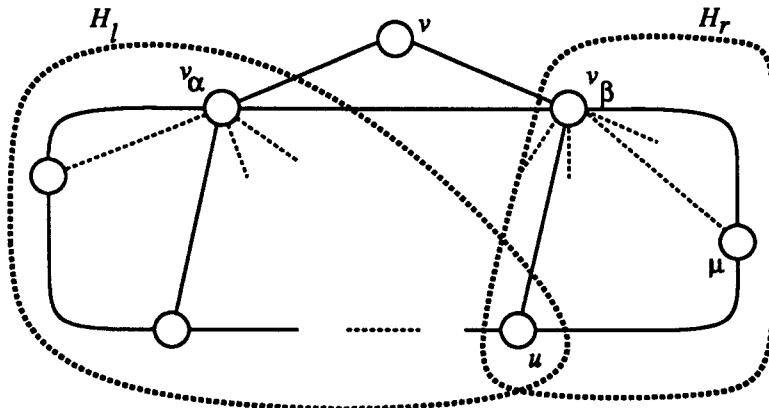


Figure 4: Both M_u and M_α miss a color γ .

on the assumption that this is not the case, i.e., all neighbours of u colored with elements of M_α are *left* neighbours of u and all neighbours of u colored with elements of M_β are *right* neighbours of u , or the other way around.

If the M_α -colored neighbours of u are right neighbours of u and the M_β -colored neighbours of u are left neighbours of u then we do the following. Let $\gamma \in M_u$, $\delta \in M_\alpha$ and $\mu \in M_\beta$. Change the coloring in H by doing a $\text{SWITCH}_{H_l}(\gamma, \delta)$ and a $\text{SWITCH}_{H_r}(\gamma, \mu)$ operation. It is easily argued that $\text{SWITCH}_{H_l}(\gamma, \delta)$ preserves the strong coloring of H_l and by Lemma 6.1 has no effect on the strong coloring of H_r . Similarly a $\text{SWITCH}_{H_r}(\gamma, \mu)$ has no effect on the strong coloring of H_l . These two recolorings have the effect that now both v_α and v_β miss color γ , which suffices for our claim.

Thus we proceed on the assumption that all M_α -colored neighbours of u are *left* neighbours of u and all M_β -colored neighbours of u are *right* neighbours of u . Let J_u be the set of colors used at u that are *not* in $M_\alpha \cup M_\beta$. Let L_u be the subset of colors of J_u , used by left neighbours of u , and let R_u be the subset of colors of J_u used by right neighbours of u . Note that J_u is disjoint from M_u, M_α and M_β , by definition. We have come some way in reconstructing the neighbourhood of v_α and v_β . We now know that v_α has (distinct) neighbours that are colored with the colors in $J_u \cup M_u \cup M_\beta$ and that v_β has (distinct) neighbours that are colored with the colors in $J_u \cup M_u \cup M_\beta$.

Lemma 6.2 *The colors can be flipped such the strong coloring requirement is preserved and some M_β -colored neighbour of v_α precedes all $R_u \cup M_u$ -colored neighbours (along the arc from v_α to u) and some M_α -colored neighbour of v_β precedes all $L_u \cup M_u$ -colored neighbours (along the arc from v_β to u).*

Proof: We only prove this for the neighbours of v_α , as the argument is similar for v_β and respective color-flips that are needed do not interfere. Suppose the

property is not satisfied yet, i.e., there exists a $\gamma \in R_u \cup M_u$ that precedes all colors μ with $\mu \in M_\beta$ along the arc from v_α to u for correspondingly colored neighbours of v_α (along the arc towards u). We let γ be the first color from $R_u \cup M_u$ that occurs along the arc and has this property. Let $\mu \in M_\beta$ be some color. Consider first the case $\gamma \in M_u$. It is easily seen that $\text{SWITCH}_{H_l}(\gamma, \mu)$ preserves the strong coloring of H_l and of H , and swaps μ into the desired “leading” position. Next, we consider the case that $\gamma \in R_u$, thus there is a right neighbour of u , colored with γ . Note that all M_β -colored neighbours of u are *right* neighbours of u , thus there is right neighbour of u colored with μ . Note that now a $\text{SWITCH}_{H_l}(\gamma, \mu)$ will again do the trick, for any $\mu \in M_\beta$ and swaps μ into the desired “leading” position. \square

(The proof of Lemma 6.2 is easily extended to show that all M_β -colored neighbours of v_α precede the $R_u \cup M_u$ -colored neighbours, but this is not needed for our argument here.) Let β_1 be the color of the first M_β -colored neighbour of v_α along the arc, and likewise α_1 the color of the first M_α -colored neighbour of v_β along the arc.

