Continuous Informations Systems

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category: it is a *semi* Cartesian closed category ([3]). In general, *semi* categorical notions arise if "functors" are used which do not preserve identities ([3]). The Karoubi envelope construction transforms *semi* notions to *ordinary* notions. For example, the Karoubi envelope of a semi Cartesian closed category is a Cartesian closed category. Hence we may define data structures such as products and function spaces in the simple category qIS, and transform them via the Karoubi envelope to cIS.

The rest of this paper is organised as follows. In section 2 a short overview of some relevant domain-theory is given. In section 3 *continuous* information systems are defined, and it is shown that they are equivalent to continuous Scott domains. Furthermore, it turns out that *algebraic* informations systems are just the *reflexive* continuous ones. In section 4 *qualitative* information systems are defined, and it is shown that continuous information systems may be constructed by means of the *Karoubi envelope.* In section 5 we show how various data types may be defined on qualitative information systems, and translated to continuous information systems. Among other things, we prove that the category of qualitative information systems is semi Cartesian closed with products. Finally, in section 6 two *universal* information systems are defined.

2 Definitions

In this section a short overview of some relevant domain-theory is given by presenting some definitions and theorems.

2.1 Domain theory

We consider posets in which the least upper bounds (lub's) of certain subsets exist.

Definition 1 *A subset S of a poset P is* directed *iff each finite subset of S has an upperbound in S.*

Note that a directed set S can not be empty, because the empty set must have an upperbound in S. So a set S is directed iff it is not empty and every pair of elements of S has an upperbound in S.

Definition 2 *A* directed complete poset (dcpo) *D is a poset D in which each directed subset has a lub.*

The appropriate kind of morphism between dcpo's preserves the relevant structure.

Definition 3 *A function $f : D \rightarrow E$ between dcpo's is* continuous *iff it preserves lubs of directed sets, i.e. if $S \subseteq D$ directed, then $f(\bigvee S) = \bigvee f(S)$.*

Note that a continuous function f is *monotone*, i.e. $d \leq d'$ implies $f(d) \leq f(d')$. Dcpo's and continuous functions form a category **Dcpo**, with normal function composition and identity. However, we are interested in full subcategories of **Dcpo**, with as objects dcpo's which are generated by a basis. First the way-below relation is defined.

Definition 4 *Let D be a dcpo. Define the* way-below *relation $\ll \subseteq D \times D$ as follows: $x \ll y$ iff for each directed subset S of D: $y \leq \bigvee S$ implies there exists an $y' \in S$ such that $x \leq y'$.*

Theorem 5 *The relation $\ll$ on a dcpo D has the following properties:*

- $x \ll y \Rightarrow x \leq y$
- $x' \leq x \ll y \leq y' \Rightarrow x' \ll y'$
- $x \ll y \ll z \Rightarrow x \ll z$
- *If $X \subseteq D$ finite, then $\forall x \in X(x \ll y) \Leftrightarrow \bigvee X \ll y$*

Definition 6 *D is a* continuous *dcpo iff D is a dcpo and there exists a subset B_D of D such that for each $x \in D$ the set $B_D(x) = \{x' \in B_D | x' \ll x\}$ is directed, and $x = \bigvee B_D(x)$.*

The set B_D is called a *basis* for D.
Define **Cont** as the full subcategory of **Dcpo** with as objects continuous dcpo's D with a fixed basis B_D.

Example 7 *The interval $[0,1]$ of real numbers is a continuous dcpo. It can easily be checked that the relation $\ll$ on $[0,1]$ is the same as $<$, except that $0 \ll 0$. Two different bases for this dcpo are $[0,1]$ itself, and the set $\{q | 0 \leq q < 1 \& q$ is a rational number $\}$.*

Theorem 8 (Strong Interpolation) *Let D, E be continuous dcpo's, $f : D \to E$ a continuous function, x an element of D, and y of E. If $y \ll f(x)$, then there exists a $x' \in B_D$ such that $y \ll f(x')$ and $x' \ll x$.*

Proof: Take $S = \{e \in B_E | \exists x' \in B_D : e \ll f(x') \& x' \ll x\}$.
S is directed: It is clear that S is non-empty, for $B_D(x)$ is not empty so take $x' \in B_D(x)$, and $B_E(f(x'))$ is not empty, so take $e \in B_E(f(x'))$. Suppose $e_1, e_2 \in S$, then there are $x'_1, x'_2 \in B_D$ such that $e_i \ll f(x'_i)$, and $x'_i \ll x$. Now $B_D(x)$ is directed, so there exists $x' \in B_D$ such that $x' \ll x$ and $x'_1, x'_2 \leq x'$. Because $e_i \ll f(x'_i) \leq f(x')$, we have $e_i \ll f(x')$. The set $B_E(f(x'))$ is directed, hence there exists $e \in B_E$ such that $e \ll f(x')$ and $e_1, e_2 \leq e$. It follows that e is an upperbound of e_1, e_2 in S.
Furthermore

$$
\begin{aligned}
\bigvee S &= \bigvee\{\bigvee B_E(f(x'))|x' \in B_D(x)\} \\
&= \bigvee\{f(x')|x' \in B_D(x)\} \\
&= f(\bigvee B_D(x)) \\
&= f(x)
\end{aligned}
$$

Suppose $y \ll f(x)$, then $y \ll f(x) \leq \bigvee S$. There exists $e \in S$ such that $y \leq e$, hence $y \leq e \ll f(x')$ for certain $x' \ll x$, $x' \in B_D$. It follows that $y \ll f(x')$ and $x' \ll x$. ■

Theorem 9 (Weak Interpolation) *Let D be a continuous dcpo, and $x, y \in D$. If $y \ll x$, then there exists a $x' \in B_D$ such that $y \ll x' \ll x$.*

Proof: Take $f = id_D$ in the strong interpolation theorem. ■

Cont itself is not a Cartesian closed category, but it contains various Cartesian closed subcategories.

Definition 10 *A dcpo D is* bounded complete *iff each bounded subset (i.e. each subset with an upperbound) has a lub.*

Define BCCont as the full subcategory of Cont with as objects bounded complete continuous dcpo's. BCCont is Cartesian closed.

