Strong Colorings of Graphs

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RUU-CS-90-16
April 1990
Revised October 1990

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Abstract

We consider the generalization of graph coloring to distance-k coloring, also termed *strong coloring* in the case $k = 2$. Some basic facts about strong coloring of graphs are given, and several auxiliary results are presented for strong colorings of special classes of graphs. A survey is given of some recent results for strong colorings of planar and outerplanar graphs.

1 Introduction

The coloring problem for graphs has a longstanding mathematical interest. In this paper we consider the generalization to distance-k coloring for any $k \geq 1$, that is, we consider the problem of coloring a graph such that all vertices with distance $\leq k$ are colored differently. The distance-k coloring problem for graphs is NP-complete for every $k \geq 1$ [17]. For $k = 1$ one has the old definition of graph coloring, and for $k = 2$ the concept is also refered to as *strong* graph coloring [2, 4, 8]. Alternatively, a strong coloring can be defined as a coloring with the property that not only adjacent vertices have different colors (the usual "coloring condition") but also all neighbors of any vertex are colored differently (the "strong coloring condition").

The strong coloring problem for graphs has several applications. For example, in computing approximations to sparse Hessian matrices [17] the following typical problem arises: Given an $n \times n$ matrix M of 0's and 1's, one wishes to partition the columns of M into a number of sets such that no two columns in the same set have a 1 in the same row. This is equal to the strong coloring problem when we view M as the adjacency matrix of a graph. Another application occurs in the design of collision-free multi-hop channel access protocols in radio-networks [15], which can be solved using strong coloring. We also mention the application to the segmentation problem for files in a network [2]. Here the colors represent different (disjoint) segments of a file F, the graphs are regular with degree d and a strong coloring is

*This work was supported by the ESPRIT II Basic Research Actions program of the EC under contract No. 3075 (project ALCOM). This report is a revision of an earlier version.

desired with exactly $d+1$ colors. (This implies that every vertex a full copy of F can be assembled from the available segments in its direct neighborhood.) A last application we mention concerns the problem of obtaining drawings of graphs G in the plane in which the minimum angle formed by any pair of edges is maximized. After a strong coloring of the graph G with u colors is determined, a unit circle in the plane is drawn and u equidistant points $p_1, \ldots, p_u$ are marked on the circle. After placing the vertices of G that are assigned color i in a ball of radius ϵ around p_i $(1 \leq i \leq u)$ and drawing the edges as straight line segments, the minimum angle can be shown to have size $\frac{\pi - O(\epsilon)}{d^2+1}$ [7].

In this paper we present an overview of some results for the strong coloring problem for graphs as they seem to be known to date. We prove a number of basic facts and present some results when the problem is restricted to special classes of graphs. Some recent methods are shown to strongly color planar and outerplanar graphs. Several open questions are identified.

The paper is organized as follows. In section 2 we give some definitions concerning graphs and (strong) graph colorings. In section 3 we give some preliminary results for the strong coloring problem for certain classes of planar and non-planar graphs. In section 4 we give some facts for strongly coloring $(r-1)$-regular graphs with r colors, which we refer to as "perfect coloring". In section 5 some results for strongly coloring outerplanar and planar graphs are given. Section 6 contains some remarks and open questions. In an appendix we present a new proof of the NP-completeness of the strong coloring problem for graphs. We assume some familiarity with basic graph theory (cf. Harary [11]).

2 Definitions

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The distance between two vertices x and y is defined as the number of edges on the shortest path between x and y. The distance between two edges of G is defined as the shortest distance between an endpoint of one and of the other. Let $\Delta = max\{deg(v) | v \in V\}$, with $deg(v)$ the degree of vertex v. The square graph of G is the graph G^2 with $V(G^2) = V(G)$ and $E(G^2) = \{(u,v) | (u,v) \in E$ or $(u,x) \in E$ and $(x,v) \in E$ for some $x\}$. Observe that $\Delta(G^2) \leq (\Delta(G))^2$. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least $K \leq n$ such that G can be K-vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \ldots, K\}$ with $f(u) \neq f(v)$ whenever $(u,v) \in E$ [11]. The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the least $K \leq m$ such that G can be K-edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \ldots, K\}$ with $f((u,v)) \neq f((u,w))$ for all $u, v, w \in V$ and $(u,v), (u,w) \in E$ [11]. In any vertex (edge) coloring, every pair of vertices (edges) that have distance one must have different colors. Whenever vertex colorings are considered, $c(v)$ will denote the color given to a vertex v.

The generalization to distance-k coloring is now straightforward. The *k-chromatic* *number* of a graph G, denoted by $\chi_k(G)$, is the least $K \leq n$ such that G can be distance-k K-vertex-colored, i.e., such that there exists a function $f : V \to \{1, 2, \ldots, K\}$ with $f(u) \neq f(v)$ whenever u and v lie within distance k in G. The *k-chromatic index* of a graph G, denoted by $\chi'_k(G)$, is the least $K \leq m$ such that G can be distance-k K-edge-colored, i.e., such that there exists a function $f : E \to \{1, 2, \ldots, K\}$ with $f((u,v)) \neq f((w,x))$ whenever (u,v) and (w,x) lie within distance k from each other. For $k = 2$, we speak of the *strong chromatic number* and the *strong chromatic index* respectively. If a $(r-1)$-regular graph can be strongly colored with exactly r colors, then this coloring is called *perfect* [12].

McCormick [17] has proved that, given a graph and an integer K, the problem of deciding whether a graph can be distance-k vertex colored with K colors is NP-complete, for every $k \geq 1$. Another proof for the NP-completeness of the strong chromatic number problem can be found in the Appendix. Checking whether a graph can be strongly colored with $K \leq 3$ colors is trivial. If a graph has $\Delta \geq 4$ or contains a C_5 or a $K_{2,3}$, then the graph is not strongly 4-colorable. It is open whether the problem of deciding $\chi'_k(G) \leq K$ for graphs G and integers K is NP-complete. In this paper we will focus entirely on the strong coloring problem ($k = 2$). We will be refering to some special classes of graphs including planar graphs, outerplanar graphs, Halin graphs, chordal graphs and partial k-trees. We assume that the first three are known but include an inductive definition of partial k-trees.

Definition 2.1 ([1]) *The class of k-trees is the smallest class of graphs that satisfies the following rules:*

1. *the complete graph K_k on k vertices is a k-tree.*
2. *if $G = (V, E)$ is a k-tree and $v_1, \ldots, v_k$ form a complete subgraph of G, then the graph $G' = (V \cup \{w\}, E \cup \{(v_i, w) | 1 \leq i \leq k\})$ with $w \notin V$ is also a k-tree.*

A graph is a partial k-tree if and only if it is a subgraph of a k-tree.

3 Preliminaries

To obtain some first bounds for $\chi_2(G)$, consider the following (SL*) ordering of a graph, determined by the following algorithm, which is almost similar to the SL-algorithm of [16].

ALGORITHM SL*

Let n be the number of vertices in G.
Initialize H to G.
for $j = n$ **downto** 1 **do**
  begin

Choose a vertex v_j in H.
Let $v_{j_1}, \ldots, v_{j_k}$ be its neighbors.
Remove v_j and all edges incident to v_j from H.
Add zero or more edges to H to achieve that
$v_{j_1}, \ldots, v_{j_k}$ have distance ≤ 2 to each other.
end
SL$^* = v_1, v_2, \ldots, v_n$.

END OF ALGORITHM

Let SL$^* = v_1, v_2, \ldots, v_n$ be the ordering of G as computed by the algorithm. Let p be the maximum of the degrees of the vertices as they appear in the **for**-loop. Let H_j be the graph operated on by the algorithm when the loop-body is executed for j. Let Δ' be the maximum degree of any vertex in an H_j. Note that $\Delta' \geq \Delta$ as $H_n = G$.

Theorem 3.1 *With p and Δ' defined as above: $\Delta + 1 \leq \chi_2(G) \leq p\Delta' + 1$.*

Proof: The lowerbound is trivial. For the upperbound, we use induction to show that the graphs H_j and thus the vertices can be strongly colored in SL*-order with no more than $p\Delta' + 1$ colors. Let C be a set of $p\Delta' + 1$ colors. Vertex v_1 can be assigned an arbitrary color from C and a strong coloring of H_1 is obtained trivially. Assume we have colored the vertices $v_1, \ldots, v_{i-1}$ (following the ordering) using colors from C such that a strong coloring of H_{i-1} is implied, for some $i \geq 2$. Now consider H_i. Assume that all vertices except v_i are colored as in H_{i-1}. v_i is connected to at most p colored neighbors and since all these neighbors have distance ≤ 2 to each other in H_{i-1}, they are colored differently. Now v_i has at most $p\Delta'$ colored vertices within distance 2, hence at most $p\Delta'$ colors from C are blocked for it. Hence v_i can be colored with a color from C to obtain a strong coloring of H_i. This completes the induction. □

It follows from theorem 3.1 that, in order to obtain strong colorings with a "small number of colors using the SL*-algorithm, an SL*-ordering must be found that gives both a small value of p and a value of Δ' that remains close to Δ.

A possible algorithm for adding additional edges between the vertices $v_{j_1}, \ldots, v_{j_k}$ in each step of the for-loop in the SL*-algorithm is the following. Assign to every vertex v_{j_i} a label (x, y), with $0 \leq x, y < \lceil\sqrt{k}\rceil$. Add an edge between two vertices with labels (x_1, y_1) and (x_2, y_2), if and only if $x_1 = x_2$ or $y_1 = y_2$. Between every two vertices, labelled (a, b) and (c, d), there is a path of length at most two via vertex (a, d) or vertex (b, c). Note that this construction increases the degree of every vertex v_{j_i} by at most $2\lceil\sqrt{k}\rceil$.

Corollary 3.2 *k-trees can be strongly colored with at most $k\Delta + 1$ colors.*

Proof: From definition 2.1 it follows that an ordering of a k-tree is obtained by removing suitable vertices of degree k in each step of the SL*-algorithm. Note that these vertices can be chosen so their neighbors are a clique in the remaining graph (a k-tree again), hence no edges need to be added. Applying theorem 3.1, the corollary follows. □

Corollary 3.3 *Every outerplanar graph can be strongly colored with at most $2\Delta + 1$ colors.*

Proof: Every outerplanar graph G has a vertex v with degree at most 2. Deleting v with its incident edges preserves the outerplanarity. Thus the SL*-algorithm can chose v_j to be a vertex of degree ≤ 2 in every iteration of the loop. Observe that if v_i had degree 2 and its neighbors where not adjacent, then we can add an edge between them in the last step of the iteration (without increasing any degrees). Applying theorem 3.1 the result follows. □

Similar bounds can be given for distance-k vertex and edge colorings. Observe that theorem 3.1 also implies that $\chi'(G) \leq \chi_2(G)$, as $\chi'(G) \leq \Delta + 1$ by Vizing's theorem. The question whether $\chi_k(G) \leq \chi'_k(G)$ remains as an interesting open problem. For trees it is clear that $\chi_2(G) = \Delta + 1$ and $\chi'_2(G) = max\{deg(u) + deg(father(u)) | u \in V\}$. Every Halin graph can be strongly colored with at most $\Delta + 6$ colors. (For the latter result one uses that every tree can be strongly colored using at most $\Delta + 1$ colors and every circuit with at most 5 colors.)