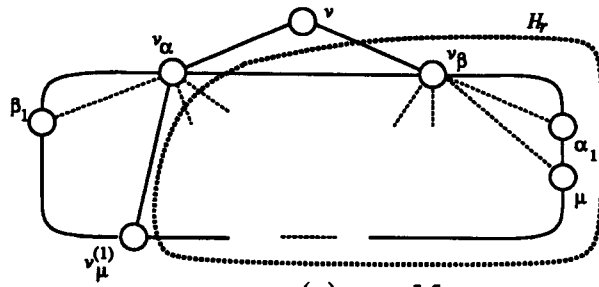
Observe that $|J_u| \geq (\deg(u) + 1) - |M_\alpha| - |M_\beta|$, $|M_u| = \Delta + c - (\deg(u) + 1)$ and $|M_\alpha| + |M_\beta| \leq \Delta + c$. This means that $|J_u| + |M_u| + |M_\alpha| + |M_\beta| \geq \Delta + c$ and (hence) $= \Delta + c$ because it are all disjoint sets, and in particular $|J_u| = (\deg(u) + 1) - |M_\alpha| - |M_\beta|$. It follows that $J_u \cup M_u \cup M_\alpha \cup M_\beta$ is the full set of colors. We can now estimate the number of neighbours of v_α and v_β as follows:

$$\begin{aligned}
|L_u| + |R_u| + 2|M_u| + |M_\alpha| + |M_\beta| &= |J_u| + 2|M_u| + |M_\alpha| + |M_\beta| \\
&= (\deg(u) + 1) - |M_\alpha| - |M_\beta| + \\
&\quad 2(\Delta + c - (\deg(u) + 1)) + |M_\alpha| + |M_\beta| \\
&= 2\Delta + 2c - (\deg(u) + 1) \\
&\geq \Delta + 2c - 1.
\end{aligned}$$

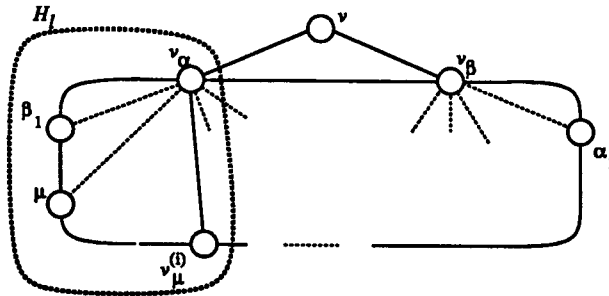
As each of these nodes must miss at least $c-1 \geq 3$ colors out of the full set of $\Delta + c$ colors and $3(\Delta + 2c - 1) > 3(\Delta + c)$, there must be some color μ that is missed in at least 4 of the nodes under consideration (i.e., neighbours between the β_1 -colored one of v_α and the α_1 -colored one of the v_β). In fact, because $3(\Delta + 2c - 3) > 3(\Delta + c)$ we can even claim that these 4 nodes occur *strictly in between* the β_1 -colored neighbour of v_α and the α_1 -colored neighbour of v_β . Let the nodes be $v_\mu^{(1)}, v_\mu^{(2)}, v_\mu^{(3)}$ and $v_\mu^{(4)}$, in this order. Assume w.l.o.g. that $v_\mu^{(i)}$ is a neighbour of v_α .

We will now complete the proof by a final case-analysis. The $v_\mu^{(1)}$ -nodes act as “separating nodes” and we can perform any sort of $\text{SWITCH}_*(\mu, *)$ operation on the arcs left or right of $v_\mu^{(i)}$ that preserves the strong coloring requirement, and try to “free” a color that can be assigned to v . We always use the same argument for it as in Lemma 6.1.

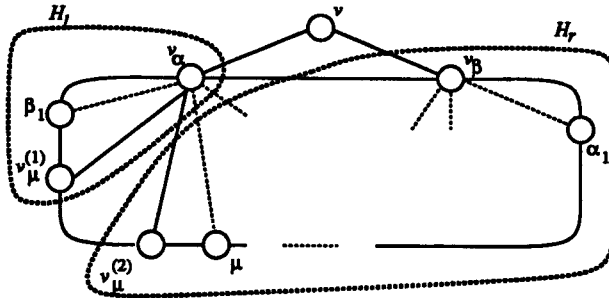
First assume that $\mu \in M_\alpha$ (see figure 5(a)). Observe that in this case v_β has a neighbour colored μ . Performing a $\text{SWITCH}_{H_r}(\mu, \beta_1)$ preserves the strong coloring