Example 11 *The interval* $[0,1]$ *of real numbers is a bounded complete continuous dcpo.*

By theorem 5 the way-below relation $\ll$ is transitive and anti-symmetric. However, it is not necessary reflexive.

Definition 12 *An element d of a dcpo D is* compact *iff $d \ll d$.*

Definition 13 *An* algebraic dcpo *D is a continuous dcpo with a basis consisting of compact elements.*

Define Alg, resp. BCAlg as the full subcategories of Cont with as objects algebraic dcpo's, resp. bounded complete algebraic dcpo's. Alg is not Cartesian closed, but BCAlg is. The objects in the last category are sometimes called *Scott domains*.
Finally we consider a full subcategory of BCAlg with as objects a very concrete kind of dcpo's ([1]), i.e. the elements of these dcpo's are sets, and the ordering is subset-inclusion.

Definition 14 *A* qualitative domain *is a set of sets A, which satisfies the following:*

1. $\emptyset \in A$.

2. *If $y \subseteq x \in A$, then $y \in A$.*

3. A is closed under directed unions.

It can easily be checked that each qualitative domain is a bounded complete algebraic dcpo. Define **Qd** as the full subcategory of **BCAlg** with as objects qualitative domains. This category is not Cartesian closed. Therefore, in [1] the Cartesian closed subcategory of **Qd** was taken with *stable* continuous functions as arrows. However, we shall see that although **Qd** is not a Cartesian closed category it is very interesting.

2.2 Algebraic information systems

Algebraic information systems are concrete representations of Scott domains.

Definition 15 *An* algebraic information system (ais) *A is a tuple* $< Dom_A, Con_A, \vdash_A >$ *where*

- Dom_A *is a set, the set of* tokens,
- $Con_A \subseteq \mathcal{P}_f(Dom_A)$*, the set of* consistent *sets of tokens,*
- $\vdash_A \subseteq Con_A \times Dom_A$*, the* entailment relation*,*

satisfying the following clauses $(X, Y \in \mathcal{P}_f(Dom_A))$*:*

1. $\emptyset \in Con_A$
2. $X \subseteq Y \in Con_A \Rightarrow X \in Con_A$
3. $a \in Dom_A \Rightarrow \{a\} \in Con_A$
4. $X \vdash_A a \Rightarrow X \cup \{a\} \in Con_A$
5. $a \in X \in Con_A \Rightarrow X \vdash_A a$
6. $\exists Y(\forall b \in Y(X \vdash_A b) \& Y \vdash_A a) \Rightarrow X \vdash_A a$

Algebraic information systems are defined in [11], where they are simply called information systems. Note that we have given here the slightly different definition of [8]. There is a notion of map between two algebraic information systems. (In the following we will abbreviate $\forall b \in Y(X \vdash_A b)$ as $X \vdash_A Y$.)

Definition 16 *An* algebraic approximable mapping (aam) *f between algebraic information systems A and B is a relation* $f \subseteq Con_A \times Con_B$ *which satisfies:*

1. $\emptyset f \emptyset$
2. $(XfY \& XfY') \Rightarrow Xf(Y \cup Y')$
3. $\exists X, Y(X' \vdash_A X \& XfY \& Y \vdash_B Y') \Rightarrow X'fY'$

3. $a \in Dom_A \Rightarrow \{a\} \in Con_A$

4. $\forall b \in Y(X \vdash_A b) \Rightarrow Y \in Con_A$

5. $(X \subseteq Y \& X \vdash_A a) \Rightarrow Y \vdash_A a$

6. $\exists Y(\forall b \in Y(X \vdash_A b) \& Y \vdash_A a) \Leftrightarrow X \vdash_A a$

We will often omit the subscripts, and write A for Dom_A. Furthermore, if R is a relation between Con and Dom, $X \in Con$, and $Y \in \mathcal{P}_f(Dom)$, then XRY stands for $\forall b \in Y(XRb)$. For example, $XR\emptyset$ holds for each consistent X, and clause 4 above can be written as $X \vdash_A Y \Rightarrow Y \in Con_A$.

Theorem 20 *Let A be a cis, then*

$$\exists Y(X \vdash Y \& Y \vdash Z) \Leftrightarrow X \vdash Z$$

Proof: From left to right it is trivial. Now suppose $X \vdash Z$. By clause 6 in the definition of a cis there exists for each $c \in Z$ an Y_c such that $X \vdash Y_c \vdash c$. Take $Y = \bigcup\{Y_c | c \in Z\}$. It is clear that Y is a finite set and that $X \vdash Y$. Furthermore, $Y_c \subseteq Y$ and $Y_c \vdash c$ hold for each $c \in Z$, hence by clause 5 in the definition of a cis it follows that for each $c \in Z$ $Y \vdash c$, and therefore $Y \vdash Z$. ■

Example 21 *The continuous information system Q is given by the following clauses:*

- $Dom_Q = \{q | 0 \leq q < 1 \& q \textit{ is a rational number }\}$
- $Con_Q = \mathcal{P}_f(Dom_Q)$
- $X \vdash_Q q = q < \bigvee X$

Example 22 *The continuous information system P is given by the following clauses:*

- $Dom_P = \mathcal{P}(\{n | n \textit{ is a natural number }\})$
- $Con_P = \mathcal{P}_f(Dom_P)$
- $X \vdash_P p = \bigcup X - p$ *infinite.*

Maps between continuous information systems are certain kinds of relations.