Observe that from a strong vertex coloring with $\chi_2(G)$ colors one can obtain a strong edge coloring with $(\chi_2(G))^2$ colors, by assigning to every edge (u, v) the color $[c(u), c(v)]$, where the colors of the vertices are taken from the strong vertex coloring. Notice that this large difference between the strong chromatic number and the strong chromatic index actually occurs in the case of the complete bipartite graphs $G = K_{n,n}$, where $\chi_2(G) = 2n$ and $\chi'_2(G) = n^2$.

There appears to be no simple connection between $\chi(G)$ and $\chi_2(G)$. A reasonable conjecture like $\chi_2(G) \leq (\Delta + 1)\chi(G) + 1$ fails, by observing the coloring of the following bipartite graph $G_p = (< V_1, V_2 >, E)$ for any $p \geq 2$. V_1 consists of p vertices $0, \ldots, p-1$. V_2 consists of $p(p-1)$ vertices $\{i, j\}, 0 \leq i < p, 0 \leq j < p - 1$, and a vertex A. Let $i \in V_1$ be connected to the vertices $\{i, x\}$ for $0 \leq x < p - 1$ and to vertex A. Add $p(p-1)$ vertices $[k, l]$ to V_1, with $0 \leq k, l < p - 1$, and the edges $([i, j], \{0, i\})$ and $([i, j], \{k, ((k-1)i + j) \bmod (p-1)\})$, with $1 \leq k < p$ and $0 \leq i, j < p - 1$. Note that G_p is p-regular bipartite and that every two vertices in V_2 have distance 2 to each other. Hence this graph has $\Delta = p, \chi(G) = 2$ and $\chi_2(G) = \Delta(\Delta - 1) + 1$. It shows that $\chi(G)$ and $\chi_2(G)$ can differ dramatically. In figure 1, an example is given for $p = 3$.

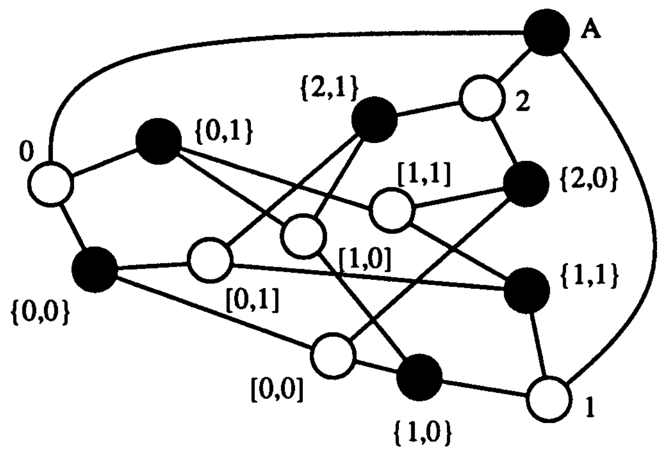

Figure 1: Example of a G_p for $p = 3$.

From the observation that $\chi_k(G) = \chi(G^k)$ and $\chi'_k(G) = \chi'(G^k)$ (using the definition of the k^{th} power graph G^k of [11]) for every $k \geq 2$, it follows that we can use the available algorithms for ordinary graph coloring for obtaining strong colorings, after calculating G^2 in $O(\Delta m)$ time. Also all lowerbounds for the chromatic number trivially hold for the strong chromatic number. This also leads to the following observation:

Theorem 3.4 *Chordal graphs can be strongly colored in a smallest possible number of colors in polynomial time. For every chordal graph G one has $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$.*

Proof: Note that the square graph of a chordal graph is a chordal graph too, and can be colored in an optimal number of colors in polynomial time (cf. Golumbic [10]).

For proving $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$ we induct on n, the size of G. For $n \leq 3$ the result trivially holds. Consider an arbitrary chordal graph G of n vertices, $n > 3$. Without loss of generality we may assume that G is connected. If G is a clique, then it can be strongly colored with $n = \Delta + 1$ colors and $\Delta + 1 \leq (\frac{1}{2}\Delta + 1)^2$. Thus let G not be a clique, S a minimal vertex separator of G and $A_1, A_2, \ldots, A_l$ the connected components of $G - S$. Let H_1 be the induced subgraph spanned by S and A_1, and let H_2 be the induced subgraph spanned by S and $A_2, \ldots, A_l$. By well-known facts for chordal graphs [10] S is a clique, and H_1 and H_2 are connected chordal graphs. Let $|S| = s$. By induction H_1 and H_2 are strongly colorable with the colors of some set C of $(\frac{1}{2}\Delta + 1)^2$ colors. A strong coloring of G can now be obtained as follows.

Permute the colors such that in the strong colorings of H_1 and H_2, the vertices of S get the same colors. (This can be done because the colors assigned to the vertices of S must all be different, by the strong coloring requirement, in both the coloring of H_1 and the coloring of H_2.) Let N_1 be the set of vertices in A_1 that are reached by an edge from S, and N_2 the set of vertices in $A_2 \cup \ldots \cup A_l$ defined similarly. Let $|N_1| = n_1$ and $|N_2| = n_2$, and observe that $n_1 + n_2 \leq s(\Delta - s + 1) \leq (\frac{1}{2}\Delta + 1)^2 - s$. Thus we have sufficiently many colors in C to arrange that s colors are fixed for the vertices in S, and the remaining colors can be permuted such that in the strong colorings of H_1 and H_2 the vertices in N_1 and N_2 are colored by disjoint sets of colors. The resulting strong colorings of H_1 and H_2 can now be combined (merged) to a correct strong coloring of G which employs no more than $(\frac{1}{2}\Delta + 1)^2$ colors. □

We conjecture that $\chi(G^2) \leq Q + 1$, with Q the number of vertices of the largest clique in the graph G^2. If G^2 is a linegraph, then this conjecture is true by noting that if the linegraph G^2 has a largest clique of size Q, then the linegraph of this linegraph has maximal degree Q and can be edge-colored with $Q + 1$ colors. Hence the linegraph G^2 can be vertex-colored with $Q + 1$ colors.

4 Perfect Colorings

A $(r - 1)$-regular graph is called perfectly colorable if it can be strongly colored with exactly r colors. This kind of strong coloring is useful for the following file distribution problem [2]: "Given a connected regular network $G = (V, E)$ and a file F, assign to each vertex $x \in V$ a segment $F_x \subseteq F$ such that for all $x \in V, \bigcup_{\{x,y\} \in E} F_y \cup F_x = F$ and in every neighborhood the distributed fragments are free of overlaps, i.e., $\forall (x, y) \in E : F_x \cap F_y = \emptyset$." When the network is $(r-1)$-regular, this problem solves the file distribution question with the smallest possible number of different disjoint segments of F. A perfect coloring describes the assignment of the segments for a valid solution of the file distribution problem. It has been shown by Bakker et al. [3] that this problem is NP-complete, even for the case $r = 4$.

In this section we give some relationships between strong colorings, perfect colorings and edge colorings.

Lemma 4.1 ([2]) *If a $(r - 1)$-regular graph with $|V| = n$ vertices can be perfectly colored, then $r|n$ and every group of equally colored vertices has $\frac{n}{r}$ vertices.*

Proof: Let $N(x)$ denote the set of vertices having distance ≤ 1 to vertex x. Consider any perfect coloring of the graph, and let c be one of the colors. Let x_1 and x_2 be two vertices colored c. There can be no vertex y in $N(x_1) \cap N(x_2)$ because, if there was, y would have two neighbors of the same color (which contradicts the strong coloring property). Thus for all x_1, x_2 with $x_1 \neq x_2$ and $c(x_1) = c(x_2) = c$: $N(x_1) \cap N(x_2) = \emptyset$. Furthermore for every vertex y there is a vertex x with $c(x) = c$ and $y \in N(x)$. Hence the neighborhoods $N(x)$ of vertices x with $c(x) = c$ form a partitioning of G. But for all $x \in V : |N(x)| = r$. Hence $r|n$ and every group of equally colored vertices has size $\frac{n}{r}$. □

Theorem 4.2 *Every strongly r-colorable graph is the induced subgraph of a perfectly colorable $(r-1)$-regular graph.*

Proof: We induct on r. For $r = 1, 2$ and 3, the theorem is trivial. Thus let $r \geq 4$ and G be a strongly r-colorable graph. Consider a strong coloring of G with the colors $c_1, \ldots, c_r$ and let H be the induced subgraph of G consisting of all vertices with a color $\in \{c_1, \ldots, c_{r-1}\}$. By induction H is an induced subgraph of some $(r-2)$-regular graph R_H that is perfectly colorable, and w.l.o.g. we can assume that it is perfectly colored with $c_1, \ldots, c_{r-1}$. Arrange the vertices of R_H into $(r-1)$ disjoint blocks $B_1, \ldots, B_{r-1}$, with B_i $(1 \leq i \leq r-1)$ containing the vertices of color c_i, and let every block contain b vertices. (By lemma 4.1 we know that the blocks must be of equal size.) Tag the vertices of R_H that correspond to the vertices of H. Let the vertices $x_1, \ldots, x_s$ (some $s \geq 1$) of $G - H$ together form the "beginning" of the r^{th} block B_r. The vertices $\{x_1, \ldots, x_s\}$ form an independent set in G (because they all have color c_r).

Now form the graph R_G as follows. Make $\lceil \frac{s}{b} \rceil$ copies of R_H and extend B_r by another $\lceil \frac{s}{b} \rceil b - s$ vertices y. We now "connect" the x- and y-vertices to the vertices in the R_H copies in two steps, as follows:

1. **for** i **from** 1 **to** s **do**
   begin
   connect x_i to a new vertex from a B_j-block for every $j, 1 \leq j \leq r-1$,
   always favoring the tagged vertex z in a block B_j if x_i is
   directly connected to z in G, but taking an untagged vertex otherwise.
   end;

Observe that we have $\lceil \frac{s}{b} \rceil b \geq s$ vertices of every color, so step 1 always works and does not "run out of vertices to connect to". But also observe that we have exactly $\lceil \frac{s}{b} \rceil b - s$ vertices left of every color after this step.