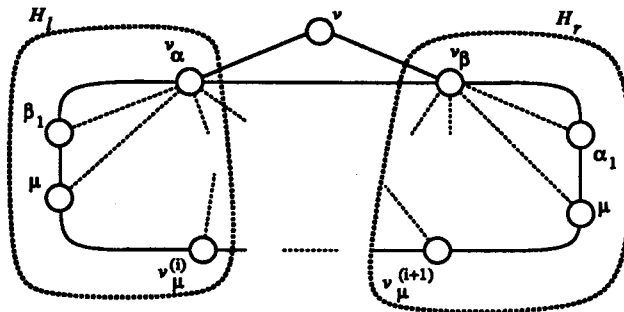
(a) $\mu \in M_\alpha$



(b) $v_\mu^{(i)}$ to the right of the μ -colored neighbour of v_α



(c) All $v_\mu^{(i)}$ nodes in between the μ -colored neighbours of v_α and v_β



(d) Two $v_\mu^{(i)}$ nodes in between the μ -colored neighbours of v_α and v_β

Figure 5: Final case analysis.

requirement in H_r and H , but it has the additional effect that in the new coloring v_β also misses μ (and v_α continues to miss it as well). Thus μ can be assigned to v and we are done.

Next assume that $\mu \in M_\beta$. In this case v_α has a neighbour colored μ . If v_β has one of the $v_\mu^{(i)}$ nodes as neighbour, then we can proceed as in the previous case. Thus let all $v_\mu^{(i)}$ be neighbours of v_α . First assume that there is a $v_\mu^{(i)}$ right of the μ -colored neighbour of v_α , as shown in figure 5(b). Now perform a $\text{SWITCH}_{H_l}(\mu, \alpha_1)$ to achieve the same effect as in the previous case. It leads to a strong coloring of H in which both v_α and v_β miss μ . Next assume that all $v_\mu^{(i)}$ are in between the β_1 - and μ -colored neighbours of v_α , see figure 5(c). (Note that H_l and H_r do not include the part of H "between" $v_\mu^{(1)}$ and $v_\mu^{(2)}$.) Now perform a $\text{SWITCH}_{H_r}(\mu, \alpha_1)$ and also a $\text{SWITCH}_{H_l}(\mu, \beta_1)$. A straightforward argument shows that this preserves the strong coloring requirement, but it has the additional effect of removing β_1 from the neighbourhood of v_α . Thus β_1 is "freed" and can be assigned to v .

Finally assume that $\mu \notin M_\alpha \cup M_\beta$. Now both v_α and v_β have μ -colored neighbours, say x and y respectively. The nodes x and y divide the arc from the β_1 -colored neighbour of v_α to the α_1 -colored neighbour of v_β into three intervals. Thus at least one of these intervals must contain two $v_\mu^{(i)}$ nodes. If either the first or the last interval contains two $v_\mu^{(i)}$ nodes, then we can reason exactly as in the previous case and are done. (Consider e.g. figure 5(b) and add a μ -colored neighbour of v_β . The argument remains unchanged.) Thus the only case left is the case in which two $v_\mu^{(i)}$ nodes occur between the μ -colored neighbours of v_α and v_β , see figure 5(d). Now perform a $\text{SWITCH}_{H_l}(\mu, \alpha_1)$ and also a $\text{SWITCH}_{H_r}(\mu, \beta_1)$. This preserves the strong coloring requirement and frees μ at both v_α and v_β . Thus the strong coloring of G can be completed by assigning μ to v . This completes the proof of Lemma 5.2.

The proof shows that we can indeed take $c = 4$.

7 Conclusions and further remarks

In this paper we have presented some basic facts about the strong coloring problem for graphs. We gave a summary of results for perfect colorings of regular graphs and for strongly coloring special classes of graphs. We proved a new upperbound, namely $3\Delta + 9$, for strongly coloring arbitrary planar graphs. For outerplanar graphs a much tighter bound for the strong coloring problem is proved, namely $\Delta + 4$. The bound for general planar graphs, while nontrivial, is not likely to be the best possible. At present the only known worst-case lowerbound is about $\frac{3}{2}\Delta + O(1)$.

Throughout the paper we have reported some results for the strong coloring problem for various classes of non-planar graphs as well. Here many interesting problems are left. For example, given a coloring algorithm A which gives a good approximate bound on the chromatic number of a graph G , does this algorithm give a good approximate bound for the strong chromatic number of G , when it is applied to the square graph G^2 ? What if G belong to a special class of graphs?

Another open question is the following. Is there an analog for strong chromatic numbers of the following theorem of Garey and Johnson [8] : "If for some constant $r < 2$ and constant d there exists a polynomial-time algorithm A which guarantees $A(G) \leq r\chi(G) + d$, then there exist a polynomial-time algorithm A which guarantees $A(G) = \chi(G)$."? The best performance ratio known for approximation algorithms for the chromatic number problem is $\frac{n \log \log n}{(\log n)^3}$ [5]. What is the corresponding best performance ratio for the strong chromatic number by applying this to the square graph G^2 ?

It would be interesting to investigate other relationships between the strong coloring problem and the well-studied coloring problem (see e.g. [12]), as well as relationships between the strong vertex coloring problem and the strong edge coloring problem.