Definition 23 *A* continuous approximable mapping (cam) f *between continuous information systems A and B is a relation $f \subseteq Con_A \times Dom_B$ which satisfies:*

1. $XfY \Rightarrow Y \in Con_B$
2. $(X \subseteq X' \& Xfb) \Rightarrow X'fb$

3. $\exists X,Y(X' \vdash_A X \& XfY \& Y \vdash_B b) \Leftrightarrow X'fb$

Theorem 24 *Let f be a cam between A and B, then*

1. $\exists X,Y(X' \vdash_A X \& XfY \& Y \vdash_B Y') \Leftrightarrow X'fY'$

2. $\exists X(X' \vdash_A X \& XfY') \Leftrightarrow X'fY'$

3. $\exists Y(X'fY \& Y \vdash_B Y') \Leftrightarrow X'fY'$

Proof:

1. From left to right it is trivial. To prove the other way round suppose $X'fY'$. By clause 3 in the definition of a cam there exist for each $b \in Y'$ consistent sets X_b and Y_b, such that $X' \vdash_A X_b \& X_b f Y_b \& Y_b \vdash_B b$. Take $X = \bigcup\{X_b | b \in Y'\}$ and $Y = \bigcup\{Y_b | b \in Y'\}$. It is clear that X, Y are finite, and that $X' \vdash_A X$. Furthermore, $X_b \subseteq X and X_b f Y_b$ hold for each $b \in Y'$, and hence by clause 2 in the definition of a cam XfY_b. Therefore XfY. Finally $Y_b \subseteq Y \& Y_b \vdash_B b$ for each $b \in Y'$, hence $Y \vdash_B b$ for each $b \in Y'$, and $Y \vdash_B Y'$.

2. $\exists X(X' \vdash X \& XfY') \Leftrightarrow$
   $\exists X, Z_1, Z_2(X' \vdash X \& X \vdash Z_1 \& Z_1 f Z_2 \& Z_2 \vdash Y') \Leftrightarrow$
   $\exists Z_1, Z_2(X' \vdash Z_1 \& Z_1 f Z_2 \& Z_2 \vdash Y') \Leftrightarrow$
   $X'fY'$

3. Analogous to 2.

■

The identity cam $I_A : A \to A$ is defined as $XI_Aa := X \vdash_A a$. Let $f : A \to B$ and $g : B \to C$ be cam, then their composition $g \circ f : A \to C$ is defined as $X(g \circ f)c := \exists Y(XfY \& Ygc)$. clS is the category with as objects continuous information systems and as arrows continuous approximable mappings.

3.2 clS $\simeq$ BCCont

It will be shown how each cis represents a bounded complete continuous dcpo. The underlying set of this dcpo will consist of certain subsets of the tokens of the cis.

Definition 25 *The* elements *$Pt(A)$ of a cis A are subsets x of tokens which satisfy:*

1. $X \subseteq x \& X$ *finite* $\Rightarrow X \in Con_A$,

2. $X \subseteq x \& X \vdash a \Rightarrow a \in x$,

3. $a \in x \Rightarrow \exists X \subseteq x(X \vdash a)$.

Hence elements are subsets of tokens which are *finitely consistent* (1) and *closed under entailment* (2). Furthermore, each token in an element *has a cause* (3), i.e. is derivable from a finite subset of the element.

Theorem 26 *Let A be a cis, and S a finitely consistent set of tokens of A. Then $[S] := \{a | \exists X \subseteq S (X \vdash a)\}$ is an element of A.*

Proof:

1. $[S]$ is finitely consistent.
   Suppose X is a finite subset of $[S]$. For each $a \in X$ there is a $X'_a \subseteq S$ such that $X'_a \vdash a$. Take $X' = \bigcup\{X'_a | a \in X\}$. The set X' is consistent, because S is finitely consistent. By clause 5 in the definition of a cis it follows that $X' \vdash X$, hence $X \in Con_A$ by clause 4.

2. $[S]$ is closed under entailment.
   Suppose X is a finite subset of $[S]$, and $X \vdash a$. Let X' be the same set as in the previous item, then $X' \subseteq S$ and $X' \vdash X \vdash a$. By clause 6 in the definition of a cis it follows that $X' \vdash a$, hence $a \in [S]$.

3. Each token in $[S]$ has a cause.
   Suppose $a \in [S]$. There is a $X \subseteq S$ such that $X \vdash a$. By clause 6 there exists an Y such that $X \vdash Y \vdash a$, hence $Y \subseteq [S]$.

■

The elements of a cis A ordered by inclusion form a bounded complete continuous dcpo.

Theorem 27 *$Pt(A)$ ordered by set-inclusion is a bounded complete continuous dcpo.*

Proof: The union of a directed set of elements is an element, and it is the lub of that set.
Let S be a bounded subset of elements. Then $\bigcup S$ is finitely consistent (because it is a subset of a finitely consistent set), hence $[\bigcup S]$ is an element. It is easy to see that $[\bigcup S]$ is smaller than each upperbound of S. Furthermore, it is itself an upperbound of S: Suppose $a \in x \in S$, then because x is an element there is $X \subseteq x$ such that $X \vdash a$. Hence $X \subseteq \bigcup S$ and $X \vdash a$, hence $a \in [\bigcup S]$.
Finally, we have to give a basis of $Pt(A)$. First, the following holds: if $X \subseteq x$, then $[X] \ll x$. For suppose S is a directed set of elements, and $x \subseteq \bigcup S$. For each $a \in X$ there is an $y_a \in S$ such that $a \in y_a$. But $\{y_a | a \in X\}$ is finite and S directed, hence there is an upperbound $y \in S$. By $X \subseteq y$ it follows that $[X] \subseteq y$.
Let x be an element and consider $\bar{x} = \{[X] | X \subseteq x, X \text{ finite }\}$. This set is directed: $[\emptyset] \in \bar{x}$, hence it is not empty. If $[X_1], [X_2] \in \bar{x}$, then $[X_1 \cup X_2] \in \bar{x}$ and this is an upperbound.

The union of $\bar{x}$ is equal to x: Suppose $a \in x$, then there is $X \subseteq x$ such that $X \vdash a$, hence $a \in [X]$. The reverse is trivial.
Therefore $\{[X] | X \in Con_A\}$ is a basis of $PT(A)$. ∎

Example 28 *Let Q be the cis from the previous subsection, then $Pt(Q) \cong [0,1]$.*

Each cam between information systems represents a continuous function between bounded complete continuous dcpo's.

Theorem 29 *Let $f : A \to B$ be a cam, then $Pt(f) : Pt(A) \to Pt(B) : x \mapsto \{b | \exists X \subseteq x(Xfb)\}$ is a continuous function.*

Proof: It is straightforward that $Pt(f)$ is well-defined and continuous. ∎

Theorem 30 *$Pt : \mathsf{cIS} \to \mathsf{BCCont}$ is a functor.*

Proof: It is straightforward that Pt preserves identities and composition. ∎

Every bounded complete continuous dcpo can be transformed into a cis.