2. **for** i **from** 1 **to** $\lceil \frac{s}{b} \rceil b - s$ **do**
   begin
   pick a new y-vertex and connect y to a vertex from a B_j-block that
   was not yet connected to, for every $j, 1 \leq j \leq r-1$.
   (Note that these vertices were not tagged.)
   end;

Note that step 2 makes the graph G_H $(r-1)$-regular. The result is a graph R_G that is $(r-1)$-regular, perfectly colorable with r colors and clearly, by design, we have that G is an induced subgraph of R_G. □

Another property is the following. Recall that by Vizing's theorem every graph is edge-colorable with Δ or $\Delta + 1$ colors.

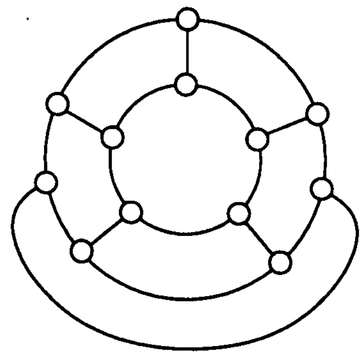

Figure 2: A 3-edge-colorable 3-regular graph that is not perfectly colorable ([13]).

Lemma 4.3 *If a graph is strongly colorable with r colors and r is even, then it is $(r-1)$-edge colorable.*

Proof: Let G be strongly colorable with r colors. If r is even, then K_r is edge-colorable with $(r-1)$ colors. Let G be strongly colored with the r names of vertices of K_r. Now G can be edge-colored as follows: color an edge from the vertex colored X to the vertex colored Y with z if the edge between X and Y in K_r is colored z. This gives a correct $(r-1)$-edge coloring of G. □

The converse is not true, see for example figure 2. Also this theorem does not hold for r odd in general, as an edge-coloring of a K_r (which is perfectly colorable with r colors) requires r colors when r is odd.

Also the spectra of perfectly colorable graphs have some interesting properties. Because a perfectly r-colorable graph G is $(r-1)$-regular, its largest eigenvalue is equal to $r-1$ and has multiplicity 1. The following more specific observation can be made as well.

Theorem 4.4 *Let G be perfectly r-colorable. Then G has an eigenvalue -1, with multiplicity $\geq (r-1)$.*

Proof: Let G be perfectly r-colored, and consider the vertices of G arranged in blocks of equally colored vertices (of size $\frac{n}{r}$ each). Let $A = A(G)$ be the adjacency matrix of G corresponding to this vertex-ordering. The symmetrix matrix A can be viewed as a block matrix, with the blocks along the main diagonal consisting of all zeroes and the off-diagonal blocks being $\frac{n}{r} \times \frac{n}{r}$ permutation matrices. (As an aside we note that, conversely, if the vertices of a graph G can be arranged so the adjacency matrix is of this form, then G is perfectly r-colorable.) Now consider the $r \times r$ matrix A' obtained from A by replacing every block on the main diagonal by a "0" and every off-diagonal block by a "1". A' is the adjacency matrix of the K_r, whose spectrum consists of one eigenvalue $(r-1)$ and $(r-1)$ eigenvalues -1 ([6]). Also, when $(x_1, \ldots, x_r)$ is an eigenvector of A', then the vector obtained by repeating each coordinate $\frac{n}{r}$-fold is an eigenvector of A and independency of eigenvectors is preserved in the process. It follows in particular that A (and hence, G) has an eigenvalue -1 with multiplicity at least $r-1$. □

From the same argument some more information can be derived. Let $n > r$ and let $\lambda_1, \ldots, \lambda_k$ and $-\mu_1, \ldots, -\mu_l$ be the remaining positive and negative eigenvalues in the spectrum of G in decreasing order different from the r eigenvalues $(r-1)$ and -1 that we have, with $k + l = n - r$. As the trace of A is zero, we have $\lambda_1 + \ldots + \lambda_k = \mu_1 + \ldots + \mu_l$. Observe also that A^2 is a symmetric matrix with all entries along the main diagonal equal to $r-1$. It follows that $\lambda_1^2 + \ldots + \lambda_k^2 + \mu_1^2 + \ldots + \mu_l^2 = \text{tr}(A^2) - (r-1)^2 - (r-1) = (n-r)(r-1)$. Now let $\lambda = \lambda_1 = \lambda_{\max}, \mu = \mu_l = \mu_{\max}$ and $\delta = max\{\lambda, \mu\}$. One easily verifies that $\delta \geq \sqrt{r-1}$ and $min\{\lambda, \mu\} \geq \frac{1}{n-r}\sqrt{r-1}$.

Another characteristic of perfectly colorable graphs is the following:

Theorem 4.5 *Let G be regular of degree ≥ 3 and perfectly colorable. Then one can partition V as $V_1 \cup V_2$ such that*

1. *the induced subgraph G_1 on V_1 is a set of chordless cycles of length divisible by 3.*

2. *the induced subgraph G_2 on V_2 is regular of degree $\Delta - 3$ and perfectly colorable.*

Proof: Let a, b, c be three colors of the perfect coloring of G. Let V_1 be the set of vertices colored a, b or c and $V_2 = V - V_1$.

Consider any vertex in V_1, say with color a. It has one neighbor colored b, this neighbor has one neighbor colored c, etc. This necessarily closes itself as a cycle at the point of departure. By the strong coloring property, this cycle must be chordless. This proves the statement, and the cycles are not connected to each other.

Consider any vertex in V_2. It has *exactly* three neighbors in V_1. Thus G_2 inherits a perfect coloring of G with the remaining $\Delta - 3$ colors. □

This theorem shows that perfectly colorable graphs decompose entirely into (disjoint) chordless cycles. Note that $\frac{|V_2|}{|V_1|} = \frac{\Delta-2}{3}$, for $\Delta \geq 3$.

For the file distribution problem perfect colorings are interesting mostly for regular networks, which includes many current processor networks. In [12] a detailed study is given of perfectly colorable processor networks. For completeness we summarize the results in the following theorem.

Theorem 4.6 ([12]) *The following processor networks are perfectly colorable:*

- *The hypercube C_n, if and only if $n = 2^i - 1$ for some $i > 0$.*

- *The d-dimensional torus of size* $l_1 \times \ldots \times l_d$ *if* $l_i \bmod q = 0$*, with* q *such that* $\sqrt[r]{2d+1} | q$ *for some integer* $r > 0$.

- *The Cube-connected Cycles* CCC_d*, if and only if* $d > 2, d \neq 5$.

- *The directed shuffle-exchange network and the directed 4-pin shuffle network.*

- *The chordal ring network with chordlength* $4p - 1$ $(p > 0)$ *and* $4kp - 4t$ $(0 \leq t < p)$ *vertices if and only if :*

  1. k *and* t *are even and (if* $t > 0$*)* $\frac{t}{\gcd(t,p)}$ *is even, or*
  2. k, $\frac{t}{\gcd(t,p)}$ *and* $\frac{p}{\gcd(t,p)}$ *are odd and* $t + p$ *is even.*

- *The hexagonal network of size* $m \times n$ *if and only if* $m, n \bmod 7 = 0$.

The reader is referred to [12] for the definition of the various networks.

5 Outerplanar and Planar Graphs

In this section we consider the strong coloring problem for outerplanar and planar graphs, respectively. By the results in section 3 we know that every outerplanar graph can be strongly colored with $2\Delta + 1$ colors. Our aim will be to improve this to a bound of $\Delta + 3$ colors (which, in turn, improves on a bound of $\Delta + 4$ colors in a precursor of this report). For this we need the following theorem.

Theorem 5.1 *A graph can be strongly colored with at most* k *colors if and only if all biconnected components of it can be strongly colored with at most* k *colors* $(k \geq \Delta + 1)$.

Proof: The "only if" part is trivial. We proceed to show the "if" part. Let G be a graph. (Without loss of generality we can confine ourselves to connected graphs.) Let all biconnected components of G be strongly colorable with at most k colors. We now show that G is strongly k-colorable. When G has no cutvertices, the theorem trivially holds. Thus assume that the theorem holds for all connected graphs with $\leq p - 1$ cutvertices, and let G have p cutvertices. Let v be a cutvertex of G, then G consists of two connected graphs H_1 and H_2 such that each contain a "copy" of the vertex v and are joined at v, but which are otherwise disjoint. W.l.o.g. we may assume that both H_1 and H_2 have $\leq p - 1$ cutvertices.

Let v have degree Δ_1 in H_1 and degree Δ_2 in H_2, where we can assume w.l.o.g. that $\Delta_1 \leq \Delta_2$ and clearly $\Delta_1 + \Delta_2 \leq \Delta$. We can assume inductively that H_1 and H_2 can be strongly colored using at most k colors. Shift color-names such that H_1 and H_2 use colors from the same set of k colors and v gets the same color "α" in H_1 and H_2. Joining H_1 and H_2 at v (while retaining the colorings of H_1 and H_2 respectively) results in a strong coloring of G with k colors, except in the one case that some neighbors of v in H_1 have the same color as some neighbors of v in H_2. We now argue how such a conflict can be removed by a permutation of the colors, if it arises.

Thus assume that the latter case arises. Note that v and its neighbors in H_2 use $\Delta_2 + 1$ colors. Let r neighbors of v in H_1 use colors different from these but l neighbors use colors $c_1, \ldots, c_l$ that are among the colors used by the Δ_2 neighbors in H_2, for certain r and l with $r+l = \Delta_1$. It means that Δ_2+1+r different colors are used in the neighborhood of v. Choose l different colors $d_1, \ldots, d_l$ from among the remaining colors. (This can be done because $k-(\Delta_2+1+k) \geq \Delta_1+\Delta_2+1-(\Delta_2+1+r) = l$.) Exchanging c_i and d_i (for i from 1 to l) in the coloring of H_1 throughout leaves a strong coloring in H_1 and removes the color conflicts at v, thus leading to a correct strong coloring of G using at most k colors. This completes the inductive argument. □

Forman et al. [7] prove the following lemma.

Lemma 5.2 *Every biconnected outerplanar graph contains a vertex of degree 2 with a neighbor of degree 2 or with adjacent neighbors, one of which is of degree at most 4.*

Using theorem 5.1 and lemma 5.2, the following theorem of [7] can be obtained.

Theorem 5.3 *Every outerplanar graph of maximum degree Δ can be strongly colored using at most $\Delta + 3$ colors.*

Proof: Let $U \subseteq V$ be the set of vertices of degree 2 with at least one neighbor of degree 2. If $U \neq \emptyset$, remove all vertices of U and strongly color the remaining outerplanar graph inductively. A strong coloring of the original graph can then be obtained by re-inserting the vertices of U and assigning a suitable color to them one after the other. Since there are at most $\Delta + 2$ vertices at distance ≤ 2 from any vertex v in U, at most $\Delta + 2$ colors are blocked for v and we can indeed complete the strong coloring within $\Delta + 3$ colors.