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Appendix

Theorem *Given a graph G and an integer K , the problem of determining whether G can be strongly colored with $\leq K$ colors is NP-complete (STRONG CHROMATIC NUMBER).*

Proof: The problem trivially belongs to NP. (One can assign $\leq K$ colors to the nodes of G and verify in polynomial time whether it is a strong coloring.) For proving the NP-completeness, we reduce 3-SAT to STRONG CHROMATIC NUMBER. Let F be a CNF formula having r clauses, with at most three literals per clause. Let x_i ($1 \leq i \leq n$) be the variables in F . We may assume $n \geq 4$. We shall construct, in polynomial time, a graph G that is strongly colorable with $rn + 2n + 2$ colors iff F is satisfiable. The graph $G = (V, E)$ is defined by:

$$V = \{x_1, x_2, \dots, x_n\} \cup \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\} \cup \{y_1, y_2, \dots, y_{n+1}\} \cup \{p_{1,1}, \dots, p_{n,r}\} \\ \cup \{p_{n+1,r}\} \cup \{z_1, z_2, \dots, z_n\} \cup \{C_1, C_2, \dots, C_r\}$$

and

$$E = \{(y_i, y_j) | i \neq j\} \cup \{(z_i, z_j) | i \neq j\} \cup \{(z_i, x_i), 1 \leq i \leq n\} \cup \\ \{(p_{i,j}, p_{k,l}) | i \neq k \text{ or } j \neq l\} \cup \{(z_i, \bar{x}_i), 1 \leq i \leq n\} \cup \{(p_{n+1,r}, y_{n+1})\} \cup \\ \{(y_i, z_j) | 1 \leq i \leq n, i \neq j\} \cup \{(p_{i,j}, C_j), 1 \leq i \leq n, 1 \leq j \leq r\} \cup \\ \{(p_{i,j}, z_k), 1 \leq i, k \leq n, 1 \leq j \leq r\} \cup \{(x_i, p_{i,k}) | x_i \notin C_k\} \cup \{(\bar{x}_i, p_{i,k}) | \bar{x}_i \notin C_k\}$$

To see that G is $rn + 2n + 2$ colorable iff F is satisfiable, we first observe that the y_i 's form a complete subgraph on $n + 1$ vertices. Hence, each y_i must be assigned a distinct color. Without loss of generality we may assume that in any coloring of G y_i is given the color i for $1 \leq i \leq n + 1$. Then we observe that the z_i 's together form a complete subgraph on n vertices. Every z_i is at most at distance two from every y_i , hence the z_i must be colored differently from the y_i . Assume w.l.o.g. that z_i is given the color $n + i + 1$ for $1 \leq i \leq n$. We also observe that the $p_{i,j}$'s together form a complete subgraph on $rn + 1$ vertices. Every $p_{i,j}$ is at most at distance two from every y_k , and every $p_{i,j}$ is at most distance two from every z_k , so the colors of the $p_{i,j}$ must be different from the colors of the y_k and different from the colors of the z_k . Thus we can assume that $p_{i,j}$ is given the color $2n + in + j + 1$ and $p_{n+1,r}$ is given the color $rn + 2n + 2$. Since y_i lies within distance two from all the x_j 's and the \bar{x}_j 's, except x_i and \bar{x}_i , the color i can only be assigned to x_i or \bar{x}_i . x_i lies within distance two from \bar{x}_i , so one of these two vertices must have a different color. x_i and \bar{x}_i lie within distance two from every z_k and $p_{k,l}$ and every other $y_j, j \leq i, j \neq i$, so only color $n + 1$ is available for one of these two vertices, for every $i, 1 \leq i \leq n$, because no x_i or \bar{x}_i lies within distance two from any other x_j or \bar{x}_j . The vertex that is assigned to color $n + 1$ will be called the *false* vertex. The other is the *true* vertex. The only way to color G using $rn + 2n + 2$ colors, is to assign color $n + 1$ to one of $\{x_i, \bar{x}_i\}$ for each $i, 1 \leq i \leq n$.

Under what conditions can the remaining vertices be colored using no further colors? Since $n \geq 4$ and each clause has at most three literals, each C_i lies within distance two from a pair x_j, \bar{x}_j , for at least one j . Consequently no C_i may be assigned the color $n + 1$. Also every C_i lies within distance two from every $p_{k,l}$ and every z_j , so C_i must be assigned a color less than $n + 1$.

Also no C_i can be assigned a color corresponding to an x_j or an \bar{x}_j that does not occur in clause C_i . These observations imply that the only colors that can be assigned to C_i correspond to vertices x_j or \bar{x}_j that are in clause C_i and are *true* vertices.

Hence G is strongly $rn + 2n + 2$ colorable iff there is a true vertex corresponding to each C_i , and thus iff F is satisfiable. \square