Theorem 31 *Let D be a bounded complete continuous dcpo, then $Rep(D)$ is a cis, where*

- *$Dom_{Rep(D)}$ is the basis of D.*
- *$Con_{Rep(D)}$ is the set of bounded, finite subsets of the basis.*
- *$X \vdash_{Rep(D)} a$ iff $a \ll \bigvee X$.*

Proof: We check the clauses in the definition of a cis.

1. $\emptyset$ is bounded by the least element of D.
2. If $X \subseteq Y$, and Y is bounded, then X is bounded.
3. If a in the basis of D, then $\{a\}$ is bounded.
4. If $\forall b \in Y(b \ll \bigvee X)$, then $\forall b \in Y(b \leq \bigvee X)$ by theorem 5, hence Y is bounded by $\bigvee X$.
5. If $X \subseteq Y$ and $a \ll \bigvee X$, then $a \ll \bigvee X \leq \bigvee Y$, hence $a \ll \bigvee Y$ by theorem 5.
6. If $a \ll \bigvee X$, then by the weak interpolation theorem there is a b such that $a \ll b \ll \bigvee X$.
   The other way round if $a \ll \bigvee Y \ll \bigvee X$, then $a \ll \bigvee X$ by theorem 5.

■

To extend *Rep* to a functor its value on continuous functions must be defined.

Theorem 32 *Let D, E be bounded complete continuous dcpo's, and $f : D \to E$ a continuous function, then the following defines a cam: $Rep(f) : Rep(D) \to Rep(E) : X Rep(f) b := b \ll f(\bigvee X)$.*

Proof: We check the clauses in the definition of a cam.

1. If $\forall b \in Y(b \ll f(\bigvee X))$, then $\forall b \in Y(b \leq f(\bigvee X))$ by theorem 5, hence Y is bounded by $f(\bigvee X)$.
2. If $X \subseteq X'$ and $b \ll f(\bigvee X)$, then $b \ll f(\bigvee X) \leq f(\bigvee X')$, hence $a \ll f(\bigvee X')$ by theorem 5.
3. If $b \ll f(\bigvee X')$, then by the weak interpolation theorem there is a b' such that $b \ll b' \ll f(\bigvee X')$. Furthermore, by the strong interpolation theorem there exists an a such that $b' \ll f(a)$ and $a \ll \bigvee X'$.
The other way round if $\bigvee X \ll \bigvee X' \& \bigvee Y \ll f(\bigvee X) \& b \ll \bigvee Y$, then $b \ll \bigvee Y \ll f(\bigvee X) \leq f(\bigvee X')$, hence $b \ll f(\bigvee X')$.

■

Theorem 33 *Rep : BCCont $\to$ clS is a functor.*

Proof: It is easy to see that *Rep* preserves the identity. We will show that it preserves composition, using the strong interpolation theorem.
$X Rep(g \circ f) a \Leftrightarrow$
$a \ll g(f(\bigvee X)) \Leftrightarrow$
$\exists Y(a \ll g(\bigvee Y) \& \bigvee Y \ll f(\bigvee X)) \Leftrightarrow$
$\exists Y(Y Rep(g) a \& X Rep(f) Y) \Leftrightarrow$
$X(Rep(g) \circ Rep(f)) a.$ ■

The two functors *Pt* and *Rep* form an equivalence of categories between clS and BCCont. This means that for every cis A: $A \cong Rep(Pt(A))$, and for every bounded complete continuous dcpo D: $D \cong Pt(Rep(D))$. In fact these isomorphisms need to be natural in A and D.
First we consider $Rep(Pt(A))$. The tokens of this cis are the elements of the basis of $Pt(A)$, which are the elements $[X]$, with $X \in Con_A$. The consistent sets α of $Rep(Pt(A))$ are bounded, finite sets of these tokens. The following lemma describes the way below relation $\ll$ in $Pt(A)$, and because $\alpha \vdash_{Rep(Pt(A))} [X]$ iff $[X] \ll \bigvee \alpha$ this describes the entailment relation in $Rep(Pt(A))$.

Lemma 34 *Suppose A is a cis, and $x, y \in Pt(A)$, then*

$$x \ll y \Leftrightarrow \exists Y(x \subseteq [Y] \& Y \subseteq y)$$

Proof: This is an easy consequence of the fact that $y = \bigvee \bar{y}$ as proven in theorem 27. ■

Theorem 35 *Let A be a cis. The cam $\mu_A : Rep(Pt(A)) \to A$ defined by $\alpha \mu_A a \Leftrightarrow a \in \bigvee \alpha$, has inverse $\nu_A : A \to Rep(Pt(A))$ defined by $Y \nu_A [X] \Leftrightarrow [X] \ll [Y]$ and is natural in A.*

Proof:

- μ_A is a cam.
  We check the clauses in the definition of a cam.

  1. Suppose $\alpha \mu_A Y$, then $Y \subseteq \bigvee \alpha$, hence because $\bigvee \alpha$ is finitely consistent $Y \in Con_A$.
  2. Suppose $\alpha \subseteq \alpha'$ and $\alpha \mu_A a$, then $a \in \bigvee \alpha \subseteq \bigvee \alpha'$, hence $a \in \bigvee \alpha'$, hence $\alpha' \mu_A a$.
  3. $\exists \alpha, Y(\alpha' \vdash \alpha \& \alpha \mu_A Y \& Y \vdash a) \Leftrightarrow$
     $\exists \alpha, Y(\bigvee \alpha \ll \bigvee \alpha' \& Y \subseteq \bigvee \alpha \& Y \vdash a) \Leftrightarrow$
     $\exists \alpha, Y, Z(\bigvee \alpha \subseteq [Z] \& Z \subseteq \bigvee \alpha' \& Y \subseteq \bigvee \alpha \& Y \vdash a) \Leftrightarrow$
     (($\Leftarrow$): If $Y \subseteq \bigvee \alpha'$, then $\forall a \in Y \exists Z_a \subseteq \bigvee \alpha'(Z_a \vdash a)$. Take $Z = \bigcup\{Z_a | a \in Y\}$, then $Z \subseteq \bigvee \alpha'$ consistent, and $Z \vdash Y$. Hence there exists Z' such that $Z \vdash Z' \vdash Y$. Take $\alpha = \{[Z']\}$.)
     $\exists Y(Y \subseteq \bigvee \alpha' \& Y \vdash a) \Leftrightarrow$
     $a \in \bigvee \alpha' \Leftrightarrow$
     $\alpha' \mu_A a$.