If $U = \emptyset$, then by lemma 5.2 there must exist a vertex v of degree 2 with adjacent neighbors, one of which has degree ≤ 4. Remove vertex and strongly color the remaining outerplanar graph inductively. Since there are at most $\Delta + 2$ vertices at distance ≤ 2 from v, the same argument can be applied to obtain a strong coloring of G with $\Delta + 3$ colors total. □

The lowerbound for strongly coloring outerplanar graphs is still open, though it is not difficult to construct outerplanar graphs with degree ≤ 6 that need $\Delta + 3$ colors.

For planar graphs, the problem is to strongly color them with at most $c.\Delta + O(1)$ colors for as small a constant c as possible. A first result is the following lowerbound for the strong chromatic number.

Theorem 5.4 *For every $\Delta \geq 1$ there exists a planar graph G with $\chi_2(G) \geq \lceil \frac{3}{2}\Delta \rceil$.*

Proof: We can assume w.l.o.g. that $\Delta > 1$. (For $\Delta = 1$ the theorem trivially holds by taking a graph that consists of a single edge. Choose $r, s \geq 0$ with $s \leq r$ such that $\Delta = r + s + 2$. It will be useful to take $r = s = \frac{1}{2}\Delta - 1$ when Δ is even and $s = r - 1 = \frac{1}{2}\Delta - \frac{3}{2}$ when Δ is odd. Construct the graph G_Δ consisting of a "triangle" of three vertices (A, B and C), r vertices that are each connected to A and to B, s vertices that are each connected to B and to C, and s more vertices that are each connected to A and to C. For Δ odd (implying $\Delta \geq 3$), a separate vertex D is inserted on the triangle-edge (A, B). This vertex is also connected to C. One easily verifies that G_Δ is planar, has maximum degree Δ and diameter 2. Because of the latter any strong coloring of G_Δ needs as many colors as there are vertices, which is precisely $\lceil \frac{3}{2}\Delta \rceil$. (By a result of Seyffart [18] this is about the largest possible number of vertices in any planar graph of diameter 2 and maximum degree Δ.) □

The lemma shows that $c \geq \frac{3}{2}$ for general planar graphs. For $\Delta \leq 5$ one can construct planar graphs that need $\geq 2\Delta$ colors in any strong coloring. For obtaining an upperbound for the strong chromatic number of planar graphs, the following lemma of [7] is useful.

Lemma 5.5 *Let T and U be disjoint sets of vertices in a planar graph and suppose that each vertex in T has at least 3 neighbors in U. Then $|T| \leq 2|U| - 4$.*

Using lemma 5.5, one can easily prove:

Lemma 5.6 *Every planar graph contains either a vertex of degree ≤ 2 or a vertex of degree ≤ 5 with at most two neighbors of degree ≤ 12.*

Proof: Assume by way of contradiction that we are given a graph violating the lemma. For $j = 0, 1, \ldots,$ denote by n_j the number of vertices of degree exactly j. Since in any planar graph the total number of degrees is at most 6 times the number of vertices,

$$\sum_{j=3}^{\infty} j n_j \leq 6 \sum_{j=3}^{\infty} n_j,$$

from which it follows that

$$3 \sum_{j=3}^{5} n_j \geq \sum_{j=3}^{5} (6 - j) n_j \geq \sum_{j=6}^{\infty} (j - 6) n_j \geq 6 \sum_{j=12}^{\infty} n_j.$$

Taking T as the set of vertices of degree ≤ 5 and U as the set of vertices of degree ≥ 12, this implies a contradiction to lemma 5.5. □

From lemma 5.6, the following theorem can be obtained.

Theorem 5.7 *Every planar graph of maximum degree Δ can be strongly colored using at most $2\Delta + O(1)$ colors.*

Proof: We prove by induction on the number of vertices that every planar graph can be strongly colored with at most $2.\max\{\Delta, 14\} + 34$ colors. Consider any planar graph G and apply lemma 5.6. If there is vertex of degree ≤ 2, choose such a vertex v and remove it. If v is of degree 2 and its neighbors are not adjacent, introduce an edge between them. If there is no vertex of degree ≤ 2, choose a vertex v of degree ≤ 5 with at most two neighbors of degree > 11 and contract v into one of its neighbors of degree ≤ 11. The resulting graph is planar and has maximum degree at most $\max\{\Delta, 14\}$.

In either case, the inductive hypothesis implies that the resulting graph can be strongly colored using at most $2.\max\{\Delta, 14\} + 34$ colors, and a strong coloring of the original graph can be obtained by re-inserting v and coloring it differently from all vertices at distance ≤ 2 from v. There are at most $2.\max\{\Delta, 11\} + 33$ such vertices, thus there is a free color for it. $\square$

Forman et al. [7] have recently improved the bound of theorem 5.7 to $\frac{13}{7}\Delta + O(\Delta^{\frac{2}{3}})$. This seems to be the best current bound for strongly coloring planar graphs.

6 Conclusions and further remarks

In this paper we have presented a survey of some basic facts for the strong coloring problem for graphs. Some results for strong coloring of various special classes of graphs like planar and outerplanar graphs were reviewed also. Several open questions were identified along the way.

There are many interesting further problems left. For example, given a coloring algorithm A which gives a good approximate bound on the chromatic number of a graph G, does this algorithm give a good approximate bound for the strong chromatic number of G, when it is applied to the square graph G^2? What if G belong to a special class of graphs?

Another open question is the following. Is there an analog for strong chromatic numbers of the following theorem of Garey and Johson [9] : "If for some constant $r < 2$ and constant d there exists a polynomial-time algorithm A which guarantees $A(G) \leq r\chi(G) + d$, then there exist a polynomial-time algorithm A which guarantees $A(G) = \chi(G)$."? The best performance ratio known for approximation algorithms for the chromatic number problem is $\frac{n \log\log n}{(\log n)^3}$ [5]. What is the corresponding best performance ratio for the strong chromatic number by applying this to the square graph G^2?

It would be interesting to investigate other relationships between the strong coloring problem and the well–studied coloring problem (see e.g. [13]), as well as relationships between the strong vertex coloring problem and the strong edge coloring problem.

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Appendix

Theorem *Given a graph G and an integer K, the problem of determining whether G can be strongly colored with $\leq K$ colors is NP-complete (STRONG CHROMATIC NUMBER).*

Proof: The problem trivially belongs to NP. (One can assign $\leq K$ colors to the vertices of G and verify in polynomial time whetter it is a strong coloring.) For proving the NP-completeness, we reduce 3-SAT to STRONG CHROMATIC NUMBER. Let F be a CNF formula having r clauses, with at most three literals per clause. Let x_i $(1 \leq i \leq n)$ be the variables in F. We may assume $n \geq 4$. We shall construct, in polynomial time, a graph G that is strongly colorable with $rn + 2n + 2$ colors iff F is satisfiable. The graph $G = (V, E)$ is defined by:

$$\begin{aligned} V = \; & \{x_i, x_2, \ldots, x_n\} \cup \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\} \cup \{y_1, y_2, \ldots, y_{n+1}\} \cup \{p_{1,1}, \ldots, p_{n,r}\} \\ & \cup \{p_{n+1,r}\} \cup \{z_1, z_2, \ldots, z_n\} \cup \{C_1, C_2, \ldots, C_r\} \end{aligned}$$

and

$$\begin{aligned} E = \; & \{(y_i, y_j) | i \neq j\} \cup \{(z_i, z_j) | i \neq j\} \cup \{(z_i, x_i), 1 \leq i \leq n\} \cup \\ & \{(p_{i,j}, p_{k,l}) | i \neq k \text{ or } j \neq l\} \cup \{(z_i, \overline{x}_i), 1 \leq i \leq n\} \cup \{(p_{n+1,r}, y_{n+1})\} \cup \\ & \{(y_i, z_j) | 1 \leq i \leq n, i \neq j\} \cup \{(p_{i,j}, C_j), 1 \leq i \leq n, 1 \leq j \leq r\} \cup \\ & \{(p_{i,j}, z_k), 1 \leq i, k \leq n, 1 \leq j \leq r\} \cup \{(x_i, p_{i,k}) | x_i \notin C_k\} \cup \{(\overline{x}_i, p_{i,k}) | \overline{x}_i \notin C_k\} \end{aligned}$$

To see that G is $rn+2n+2$ colorable iff F is satisfiable, we first observe that the y_i's form a complete subgraph on $n+1$ vertices. Hence, each y_i must be asssigned a distinct color. Without loss of generality we may assume that in any coloring of G y_i is given the color i for $1 \leq i \geq n+1$. Then we observe that the z_i's together form a complete subgraph on n vertices. Every z_i is at most at distance two from every y_i, hence the z_i must be colored differently from the y_i. Assume w.l.o.g. that z_i is given the color $n+i+1$ for $1 \leq i \geq n$. We also observe that the $p_{i,j}$'s together form a complete subgraph on $rn+1$ vertices. Every $p_{i,j}$ is at most at distance two from every y_k, and every $p_{i,j}$ is at most distance two from every z_k, so the colors of the $p_{i,j}$ must be different from the colors of the y_k and different from the colors of the z_l. Thus we can assume that $p_{i,j}$ is given the color $2n+in+j+1$ and $p_{n+1,r}$ is given the color $rn+2n+2$. Since y_i lies within distance two from all the x_j's and the $\overline{x}_j$'s, except x_i and $\overline{x}_i$, the color i can only be assigned to x_i or $\overline{x}_i$. x_i lies within distance two from $\overline{x}_i$, so one of these two vertices must have a different color. x_i and $\overline{x}_i$ lie within distance two from every z_k and $p_{k,l}$ and every other $y_j, j \leq i, j \neq i$, so only color $n+1$ is available for one of these two vertices, for every $i, 1 \leq i \leq n$, because no x_i or $\overline{x}_i$ lies within distance two from any other x_j or $\overline{x}_j$. The vertex that is assigned to color $n+1$ will be called the *false* vertex. The other is the *true* vertex. The only way to color G using $rn+2n+2$ colors, is to assign color $n+1$ to one of $\{x_i, \overline{x}_i\}$ for each i, $1 \leq i \leq n$.

Under what conditions can the remaining vertices be colored using no further colors? Since $n \geq 4$ and each clause has at most three literals, each C_i lies within distance two from a pair $x_j, \overline{x}_j$, for at least one j. Consequently no C_i may be assigned the color $n+1$. Also every C_i lies within distance two from every $p_{k,l}$ and every z_j, so C_i must be assigned a color less than $n+1$.

Also no C_i can be assigned a color corresponding to an x_j or an $\overline{x}_j$ that does not occur in clause C_i. These observations imply that the only colors that can be assigned to C_i correspond to vertices x_j or $\overline{x}_j$ that are in clause C_i and are *true* vertices.

Hence G is strongly $rn+2n+2$ colorable iff there is a true vertex corresponding to each C_i, and thus iff F is satisfiable. □

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RUU-CS-90-16
April 1990
Revised October 1990

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Strong Colorings of Graphs*

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Abstract

We consider the generalization of graph coloring to distance-k coloring, also termed *strong coloring* in the case $k = 2$. Some basic facts about strong coloring of graphs are given, and several auxiliary results are presented for strong colorings of special classes of graphs. A survey is given of some recent results for strong colorings of planar and outerplanar graphs.

1 Introduction

The coloring problem for graphs has a longstanding mathematical interest. In this paper we consider the generalization to distance-k coloring for any $k \geq 1$, that is, we consider the problem of coloring a graph such that all vertices with distance $\leq k$ are colored differently. The distance-k coloring problem for graphs is NP-complete for every $k \geq 1$ [17]. For $k = 1$ one has the old definition of graph coloring, and for $k = 2$ the concept is also refered to as *strong* graph coloring [2, 4, 8]. Alternatively, a strong coloring can be defined as a coloring with the property that not only adjacent vertices have different colors (the usual "coloring condition") but also all neighbors of any vertex are colored differently (the "strong coloring condition").

The strong coloring problem for graphs has several applications. For example, in computing approximations to sparse Hessian matrices [17] the following typical problem arises: Given an $n \times n$ matrix M of 0's and 1's, one wishes to partition the columns of M into a number of sets such that no two columns in the same set have a 1 in the same row. This is equal to the strong coloring problem when we view M as the adjacency matrix of a graph. Another application occurs in the design of collision-free multi-hop channel access protocols in radio-networks [15], which can be solved using strong coloring. We also mention the application to the segmentation problem for files in a network [2]. Here the colors represent different (disjoint) segments of a file F, the graphs are regular with degree d and a strong coloring is

*This work was supported by the ESPRIT II Basic Research Actions program of the EC under contract No. 3075 (project ALCOM). This report is a revision of an earlier version.

desired with exactly $d+1$ colors. (This implies that every vertex a full copy of F can be assembled from the available segments in its direct neighborhood.) A last application we mention concerns the problem of obtaining drawings of graphs G in the plane in which the minimum angle formed by any pair of edges is maximized. After a strong coloring of the graph G with u colors is determined, a unit circle in the plane is drawn and u equidistant points $p_1, \ldots, p_u$ are marked on the circle. After placing the vertices of G that are assigned color i in a ball of radius ϵ around p_i $(1 \leq i \leq u)$ and drawing the edges as straight line segments, the minimum angle can be shown to have size $\frac{\pi - O(\epsilon)}{d^2+1}$ [7].

In this paper we present an overview of some results for the strong coloring problem for graphs as they seem to be known to date. We prove a number of basic facts and present some results when the problem is restricted to special classes of graphs. Some recent methods are shown to strongly color planar and outerplanar graphs. Several open questions are identified.

The paper is organized as follows. In section 2 we give some definitions concerning graphs and (strong) graph colorings. In section 3 we give some preliminary results for the strong coloring problem for certain classes of planar and non-planar graphs. In section 4 we give some facts for strongly coloring $(r-1)$-regular graphs with r colors, which we refer to as "perfect coloring". In section 5 some results for strongly coloring outerplanar and planar graphs are given. Section 6 contains some remarks and open questions. In an appendix we present a new proof of the NP-completeness of the strong coloring problem for graphs. We assume some familiarity with basic graph theory (cf. Harary [11]).

2 Definitions

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The distance between two vertices x and y is defined as the number of edges on the shortest path between x and y. The distance between two edges of G is defined as the shortest distance between an endpoint of one and of the other. Let $\Delta = max\{deg(v) | v \in V\}$, with $deg(v)$ the degree of vertex v. The square graph of G is the graph G^2 with $V(G^2) = V(G)$ and $E(G^2) = \{(u,v) | (u,v) \in E$ or $(u,x) \in E$ and $(x,v) \in E$ for some $x\}$. Observe that $\Delta(G^2) \leq (\Delta(G))^2$. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least $K \leq n$ such that G can be K-vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \ldots, K\}$ with $f(u) \neq f(v)$ whenever $(u,v) \in E$ [11]. The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the least $K \leq m$ such that G can be K-edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \ldots, K\}$ with $f((u,v)) \neq f((u,w))$ for all $u, v, w \in V$ and $(u,v), (u,w) \in E$ [11]. In any vertex (edge) coloring, every pair of vertices (edges) that have distance one must have different colors. Whenever vertex colorings are considered, $c(v)$ will denote the color given to a vertex v.

The generalization to distance-k coloring is now straightforward. The *k-chromatic* *number* of a graph G, denoted by $\chi_k(G)$, is the least $K \leq n$ such that G can be distance-k K-vertex-colored, i.e., such that there exists a function $f : V \rightarrow \{1, 2, \ldots, K\}$ with $f(u) \neq f(v)$ whenever u and v lie within distance k in G. The *k-chromatic index* of a graph G, denoted by $\chi'_k(G)$, is the least $K \leq m$ such that G can be distance-k K-edge-colored, i.e., such that there exists a function $f : E \rightarrow \{1, 2, \ldots, K\}$ with $f((u, v)) \neq f((w, x))$ whenever (u, v) and (w, x) lie within distance k from each other. For $k = 2$, we speak of the *strong chromatic number* and the *strong chromatic index* respectively. If a $(r-1)$-regular graph can be strongly colored with exactly r colors, then this coloring is called *perfect* [12].

McCormick [17] has proved that, given a graph and an integer K, the problem of deciding whether a graph can be distance-k vertex colored with K colors is NP-complete, for every $k \geq 1$. Another proof for the NP-completeness of the strong chromatic number problem can be found in the Appendix. Checking whether a graph can be strongly colored with $K \leq 3$ colors is trivial. If a graph has $\Delta \geq 4$ or contains a C_5 or a $K_{2,3}$, then the graph is not strongly 4-colorable. It is open whether the problem of deciding $\chi'_k(G) \leq K$ for graphs G and integers K is NP-complete. In this paper we will focus entirely on the strong coloring problem ($k = 2$). We will be refering to some special classes of graphs including planar graphs, outerplanar graphs, Halin graphs, chordal graphs and partial k-trees. We assume that the first three are known but include an inductive definition of partial k-trees.

Definition 2.1 ([1]) *The class of k-trees is the smallest class of graphs that satisfies the following rules:*

1. *the complete graph K_k on k vertices is a k-tree.*
2. *if $G = (V, E)$ is a k-tree and $v_1, \ldots, v_k$ form a complete subgraph of G, then the graph $G' = (V \cup \{w\}, E \cup \{(v_i, w) | 1 \leq i \leq k\})$ with $w \notin V$ is also a k-tree.*

A graph is a partial k-tree if and only if it is a subgraph of a k-tree.

3 Preliminaries

To obtain some first bounds for $\chi_2(G)$, consider the following (SL*) ordering of a graph, determined by the following algorithm, which is almost similar to the SL-algorithm of [16].

ALGORITHM SL*

Let n be the number of vertices in G.
Initialize H to G.
for $j = n$ **downto** 1 **do**
  begin

Choose a vertex v_j in H.
Let $v_{j_1}, \ldots, v_{j_k}$ be its neighbors.
Remove v_j and all edges incident to v_j from H.
Add zero or more edges to H to achieve that
$v_{j_1}, \ldots, v_{j_k}$ have distance ≤ 2 to each other.
end
$\mathrm{SL}^* = v_1, v_2, \ldots, v_n$.

END OF ALGORITHM

Let $\mathrm{SL}^* = v_1, v_2, \ldots, v_n$ be the ordering of G as computed by the algorithm. Let p be the maximum of the degrees of the vertices as they appear in the **for**-loop. Let H_j be the graph operated on by the algorithm when the loop-body is executed for j. Let Δ' be the maximum degree of any vertex in an H_j. Note that $\Delta' \geq \Delta$ as $H_n = G$.

Theorem 3.1 *With p and Δ' defined as above:* $\Delta + 1 \leq \chi_2(G) \leq p\Delta' + 1$.

Proof: The lowerbound is trivial. For the upperbound, we use induction to show that the graphs H_j and thus the vertices can be strongly colored in SL^*-order with no more than $p\Delta' + 1$ colors. Let C be a set of $p\Delta' + 1$ colors. Vertex v_1 can be assigned an arbitrary color from C and a strong coloring of H_1 is obtained trivially. Assume we have colored the vertices $v_1, \ldots, v_{i-1}$ (following the ordering) using colors from C such that a strong coloring of H_{i-1} is implied, for some $i \geq 2$. Now consider H_i. Assume that all vertices except v_i are colored as in H_{i-1}. v_i is connected to at most p colored neighbors and since all these neighbors have distance ≤ 2 to each other in H_{i-1}, they are colored differently. Now v_i has at most $p\Delta'$ colored vertices within distance 2, hence at most $p\Delta'$ colors from C are blocked for it. Hence v_i can be colored with a color from C to obtain a strong coloring of H_i. This completes the induction. □

It follows from theorem 3.1 that, in order to obtain strong colorings with a "small number of colors using the SL^*-algorithm, an SL^*-ordering must be found that gives both a small value of p and a value of Δ' that remains close to Δ.

A possible algorithm for adding additional edges between the vertices $v_{j_1}, \ldots, v_{j_k}$ in each step of the for-loop in the SL^*-algorithm is the following. Assign to every vertex v_{j_i} a label (x, y), with $0 \leq x, y < \lceil\sqrt{k}\rceil$. Add an edge between two vertices with labels (x_1, y_1) and (x_2, y_2), if and only if $x_1 = x_2$ or $y_1 = y_2$. Between every two vertices, labelled (a, b) and (c, d), there is a path of length at most two via vertex (a, d) or vertex (b, c). Note that this construction increases the degree of every vertex v_{j_i} by at most $2\lceil\sqrt{k}\rceil$.

Corollary 3.2 *k-trees can be strongly colored with at most $k\Delta + 1$ colors.*

Proof: From definition 2.1 it follows that an ordering of a k-tree is obtained by removing suitable vertices of degree k in each step of the SL*-algorithm. Note that these vertices can be chosen so their neighbors are a clique in the remaining graph (a k-tree again), hence no edges need to be added. Applying theorem 3.1, the corollary follows. □

Corollary 3.3 *Every outerplanar graph can be strongly colored with at most $2\Delta + 1$ colors.*

Proof: Every outerplanar graph G has a vertex v with degree at most 2. Deleting v with its incident edges preserves the outerplanarity. Thus the SL*-algorithm can chose v_j to be a vertex of degree ≤ 2 in every iteration of the loop. Observe that if v_i had degree 2 and its neighbors where not adjacent, then we can add an edge between them in the last step of the iteration (without increasing any degrees). Applying theorem 3.1 the result follows. □

Similar bounds can be given for distance-k vertex and edge colorings. Observe that theorem 3.1 also implies that $\chi'(G) \leq \chi_2(G)$, as $\chi'(G) \leq \Delta + 1$ by Vizing's theorem. The question whether $\chi_k(G) \leq \chi'_k(G)$ remains as an interesting open problem. For trees it is clear that $\chi_2(G) = \Delta + 1$ and $\chi'_2(G) = max\{deg(u) + deg(father(u)) | u \in V\}$. Every Halin graph can be strongly colored with at most $\Delta + 6$ colors. (For the latter result one uses that every tree can be strongly colored using at most $\Delta + 1$ colors and every circuit with at most 5 colors.)