- ν_A is a cam.
  We check the clauses in the definition of a cam.

  1. Suppose $Y \nu_A \alpha$, then $[X] \ll [Y]$ for all $[X] \in \alpha$, hence $[X] \subseteq [Y]$. It follows that α is bounded by $[Y]$.
  2. Suppose $Y \subseteq Y'$ and $Y \nu_A [X]$, then $[X] \ll [Y] \subseteq [Y']$, hence $[X] \ll [Y']$ by theorem 5. It follows that $Y' \nu_A [X]$.
  3. $\exists Y, \alpha(Y' \vdash Y \& Y \nu_A \alpha \& \alpha \vdash [X]) \Leftrightarrow$
     $\exists Y, \alpha(Y' \vdash Y \& \bigvee \alpha \ll [Y] \& [X] \ll \bigvee \alpha) \Leftrightarrow$
     $[X] \ll [Y'] \Leftrightarrow$
     $Y' \nu_A [X]$

4 The Karoubi-envelope

Considering the definitions of continuous information system and approximable mapping, we see that the requirements on $\vdash$ and on an arbitrary cam f look very much alike. We make this formal by showing that clS can be constructed out of a category in which both $\vdash$ and f are special kind of arrows: clS is equivalent to the Karoubi-envelope of the category of qualitative information systems.

4.1 The category qIS

A qualitative information system A is a cis which satisfies $X \vdash_A a \Leftrightarrow a \in X$. However, we will give a direct definition.

Definition 43 *A* qualitative information system (qis) *A is a tuple* $< Dom_A, Con_A >$ *where*

- Dom_A *is a set, the set of* tokens,
- $Con_A \subseteq \mathcal{P}_f(Dom_A)$*, the set of* consistent sets,

satisfying the following clauses $(X, Y \in \mathcal{P}_f(Dom_A))$*:*

1. $\emptyset \in Con_A$
2. $X \subseteq Y \in Con_A \Rightarrow X \in Con_A$
3. $a \in Dom_A \Rightarrow \{a\} \in Con_A$

Maps between qualitative information systems also become very simple.

Definition 44 *A* qualitative approximable mapping (qam) *f between qualitative information systems A and B is a relation* $f \subseteq Con_A \times Dom_B$ *which satisfies:*

1. $XfY \Rightarrow Y \in Con_B$
2. $(X \subseteq X' \& Xfb) \Rightarrow X'fb$

Define qIS as the category with as objects qis and as arrows qam. The functors *Pt* and *Rep* from the previous section cut down to functors between qIS and Qd. Because qualitative domains are so concrete, this forms an isomorfy rather than an equivalence of categories.

Theorem 45 qIS $\cong$ Qd

4.2 The Karoubi-envelope of qIS

Given a category C a new category can be formed with as objects certain arrows of C.

Definition 46 *Let* C *be a category. Define the* Karoubi envelope $K(\mathsf{C})$ *of* C *as the category with as objects* idempotent *arrows* $f : A \to A$ *of* C *(i.e.* $f \circ f = f$*), and as arrows* $\phi : (f : A \to A) \to (g : B \to B)$ *arrows* $\phi : A \to B$ *of* C *such that* $g \circ \phi \circ f = \phi$*, or equivalently* $g \circ \phi = \phi$ *and* $\phi \circ f = \phi$*.*

Consider the category $K(\mathsf{qIS})$. It has as objects idempotent arrows $f : A \to A$ of qIS. Hence $f \circ f = f$, or by definition of composition in qIS: $\exists Y(XfY \& Yfb) \Leftrightarrow Xfb$. This is exactly clause 6 in the definition of a cis writing $\vdash_A$ for f. Clause 4 and 5 hold because f is an arrow in qIS, and clause 1,2, and 3 because A is a qis. Hence $f : A \to A$ is a cis. The other way round each cis gives an idempotent $\vdash_A: A \to A$ in qIS.
An arrow $\phi : (f : A \to A) \to (g : B \to B)$ in $K(\mathsf{qIS})$ is a qam $\phi : A \to B$ such that $g \circ \phi \circ f = \phi$. Writing this out, we find that ϕ satisfies $\exists X, Y(X'fX \& X\phi Y \& Ygb) \Leftrightarrow X'\phi b$, which is exactly clause 3 in the definition of a cam (writing $\vdash_A, \vdash_B$ for f, g). Clause 1 and 2 are satisfied by ϕ because it is a qam. Hence ϕ is a cam. The other way round it is easy to see that each cam gives an arrow in $K(\mathsf{qIS})$.

Theorem 47 $\mathsf{cIS} = K(\mathsf{qIS})$.

Corollary 48 $\mathsf{BCCont} \simeq K(\mathsf{Qd})$.

A nice characterisation of aIS can also be given. The arrows in qIS are ordered by $f \leq f' \Leftrightarrow (Xfb \Rightarrow Xf'b)$. It is clear that a qam $f : A \to A$ satisfies the axiom of reflexity iff $id_A \subseteq f$. Hence the full subcategory of $K(\mathsf{qIS})$ with as objects the idempotents which satisfy $id_A \leq f$ is equal to aIS.

Definition 49 *Let* C *be a category in which each hom-set is a poset. Define the* Closure Karoubi-envelope $K_c(\mathsf{C})$ *of* C *as the full subcategory of* $K(\mathsf{C})$ *with as objects closures, i.e. idempotent arrows* $f : A \to A$ *such that* $id_A \leq f$*.*

Theorem 50 $\mathsf{aIS} = K_c(\mathsf{qIS})$

Corollary 51 $\mathsf{BCAlg} \simeq K_c(\mathsf{Qd})$

5 Constructions

In this section some constructions in cIS are given. We shall take advantage of the result proved in the previous section by defining some datatypes (such as products and function types) in the "easy" category qIS, and then translating them to cIS by the mechanism of the Karoubi-envelope.