Observe that from a strong vertex coloring with $\chi_2(G)$ colors one can obtain a strong edge coloring with $(\chi_2(G))^2$ colors, by assigning to every edge (u, v) the color $[c(u), c(v)]$, where the colors of the vertices are taken from the strong vertex coloring. Notice that this large difference between the strong chromatic number and the strong chromatic index actually occurs in the case of the complete bipartite graphs $G = K_{n,n}$, where $\chi_2(G) = 2n$ and $\chi'_2(G) = n^2$.

There appears to be no simple connection between $\chi(G)$ and $\chi_2(G)$. A reasonable conjecture like $\chi_2(G) \leq (\Delta + 1)\chi(G) + 1$ fails, by observing the coloring of the following bipartite graph $G_p = (< V_1, V_2 >, E)$ for any $p \geq 2$. V_1 consists of p vertices $0, \ldots, p-1$. V_2 consists of $p(p-1)$ vertices $\{i, j\}, 0 \leq i < p, 0 \leq j < p-1$, and a vertex A. Let $i \in V_1$ be connected to the vertices $\{i, x\}$ for $0 \leq x < p-1$ and to vertex A. Add $p(p-1)$ vertices $[k, l]$ to V_1, with $0 \leq k, l < p-1$, and the edges $([i, j], \{0, i\})$ and $([i, j], \{k, ((k-1)i + j) \bmod (p-1)\})$, with $1 \leq k < p$ and $0 \leq i, j < p-1$. Note that G_p is p-regular bipartite and that every two vertices in V_2 have distance 2 to each other. Hence this graph has $\Delta = p, \chi(G) = 2$ and $\chi_2(G) = \Delta(\Delta - 1) + 1$. It shows that $\chi(G)$ and $\chi_2(G)$ can differ dramatically. In figure 1, an example is given for $p = 3$.

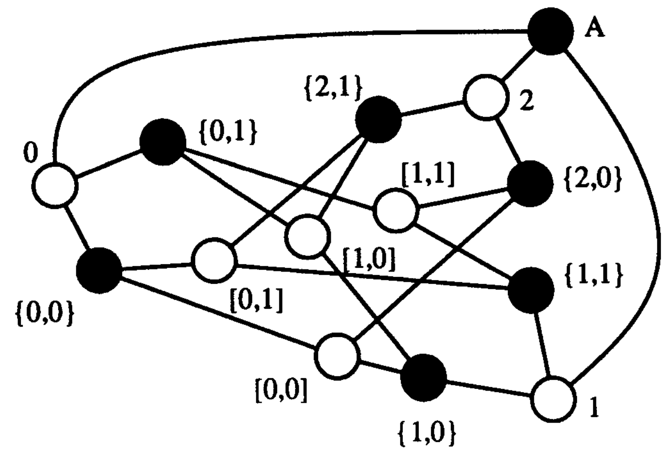

Figure 1: Example of a G_p for $p = 3$.

From the observation that $\chi_k(G) = \chi(G^k)$ and $\chi'_k(G) = \chi'(G^k)$ (using the definition of the k^{th} power graph G^k of [11]) for every $k \geq 2$, it follows that we can use the available algorithms for ordinary graph coloring for obtaining strong colorings, after calculating G^2 in $O(\Delta m)$ time. Also all lowerbounds for the chromatic number trivially hold for the strong chromatic number. This also leads to the following observation:

Theorem 3.4 *Chordal graphs can be strongly colored in a smallest possible number of colors in polynomial time. For every chordal graph G one has $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$.*

Proof: Note that the square graph of a chordal graph is a chordal graph too, and can be colored in an optimal number of colors in polynomial time (cf. Golumbic [10]).

For proving $\chi_2(G) \leq (\frac{1}{2}\Delta + 1)^2$ we induct on n, the size of G. For $n \leq 3$ the result trivially holds. Consider an arbitrary chordal graph G of n vertices, $n > 3$. Without loss of generality we may assume that G is connected. If G is a clique, then it can be strongly colored with $n = \Delta + 1$ colors and $\Delta + 1 \leq (\frac{1}{2}\Delta + 1)^2$. Thus let G not be a clique, S a minimal vertex separator of G and $A_1, A_2, \ldots, A_l$ the connected components of $G - S$. Let H_1 be the induced subgraph spanned by S and A_1, and let H_2 be the induced subgraph spanned by S and $A_2, \ldots, A_l$. By well-known facts for chordal graphs [10] S is a clique, and H_1 and H_2 are connected chordal graphs. Let $|S| = s$. By induction H_1 and H_2 are strongly colorable with the colors of some set C of $(\frac{1}{2}\Delta + 1)^2$ colors. A strong coloring of G can now be obtained as follows.

Permute the colors such that in the strong colorings of H_1 and H_2, the vertices of S get the same colors. (This can be done because the colors assigned to the vertices of S must all be different, by the strong coloring requirement, in both the coloring of H_1 and the coloring of H_2.) Let N_1 be the set of vertices in A_1 that are reached by an edge from S, and N_2 the set of vertices in $A_2 \cup \ldots \cup A_l$ defined similarly. Let $|N_1| = n_1$ and $|N_2| = n_2$, and observe that $n_1 + n_2 \leq s(\Delta - s + 1) \leq (\frac{1}{2}\Delta + 1)^2 - s$. Thus we have sufficiently many colors in C to arrange that s colors are fixed for the vertices in S, and the remaining colors can be permuted such that in the strong colorings of H_1 and H_2 the vertices in N_1 and N_2 are colored by disjoint sets of colors. The resulting strong colorings of H_1 and H_2 can now be combined (merged) to a correct strong coloring of G which employs no more than $(\frac{1}{2}\Delta + 1)^2$ colors. □

We conjecture that $\chi(G^2) \leq Q + 1$, with Q the number of vertices of the largest clique in the graph G^2. If G^2 is a linegraph, then this conjecture is true by noting that if the linegraph G^2 has a largest clique of size Q, then the linegraph of this linegraph has maximal degree Q and can be edge-colored with $Q + 1$ colors. Hence the linegraph G^2 can be vertex-colored with $Q + 1$ colors.

4 Perfect Colorings

A $(r-1)$-regular graph is called perfectly colorable if it can be strongly colored with exactly r colors. This kind of strong coloring is useful for the following file distribution problem [2]: "Given a connected regular network $G = (V, E)$ and a file F, assign to each vertex $x \in V$ a segment $F_x \subseteq F$ such that for all $x \in V, \bigcup_{\{x,y\} \in E} F_y \cup F_x = F$ and in every neighborhood the distributed fragments are free of overlaps, i.e., $\forall (x, y) \in E : F_x \cap F_y = \emptyset$." When the network is $(r-1)$-regular, this problem solves the file distribution question with the smallest possible number of different disjoint segments of F. A perfect coloring describes the assignment of the segments for a valid solution of the file distribution problem. It has been shown by Bakker et al. [3] that this problem is NP-complete, even for the case $r = 4$.

In this section we give some relationships between strong colorings, perfect colorings and edge colorings.

Lemma 4.1 ([2]) *If a $(r-1)$-regular graph with $|V| = n$ vertices can be perfectly colored, then $r|n$ and every group of equally colored vertices has $\frac{n}{r}$ vertices.*

Proof: Let $N(x)$ denote the set of vertices having distance ≤ 1 to vertex x. Consider any perfect coloring of the graph, and let c be one of the colors. Let x_1 and x_2 be two vertices colored c. There can be no vertex y in $N(x_1) \cap N(x_2)$ because, if there was, y would have two neighbors of the same color (which contradicts the strong coloring property). Thus for all x_1, x_2 with $x_1 \neq x_2$ and $c(x_1) = c(x_2) = c$: $N(x_1) \cap N(x_2) = \emptyset$. Furthermore for every vertex y there is a vertex x with $c(x) = c$ and $y \in N(x)$. Hence the neighborhoods $N(x)$ of vertices x with $c(x) = c$ form a partitioning of G. But for all $x \in V : |N(x)| = r$. Hence $r|n$ and every group of equally colored vertices has size $\frac{n}{r}$. □

Theorem 4.2 *Every strongly r-colorable graph is the induced subgraph of a perfectly colorable $(r-1)$-regular graph.*

Proof: We induct on r. For $r = 1, 2$ and 3, the theorem is trivial. Thus let $r \geq 4$ and G be a strongly r-colorable graph. Consider a strong coloring of G with the colors $c_1, \ldots, c_r$ and let H be the induced subgraph of G consisting of all vertices with a color $\in \{c_1, \ldots, c_{r-1}\}$. By induction H is an induced subgraph of some $(r-2)$-regular graph R_H that is perfectly colorable, and w.l.o.g. we can assume that it is perfectly colored with $c_1, \ldots, c_{r-1}$. Arrange the vertices of R_H into $(r-1)$ disjoint blocks $B_1, \ldots, B_{r-1}$, with B_i $(1 \leq i \leq r-1)$ containing the vertices of color c_i, and let every block contain b vertices. (By lemma 4.1 we know that the blocks must be of equal size.) Tag the vertices of R_H that correspond to the vertices of H. Let the vertices $x_1, \ldots, x_s$ (some $s \geq 1$) of $G - H$ together form the "beginning" of the r^{th} block B_r. The vertices $\{x_1, \ldots, x_s\}$ form an independent set in G (because they all have color c_r).

Now form the graph R_G as follows. Make $\lceil \frac{s}{b} \rceil$ copies of R_H and extend B_r by another $\lceil \frac{s}{b} \rceil b - s$ vertices y. We now "connect" the x- and y-vertices to the vertices in the R_H copies in two steps, as follows:

1. **for** i **from** 1 **to** s **do**
   begin
   connect x_i to a new vertex from a B_j-block for every $j, 1 \leq j \leq r-1$,
   always favoring the tagged vertex z in a block B_j if x_i is
   directly connected to z in G, but taking an untagged vertex otherwise.
   end;

Observe that we have $\lceil \frac{s}{b} \rceil b \geq s$ vertices of every color, so step 1 always works and does not "run out of vertices to connect to". But also observe that we have exactly $\lceil \frac{s}{b} \rceil b - s$ vertices left of every color after this step.

2. **for** i **from** 1 **to** $\lceil \frac{s}{b} \rceil b - s$ **do**
   begin
   pick a new y-vertex and connect y to a vertex from a B_j-block that
   was not yet connected to, for every $j, 1 \leq j \leq r-1$.
   (Note that these vertices were not tagged.)
   end;

Note that step 2 makes the graph G_H $(r-1)$-regular. The result is a graph R_G that is $(r-1)$-regular, perfectly colorable with r colors and clearly, by design, we have that G is an induced subgraph of R_G. □

Another property is the following. Recall that by Vizing's theorem every graph is edge-colorable with Δ or $\Delta + 1$ colors.

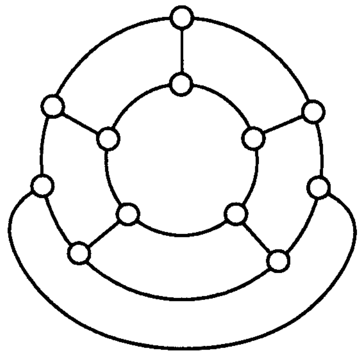

Figure 2: A 3-edge-colorable 3-regular graph that is not perfectly colorable ([13]).

Lemma 4.3 *If a graph is strongly colorable with r colors and r is even, then it is* $(r-1)$*-edge colorable.*

Proof: Let G be strongly colorable with r colors. If r is even, then K_r is edge-colorable with $(r-1)$ colors. Let G be strongly colored with the r names of vertices of K_r. Now G can be edge-colored as follows: color an edge from the vertex colored X to the vertex colored Y with z if the edge between X and Y in K_r is colored z. This gives a correct $(r-1)$-edge coloring of G. □

The converse is not true, see for example figure 2. Also this theorem does not hold for r odd in general, as an edge-coloring of a K_r (which is perfectly colorable with r colors) requires r colors when r is odd.

Also the spectra of perfectly colorable graphs have some interesting properties. Because a perfectly r-colorable graph G is $(r-1)$-regular, its largest eigenvalue is equal to $r-1$ and has multiplicity 1. The following more specific observation can be made as well.

Theorem 4.4 *Let G be perfectly r-colorable. Then G has an eigenvalue -1, with multiplicity* $\geq (r-1)$.

Proof: Let G be perfectly r-colored, and consider the vertices of G arranged in blocks of equally colored vertices (of size $\frac{n}{r}$ each). Let $A = A(G)$ be the adjacency matrix of G corresponding to this vertex-ordering. The symmetrix matrix A can be viewed as a block matrix, with the blocks along the main diagonal consisting of all zeroes and the off-diagonal blocks being $\frac{n}{r} \times \frac{n}{r}$ permutation matrices. (As an aside we note that, conversely, if the vertices of a graph G can be arranged so the adjacency matrix is of this form, then G is perfectly r-colorable.) Now consider the $r \times r$ matrix A' obtained from A by replacing every block on the main diagonal by a "0" and every off-diagonal block by a "1". A' is the adjacency matrix of the K_r, whose spectrum consists of one eigenvalue $(r-1)$ and $(r-1)$ eigenvalues -1 ([6]). Also, when $(x_1, \ldots, x_r)$ is an eigenvector of A', then the vector obtained by repeating each coordinate $\frac{n}{r}$-fold is an eigenvector of A and independency of eigenvectors is preserved in the process. It follows in particular that A (and hence, G) has an eigenvalue -1 with multiplicity at least $r-1$. □

From the same argument some more information can be derived. Let $n > r$ and let $\lambda_1, \ldots, \lambda_k$ and $-\mu_1, \ldots, -\mu_l$ be the remaining positive and negative eigenvalues in the spectrum of G in decreasing order different from the r eigenvalues $(r-1)$ and -1 that we have, with $k + l = n - r$. As the trace of A is zero, we have $\lambda_1 + \ldots + \lambda_k = \mu_1 + \ldots + \mu_l$. Observe also that A^2 is a symmetric matrix with all entries along the main diagonal equal to $r-1$. It follows that $\lambda_1^2 + \ldots + \lambda_k^2 + \mu_1^2 + \ldots + \mu_l^2 = \text{tr}(A^2) - (r-1)^2 - (r-1) = (n-r)(r-1)$. Now let $\lambda = \lambda_1 = \lambda_{\max}$, $\mu = \mu_l = \mu_{\max}$ and $\delta = max\{\lambda, \mu\}$. One easily verifies that $\delta \geq \sqrt{r-1}$ and $min\{\lambda, \mu\} \geq \frac{1}{n-r}\sqrt{r-1}$.

Another characteristic of perfectly colorable graphs is the following:

Theorem 4.5 *Let G be regular of degree ≥ 3 and perfectly colorable. Then one can partition V as $V_1 \cup V_2$ such that*

1. *the induced subgraph G_1 on V_1 is a set of chordless cycles of length divisible by 3.*
2. *the induced subgraph G_2 on V_2 is regular of degree $\Delta - 3$ and perfectly colorable.*

Proof: Let a, b, c be three colors of the perfect coloring of G. Let V_1 be the set of vertices colored a, b or c and $V_2 = V - V_1$.

Consider any vertex in V_1, say with color a. It has one neighbor colored b, this neighbor has one neighbor colored c, etc. This necessarily closes itself as a cycle at the point of departure. By the strong coloring property, this cycle must be chordless. This proves the statement, and the cycles are not connected to each other.

Consider any vertex in V_2. It has *exactly* three neighbors in V_1. Thus G_2 inherits a perfect coloring of G with the remaining $\Delta - 3$ colors. □

This theorem shows that perfectly colorable graphs decompose entirely into (disjoint) chordless cycles. Note that $\frac{|V_2|}{|V_1|} = \frac{\Delta-2}{3}$, for $\Delta \geq 3$.

For the file distribution problem perfect colorings are interesting mostly for regular networks, which includes many current processor networks. In [12] a detailed study is given of perfectly colorable processor networks. For completeness we summarize the results in the following theorem.

Theorem 4.6 ([12]) *The following processor networks are perfectly colorable:*

- *The hypercube C_n, if and only if $n = 2^i - 1$ for some $i > 0$.*

- *The d-dimensional torus of size* $l_1 \times \ldots \times l_d$ *if* $l_i \bmod q = 0$*, with* q *such that* $\sqrt[r]{2d+1} | q$ *for some integer* $r > 0$*.*
- *The Cube-connected Cycles* CCC_d*, if and only if* $d > 2, d \neq 5$*.*
- *The directed shuffle-exchange network and the directed 4-pin shuffle network.*
- *The chordal ring network with chordlength* $4p - 1$ $(p > 0)$ *and* $4kp - 4t$ $(0 \leq t < p)$ *vertices if and only if :*
  1. k *and* t *are even and (if* $t > 0$*)* $\frac{t}{\gcd(t,p)}$ *is even, or*
  2. $k, \frac{t}{\gcd(t,p)}$ *and* $\frac{p}{\gcd(t,p)}$ *are odd and* $t + p$ *is even.*
- *The hexagonal network of size* $m \times n$ *if and only if* $m, n \bmod 7 = 0$*.*

The reader is referred to [12] for the definition of the various networks.

5 Outerplanar and Planar Graphs

In this section we consider the strong coloring problem for outerplanar and planar graphs, respectively. By the results in section 3 we know that every outerplanar graph can be strongly colored with $2\Delta + 1$ colors. Our aim will be to improve this to a bound of $\Delta + 3$ colors (which, in turn, improves on a bound of $\Delta + 4$ colors in a precursor of this report). For this we need the following theorem.

Theorem 5.1 *A graph can be strongly colored with at most* k *colors if and only if all biconnected components of it can be strongly colored with at most* k *colors* $(k \geq \Delta + 1)$*.*

Proof: The "only if" part is trivial. We proceed to show the "if" part. Let G be a graph. (Without loss of generality we can confine ourselves to connected graphs.) Let all biconnected components of G be strongly colorable with at most k colors. We now show that G is strongly k-colorable. When G has no cutvertices, the theorem trivially holds. Thus assume that the theorem holds for all connected graphs with $\leq p - 1$ cutvertices, and let G have p cutvertices. Let v be a cutvertex of G, then G consists of two connected graphs H_1 and H_2 such that each contain a "copy" of the vertex v and are joined at v, but which are otherwise disjoint. W.l.o.g. we may assume that both H_1 and H_2 have $\leq p - 1$ cutvertices.

Let v have degree Δ_1 in H_1 and degree Δ_2 in H_2, where we can assume w.l.o.g. that $\Delta_1 \leq \Delta_2$ and clearly $\Delta_1 + \Delta_2 \leq \Delta$. We can assume inductively that H_1 and H_2 can be strongly colored using at most k colors. Shift color-names such that H_1 and H_2 use colors from the same set of k colors and v gets the same color "α" in H_1 and H_2. Joining H_1 and H_2 at v (while retaining the colorings of H_1 and H_2 respectively) results in a strong coloring of G with k colors, except in the one case that some neighbors of v in H_1 have the same color as some neighbors of v in H_2. We now argue how such a conflict can be removed by a permutation of the colors, if it arises.

Thus assume that the latter case arises. Note that v and its neighbors in H_2 use $\Delta_2 + 1$ colors. Let r neighbors of v in H_1 use colors different from these but l neighbors use colors $c_1, \ldots, c_l$ that are among the colors used by the Δ_2 neighbors in H_2, for certain r and l with $r+l = \Delta_1$. It means that Δ_2+1+r different colors are used in the neighborhood of v. Choose l different colors $d_1, \ldots, d_l$ from among the remaining colors. (This can be done because $k-(\Delta_2+1+k) \geq \Delta_1+\Delta_2+1-(\Delta_2+1+r) = l$.) Exchanging c_i and d_i (for i from 1 to l) in the coloring of H_1 throughout leaves a strong coloring in H_1 and removes the color conflicts at v, thus leading to a correct strong coloring of G using at most k colors. This completes the inductive argument. □

Forman et al. [7] prove the following lemma.

Lemma 5.2 *Every biconnected outerplanar graph contains a vertex of degree 2 with a neighbor of degree 2 or with adjacent neighbors, one of which is of degree at most 4.*

Using theorem 5.1 and lemma 5.2, the following theorem of [7] can be obtained.

Theorem 5.3 *Every outerplanar graph of maximum degree Δ can be strongly colored using at most $\Delta + 3$ colors.*

Proof: Let $U \subseteq V$ be the set of vertices of degree 2 with at least one neighbor of degree 2. If $U \neq \emptyset$, remove all vertices of U and strongly color the remaining outerplanar graph inductively. A strong coloring of the original graph can then be obtained by re-inserting the vertices of U and assigning a suitable color to them one after the other. Since there are at most $\Delta + 2$ vertices at distance ≤ 2 from any vertex v in U, at most $\Delta + 2$ colors are blocked for v and we can indeed complete the strong coloring within $\Delta + 3$ colors.