Proof: The terminal object in $K(\mathsf{C})$ is $!_T$, and the unique arrow from an object $f : A \to A$ of $K(\mathsf{C})$ to $!_T$ is $!_A$.
Let $f : A \to A$ and $g : B \to B$ be objects in $K(\mathsf{C})$. The product $f \times g$ is $< f \circ \pi, g \circ \pi' >$, and the projections $p_{f,g} : f \times g \to f$ and $p'_{f,g} : f \times g \to g$ are $f \circ \pi$, resp. $g \circ \pi'$. If $\phi : h \to f$ and $\psi : h \to g$ are arrows in $K(\mathsf{C})$, then $< \phi, \psi >$ is an arrow in $K(\mathsf{C})(h, f \times g)$.
The exponent f^g is $\Lambda(f \circ \varepsilon \circ (id_{AB} \times g))$, and the evaluation $e : f^g \times g \to f$ is $f \circ \varepsilon \circ (id_{AB} \times g)$. If $\phi : h \times g \to f$ is an arrow in $K(\mathsf{C})$, then $\Lambda(\phi)$ is an arrow in $K(\mathsf{C})(h, f^g)$. ∎

Definition 54 *An* ordered *semi-CCC is a semi-CCC such that each Hom-set is ordered, composition is monotone, and the following clauses are satisfied:*

1. *If* $f \leq f'$ *and* $g \leq g'$, *then* $< f, g > \leq < f', g' >$.
2. *If* $f \leq f'$, *then* $\Lambda(f) \leq \Lambda(f')$.
3. $id \leq < \pi, \pi' >$
4. $id \leq \Lambda(\varepsilon)$

Theorem 55 *If* C *is an ordered semi-CCC, then* $K_c(\mathsf{C})$ *is a CCC.*

Proof: It is easy to show that if f, g are closures, then $f \times g$ and f^g as defined in the proof of theorem 53 are closures. ∎

5.2 Constructions in qIS

We prove that **qIS** is a semi-CCC with finite products. Hence **qIS** has, among other things, a terminal object and binair products.

Theorem 56 qIS *has a terminal object.*

Proof: Let T be the qis defined by the following clauses:

- $Dom_T = \emptyset$
- $Con_T = \{\emptyset\}$

If A is an other qis, then there is an unique cam $\emptyset : A \to T$. ∎

Theorem 57 qIS *has products.*

Proof: Let A, B be qis. Define a new qis $A \times B$ as follows:

- $Dom_{A\times B} = Dom_A \uplus Dom_B$
- $Con_{A\times B} = \{X \uplus Y | X \in Con_A, Y \in Con_B\}$

where $\uplus$ is disjoint union: $S \uplus R = \{(s,1)|s \in S\} \cup \{(r,2)|r \in R\}$.
Define projections $\pi : A \times B \to A$ and $\pi' : A \times B \to B$ as $X \uplus Y \pi a \Leftrightarrow a \in X$, resp. $X \uplus Y \pi' b \Leftrightarrow b \in Y$. If $f : D \to A$ and $g : D \to B$ are cam, then define $< f, g >: D \to A \times B$ as $Z < f, g > c \Leftrightarrow ((c = (a,1) \wedge Zfa) \vee (c = (b,2) \wedge Zgb))$.
It is easy to check that everything is well-defined, and that $\pi \circ < f, g >= f$, $\pi' \circ < f, g >= g$. We shall prove that the equation $< \pi \circ f, \pi' \circ f >= f$ holds.
$Z < \pi \circ f, \pi' \circ f > X \uplus Y \Leftrightarrow$
$Z\pi \circ fX \& Z\pi' \circ fY \Leftrightarrow$
$\exists X_1, X_2, Y_1, Y_2 (ZfX_1 \uplus Y_1 \pi X \& ZfX_2 \uplus Y_2 \pi' Y) \Leftrightarrow$
$\exists X_1, X_2, Y_1, Y_2 (ZfX_1 \uplus Y_1 \& X \subseteq X_1 \& ZfX_2 \uplus Y_2 \& Y \subseteq Y_2) \Leftrightarrow$
$ZfX \uplus \emptyset \& Zf\emptyset \uplus Y \Leftrightarrow$
$ZfX \uplus Y$ ■

Theorem 58 qIS *is a semi-CCC with finite products.*

Proof: We already know that qIS has binair products and a terminal object.
Let A, B be qis. Define a new qis B^A as follows:

- $Dom_{B^A} = \{(X, b) | X \in Con_A, b \in Dom_B\}$
- $Con_{B^A} =$ the set of all finite subsets $\{(X_0, b_0), \ldots, (X_n, b_n)\}$ of Dom_{B^A} which satisfy $\forall I \subseteq \{0, \ldots, n\}(\bigcup\{X_i | i \in I\} \in Con_A \Rightarrow \{b_i | i \in I\} \in Con_B)$.

Define evaluation $\varepsilon_{A,B} : B^A \times A \to B$ as $F \uplus X \varepsilon_{A,B} b \Leftrightarrow \exists (X', b) \in F (X' \subseteq X)$. If $f : D \times A \to B$ is a qam, then define $\Lambda(f) : D \to B^A$ as $Z\Lambda(f)(X, b) \Leftrightarrow Z \uplus Xfb$.
It is easy to check that everything is well-defined. For example, we shall prove that $\Lambda(f)$ satisfies clause 1 in the definition of a qam: Suppose $Z\Lambda(f)F$, then for all $(X, b) \in F$ we have that $Z \uplus Xfb$. Take an arbitrary $F' \subseteq F$ such that $X' = \bigcup\{X | \exists b((X, b) \in F'\} \in Con_A$. Define $Y = \{b | \exists X((X, b) \in F')\}$. We show that $Y \in Con_B$. For an arbitrary $(X, b) \in F'$ we have $X \subseteq X'$, hence $Z \uplus X \subseteq Z \uplus X'$. For every $b \in Y$ there is a X such that $(X, b) \in F'$, hence $Z \uplus Xfb$. Therefore for every $b \in Y$ it holds that $Z \uplus X'fb$, hence $Z \uplus X'fY$ and $Y \in Con_B$.
Finally there are some equations to check.