If $U = \emptyset$, then by lemma 5.2 there must exist a vertex v of degree 2 with adjacent neighbors, one of which has degree ≤ 4. Remove vertex and strongly color the remaining outerplanar graph inductively. Since there are at most $\Delta + 2$ vertices at distance ≤ 2 from v, the same argument can be applied to obtain a strong coloring of G with $\Delta + 3$ colors total. □

The lowerbound for strongly coloring outerplanar graphs is still open, though it is not difficult to construct outerplanar graphs with degree ≤ 6 that need $\Delta + 3$ colors.

For planar graphs, the problem is to strongly color them with at most $c.\Delta + O(1)$ colors for as small a constant c as possible. A first result is the following lowerbound for the strong chromatic number.

Theorem 5.4 *For every $\Delta \geq 1$ there exists a planar graph G with $\chi_2(G) \geq \lceil \frac{3}{2}\Delta \rceil$.*

Proof: We can assume w.l.o.g. that $\Delta > 1$. (For $\Delta = 1$ the theorem trivially holds by taking a graph that consists of a single edge. Choose $r, s \geq 0$ with $s \leq r$ such that $\Delta = r + s + 2$. It will be useful to take $r = s = \frac{1}{2}\Delta - 1$ when Δ is even and $s = r - 1 = \frac{1}{2}\Delta - \frac{3}{2}$ when Δ is odd. Construct the graph G_Δ consisting of a "triangle" of three vertices (A, B and C), r vertices that are each connected to A and to B, s vertices that are each connected to B and to C, and s more vertices that are each connected to A and to C. For Δ odd (implying $\Delta \geq 3$), a separate vertex D is inserted on the triangle-edge (A, B). This vertex is also connected to C. One easily verifies that G_Δ is planar, has maximum degree Δ and diameter 2. Because of the latter any strong coloring of G_Δ needs as many colors as there are vertices, which is precisely $\lceil \frac{3}{2}\Delta \rceil$. (By a result of Seyffart [18] this is about the largest possible number of vertices in any planar graph of diameter 2 and maximum degree Δ.) □

The lemma shows that $c \geq \frac{3}{2}$ for general planar graphs. For $\Delta \leq 5$ one can construct planar graphs that need $\geq 2\Delta$ colors in any strong coloring. For obtaining an upperbound for the strong chromatic number of planar graphs, the following lemma of [7] is useful.

Lemma 5.5 *Let T and U be disjoint sets of vertices in a planar graph and suppose that each vertex in T has at least 3 neighbors in U. Then $|T| \leq 2|U| - 4$.*

Using lemma 5.5, one can easily prove:

Lemma 5.6 *Every planar graph contains either a vertex of degree ≤ 2 or a vertex of degree ≤ 5 with at most two neighbors of degree ≤ 12.*

Proof: Assume by way of contradiction that we are given a graph violating the lemma. For $j = 0, 1, \ldots,$ denote by n_j the number of vertices of degree exactly j. Since in any planar graph the total number of degrees is at most 6 times the number of vertices,

$$\sum_{j=3}^{\infty} j n_j \leq 6 \sum_{j=3}^{\infty} n_j,$$

from which it follows that

$$3 \sum_{j=3}^{5} n_j \geq \sum_{j=3}^{5} (6 - j) n_j \geq \sum_{j=6}^{\infty} (j - 6) n_j \geq 6 \sum_{j=12}^{\infty} n_j.$$

Taking T as the set of vertices of degree ≤ 5 and U as the set of vertices of degree ≥ 12, this implies a contradiction to lemma 5.5. □

From lemma 5.6, the following theorem can be obtained.

Theorem 5.7 *Every planar graph of maximum degree Δ can be strongly colored using at most $2\Delta + O(1)$ colors.*

Proof: We prove by induction on the number of vertices that every planar graph can be strongly colored with at most $2.\max\{\Delta, 14\} + 34$ colors. Consider any planar graph G and apply lemma 5.6. If there is vertex of degree ≤ 2, choose such a vertex v and remove it. If v is of degree 2 and its neighbors are not adjacent, introduce an edge between them. If there is no vertex of degree ≤ 2, choose a vertex v of degree ≤ 5 with at most two neighbors of degree > 11 and contract v into one of its neighbors of degree ≤ 11. The resulting graph is planar and has maximum degree at most $\max\{\Delta, 14\}$.

In either case, the inductive hypothesis implies that the resulting graph can be strongly colored using at most $2.\max\{\Delta, 14\} + 34$ colors, and a strong coloring of the original graph can be obtained by re-inserting v and coloring it differently from all vertices at distance ≤ 2 from v. There are at most $2.\max\{\Delta, 11\} + 33$ such vertices, thus there is a free color for it. □

Forman et al. [7] have recently improved the bound of theorem 5.7 to $\frac{13}{7}\Delta + O(\Delta^{\frac{2}{3}})$. This seems to be the best current bound for strongly coloring planar graphs.

6 Conclusions and further remarks

In this paper we have presented a survey of some basic facts for the strong coloring problem for graphs. Some results for strong coloring of various special classes of graphs like planar and outerplanar graphs were reviewed also. Several open questions were identified along the way.

There are many interesting further problems left. For example, given a coloring algorithm A which gives a good approximate bound on the chromatic number of a graph G, does this algorithm give a good approximate bound for the strong chromatic number of G, when it is applied to the square graph G^2? What if G belong to a special class of graphs?

Another open question is the following. Is there an analog for strong chromatic numbers of the following theorem of Garey and Johson [9] : "If for some constant $r < 2$ and constant d there exists a polynomial-time algorithm A which guarantees $A(G) \leq r\chi(G)+d$, then there exist a polynomial-time algorithm A which guarantees $A(G) = \chi(G)$."? The best performance ratio known for approximation algorithms for the chromatic number problem is $\frac{n \log\log n}{(\log n)^3}$ [5]. What is the corresponding best performance ratio for the strong chromatic number by applying this to the square graph G^2?

It would be interesting to investigate other relationships between the strong coloring problem and the well–studied coloring problem (see e.g. [13]), as well as relationships between the strong vertex coloring problem and the strong edge coloring problem.

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Appendix

Theorem *Given a graph G and an integer K, the problem of determining whether G can be strongly colored with $\leq K$ colors is NP-complete (STRONG CHROMATIC NUMBER).*

Proof: The problem trivially belongs to NP. (One can assign $\leq K$ colors to the vertices of G and verify in polynomial time whetter it is a strong coloring.) For proving the NP-completeness, we reduce 3-SAT to STRONG CHROMATIC NUMBER. Let F be a CNF formula having r clauses, with at most three literals per clause. Let x_i $(1 \leq i \leq n)$ be the variables in F. We may assume $n \geq 4$. We shall construct, in polynomial time, a graph G that is strongly colorable with $rn + 2n + 2$ colors iff F is satisfiable. The graph $G = (V, E)$ is defined by:

$$V = \{x_i, x_2, \ldots, x_n\} \cup \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\} \cup \{y_1, y_2, \ldots, y_{n+1}\} \cup \{p_{1,1}, \ldots, p_{n,r}\} \\ \cup \{p_{n+1,r}\} \cup \{z_1, z_2, \ldots, z_n\} \cup \{C_1, C_2, \ldots, C_r\}$$

and

$$E = \{(y_i, y_j) | i \neq j\} \cup \{(z_i, z_j) | i \neq j\} \cup \{(z_i, x_i), 1 \leq i \leq n\} \cup \\ \{(p_{i,j}, p_{k,l}) | i \neq k \text{ or } j \neq l\} \cup \{(z_i, \overline{x}_i), 1 \leq i \leq n\} \cup \{(p_{n+1,r}, y_{n+1})\} \cup \\ \{(y_i, z_j) | 1 \leq i \leq n, i \neq j\} \cup \{(p_{i,j}, C_j), 1 \leq i \leq n, 1 \leq j \leq r\} \cup \\ \{(p_{i,j}, z_k), 1 \leq i, k \leq n, 1 \leq j \leq r\} \cup \{(x_i, p_{i,k}) | x_i \notin C_k\} \cup \{(\overline{x}_i, p_{i,k}) | \overline{x}_i \notin C_k\}$$

To see that G is $rn+2n+2$ colorable iff F is satisfiable, we first observe that the y_i's form a complete subgraph on $n+1$ vertices. Hence, each y_i must be asssigned a distinct color. Without loss of generality we may assume that in any coloring of G y_i is given the color i for $1 \leq i \geq n+1$. Then we observe that the z_i's together form a complete subgraph on n vertices. Every z_i is at most at distance two from every y_i, hence the z_i must be colored differently from the y_i. Assume w.l.o.g. that z_i is given the color $n+i+1$ for $1 \leq i \geq n$. We also observe that the $p_{i,j}$'s together form a complete subgraph on $rn+1$ vertices. Every $p_{i,j}$ is at most at distance two from every y_k, and every $p_{i,j}$ is at most distance two from every z_k, so the colors of the $p_{i,j}$ must be different from the colors of the y_k and different from the colors of the z_l. Thus we can assume that $p_{i,j}$ is given the color $2n+in+j+1$ and $p_{n+1,r}$ is given the color $rn+2n+2$. Since y_i lies within distance two from all the x_j's and the $\overline{x}_j$'s, except x_i and $\overline{x}_i$, the color i can only be assigned to x_i or $\overline{x}_i$. x_i lies within distance two from $\overline{x}_i$, so one of these two vertices must have a different color. x_i and $\overline{x}_i$ lie within distance two from every z_k and $p_{k,l}$ and every other $y_j, j \leq i, j \neq i$, so only color $n+1$ is available for one of these two vertices, for every $i, 1 \leq i \leq n$, because no x_i or $\overline{x}_i$ lies within distance two from any other x_j or $\overline{x}_j$. The vertex that is assigned to color $n+1$ will be called the *false* vertex. The other is the *true* vertex. The only way to color G using $rn+2n+2$ colors, is to assign color $n+1$ to one of $\{x_i, \overline{x}_i\}$ for each i, $1 \leq i \leq n$.

Under what conditions can the remaining vertices be colored using no further colors? Since $n \geq 4$ and each clause has at most three literals, each C_i lies within distance two from a pair $x_j, \overline{x}_j$, for at least one j. Consequently no C_i may be assigned the color $n+1$. Also every C_i lies within distance two from every $p_{k,l}$ and every z_j, so C_i must be assigned a color less than $n+1$.

Also no C_i can be assigned a color corresponding to an x_j or an $\overline{x}_j$ that does not occur in clause C_i. These observations imply that the only colors that can be assigned to C_i correspond to vertices x_j or $\overline{x}_j$ that are in clause C_i and are *true* vertices.

Hence G is strongly $rn+2n+2$ colorable iff there is a true vertex corresponding to each C_i, and thus iff F is satisfiable. □