- $\varepsilon \circ < \Lambda(f) \circ g, h >= f \circ < g, h >$
  We have

$Z\varepsilon < \Lambda(f)g, h > b \Leftrightarrow$
$\exists F, X(Z < \Lambda(f)g, h > F \uplus X\varepsilon b) \Leftrightarrow$
$\exists F, X(Z\Lambda(f)gF \& ZhX \& F \uplus X\varepsilon b) \Leftrightarrow$
$\exists F, X, X', Z'(ZgZ'\Lambda(f)F \& ZhX \& (X', b) \in F \& X' \subseteq X) \Leftrightarrow$
$\exists F, X, X', Z'(ZgZ' \& \forall (X'', b'') \in F(Z' \uplus X'' f b'') \& ZhX \& (X', b) \in F \& X' \subseteq X)$
$\Leftrightarrow$
$\exists X, X', Z'(ZgZ' \& ZhX \& Z' \uplus X' fb \& X' \subseteq X) \Leftrightarrow$
$\exists X, Z'(ZgZ' \& ZhX \& Z' \uplus Xfb) \Leftrightarrow$
$\exists X, Z'(Z < g, h > Z' \uplus X \& Z' \uplus Xfb) \Leftrightarrow$
$Zf < g, h > b$

- $\Lambda(f\circ < g \circ \pi, \pi' >) = \Lambda(f) \circ g$
  We have
  $Z\Lambda(f < g\pi, \pi' >)(X, b) \Leftrightarrow$
  $Z \uplus Xf < g\pi, \pi' > b \Leftrightarrow$
  $\exists Z', X'(Z \uplus X < g\pi, \pi' > Z' \uplus X' fb) \Leftrightarrow$
  $\exists Z', X'(Z \uplus X g\pi Z' \& Z \uplus X\pi' X' \& Z' \uplus X' fb) \Leftrightarrow$
  $\exists Z', Z'', X'(Z \uplus X\pi Z'' \& Z'' gZ' \& Z \uplus X\pi' X' \& Z' \uplus X' fb) \Leftrightarrow$
  $\exists Z', Z'', X'(Z'' \subseteq Z \& Z'' gZ' \& X' \subseteq X \& Z' \uplus X' fb) \Leftrightarrow$
  $\exists Z'(ZgZ' \& Z' \uplus Xfb) \Leftrightarrow$
  $\exists Z'(ZgZ' \& Z'\Lambda(f)(X, b)) \Leftrightarrow$
  $Z\Lambda(f)g(X, b)$

- $\varepsilon\circ < \pi, \pi' >= \varepsilon$
  This is trivial because qIS has products, and hence $< \pi, \pi' >= id$.

■

Ordering the arrows in qIS as before, this theorem can be strengthened.

Theorem 59 qIS *is an ordered semi-CCC with surjective pairing.*

Proof: Composition in qIS is monotone. It is easy to check that $< -, - >$ and $\Lambda(-)$ are monotone. Because qIS has products we have $< \pi, \pi' >= id$. Finally $id \leq \Lambda(\varepsilon)$ is easy. ■

5.3 Constructions in cIS

Because qIS is an ordered semi-CCC we know that K(qIS) and K_c(qIS) are CCC's by theorem 53. Moreover, the proof of this theorem is constructive, and we can translate the constructions in qIS to those of cIS and aIS.

Theorem 63 *If U is universal for C, then id_U is universal for $K(\mathsf{C})$.*

Proof: Suppose U is universal for C, and let $f : A \to A$ be an object of $K(\mathsf{C})$. There are arrows $r \in \mathsf{C}(U, A)$ and $s \in \mathsf{C}(A, U)$ such that $r \circ s = id_A$. It is clear that $f \circ r \in K(\mathsf{C})(id_U, f)$ and $s \circ f \in K(\mathsf{C})(f, id_U)$. Furthermore, $f \circ r \circ s \circ f = f \circ f = f = id_f$. ■

We shall define a countable qis U_1 such that each countable qis is a retract of U_1.
In general, there are two types of judgements which can be made about a qis A:

- p_a, where $p_a(A)$ is true iff $a \in Dom_A$
- q_X, where $q_X(A)$ is true iff $X \notin Con_A$

A qis A can be completely described by judgements of these two kinds.
Now the tokens of U_1 (i.e. the elements of Dom_{U_1}) will be these judgements. However, the tokens of an arbitrary $A \in \mathsf{qIS}_\omega$ might be looked at as natural numbers, because there is always an injective function $\mu : Dom_A \to \omega$. Hence, the tokens of U_1 are of the following two sorts:

- p_n, for $n \in \omega$
- q_N, for $N \in \mathcal{P}_f(\omega)$

Technically, we take $Dom_{U_1} = \omega \uplus \mathcal{P}_f(\omega)$, where $(n, 1)$ stands for p_n, and $(N, 2)$ for q_N.
Let $N \uplus \alpha$ be a finite subset of $\omega \uplus \mathcal{P}_f(\omega)$, then N should represent a consistent set, and α a set of inconsistent sets. Hence, $N \uplus \alpha$ is consistent in U_1 iff these two pieces of information do not contradict each other, i.e. $\forall N' \in \alpha(N' \not\subseteq N)$.

Theorem 64 *The qis U_1 defined by*

- $Dom_{U_1} = \omega \uplus (\mathcal{P}_f(\omega) - \{\emptyset\})$
- $N \uplus \alpha \in Con_{U_1} \Leftrightarrow \forall N' \in \alpha(N' \not\subseteq N)$

is universal for qIS_ω [1].

Proof: It is clear that U_1 is a countable qis (note that the singleton $\{(\emptyset, 2)\}$ would not be consistent!).
Let A be an object of qIS_ω, and $\mu : Dom_A \to \omega$ an injective function. Define $E : A \to U_1$ by

- $XE(n, 1) \Leftrightarrow \exists a \in X(\mu(a) = n)$
- $XE(N, 2) \Leftrightarrow \mu^{-1}(N) \notin Con_A$

[1] In fact U_1 is similar to the well-known universal domain T^ω ([10])

Define $R : U_1 \to A$ by

- $N \uplus \alpha R a \Leftrightarrow (\mu(a) \in N) \& \{N' | max(N') \leq \mu(a) \& \mu^{-1}(N') \notin Con_A\} \subseteq \alpha$

It is easy to prove that E, R are qam, and that $R \circ E = id_A$. ∎

Corollary 65 *The qis U_1 is universal for* clS_ω.

It is more difficult to find an universal information system which is based on judgements giving *positive* information, i.e. judgements that state that certain sets *are* consistent. We consider the following kind of judgements:

- $p_{a,\alpha}$, where $p_{a,\alpha}(A)$ is true iff $a \in Dom_A$ and $\forall X \in \alpha(X \in Con_A)$

Note that in one judgement we can declare more than one set to be consistent. Basically, the qis U_2 has these judgements as tokens. We take $Dom_{U_2} = \omega \times \mathcal{P}_f \mathcal{P}_f(\omega)$, where $< n, \alpha >$ stands for $p_{n,\alpha}$. A finite subset $\{< n_1, \alpha_1 >, \ldots, < n_m, \alpha_m >\}$ of $\omega \times \mathcal{P}_f \mathcal{P}_f(\omega)$ denotes the set $N = \{n_1, \ldots, n_m\}$. Hence, it is consistent iff each subset of N is declared to be consistent, i.e. $\forall I \subseteq \{1, \ldots, m\}, |I| > 1\ \exists i \in I(\{n_j | j \in I\} \in \alpha_i)$.

Theorem 66 *The qis U_2 defined by*

- $Dom_{U_2} = \omega \times \mathcal{P}_f \mathcal{P}_f(\omega)$
- $\{< n_1, \alpha_1 >, \ldots, < n_m, \alpha_m >\} \in Con_{U_2} \Leftrightarrow$
  $\forall I \subseteq \{1, \ldots, m\}, |I| > 1\ \exists i \in I(\{n_j | j \in I\} \in \alpha_i)$.

is universal for qIS_ω.

Proof: It is clear that U_2 is a countable qis. Let A be a countable qis, and $\mu : Dom_A \to \omega$ an injective function. Define $E : A \to U_1$ by

- $XE < n, \alpha > \Leftrightarrow \exists a \in X(n = \mu(a) \& \alpha = \{N' | max(N') \leq n \& \mu^{-1}(N') \in Con_A\}$

Define $R : U_2 \to A$ by

- $\{< n_1, \alpha_1 >, \ldots, < n_m, \alpha_m >\} Ra \Leftrightarrow$
  $\exists i(\mu(a) = n_i \& \alpha_i = \{N' | max(N') \leq n_i \& \mu^{-1}(N') \in Con_A\}$

It is easy to check that E, R are qam, and that $R \circ E = id_A$. ∎

Corollary 67 *The qis U_2 is universal for* clS_ω.

Appendix

A Notation

Category	Objects	Arrows
Dcpo	dcpo's	continuous functions
Cont	continuous dcpo's	continuous functions
BCCont	bounded complete continuous dcpo's	continuous functions
Alg	algebraic dcpo's	continuous functions
BCAlg	bounded complete algebraic dcpo's	continuous functions
Qd	qualitative domains	continuous functions
cIS	continuous information systems	continuous approximable mappings
aIS	algebraic information systems	algebraic approximable mappings
qIS	qualitative information systems	qualitative approximable mappings

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- Terminal object
  The terminal object in clS is given by the terminal object in qlS, with as entailment relation the arrow $!_T$ which is the empty relation.

- Product
  The product of two cis A, B in clS is given by their product in qlS, together with the entailment relation given by $<\vdash_A o\pi, \vdash_B o\pi'>$. If we write this out we find that $X \uplus Y \vdash_{A,B} c \Leftrightarrow ((c = (a,1) \wedge X \vdash_A a) \vee (c = (b,2) \wedge Y \vdash_B b))$.
  The first projection $p : A \times B \rightarrow A$ in clS is given by $\vdash_A o\pi$, hence $X \uplus Ypa \Leftrightarrow X \vdash_A a$. The second projection is defined analogously.

- Exponents
  The exponent of two cis A, B in cIS is given by their (semi-)exponent in qlS, together with the entailment relation given by $\Lambda(\vdash_A o\varepsilon \circ (id_{A^B} \times \vdash_B))$. If we write this out we find that $F \vdash_{B^A} (X, b) \Leftrightarrow \{b' | \exists X'((X', b) \in F \& X \vdash_A X')\} \vdash_B b$.
  The evaluation $e : B^A \times A \rightarrow B$ is given by $\vdash_A o\varepsilon \circ (id_{A^B} \times \vdash_B)$, hence $F \uplus Xeb \Leftrightarrow F \vdash_{b^A} (X, b)$. The operation $\Lambda(-)$ in clS is the same as in qlS.

The constructions in alS are the same as those in clS.

6 Universal Information Systems

Intuitively, an information system U is *universal* for a certain category C of information systems iff $U \in$ C and each $A \in$ C can be "embedded" in U.

Definition 60 *Let A, B be objects in a category* C*, then B is a* retract *of A iff there are arrows $r \in$ C(A, B) and $s \in$ C(B, A) such that $r \circ s = id_B$. In this case r is called a* retraction.

Definition 61 *Let $U \in$ C, then U is* universal *for* C *iff each $A \in$ C is a retract of U.*

Example 62 *Let* Set_ω *be the category of countable sets and functions. A function is a retract iff it is injective. Hence, the set of natural numbers ω (and in general every infinite countable set) is universal for* Set_ω.

It is clear that to find universal information systems we have to set a bound on the cardinality. If C $\in$ {clS, alS, qlS}, then C_ω denotes the full subcategory of C with as objects information systems A such that Dom_A is countable. As in the previous sections, these countable information systems correspond to the various kinds of domains with a countable basis B_D.
We are interested in universal information systems for the category clS_ω. However, just as in the previous section, we can work in the "easy" category qlS_ω, and translate to clS_ω. In fact, the following theorem says that it is enough to find an universal information system for qlS_ω.

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5.1 Semi Cartesian Closed Categories

A *semifunctor* is defined just like a functor, except that it need not preserve identities. Various definitions of category theory which apply to functors, can be generalised to semifunctors. For example [3] generalises the notion of adjunction to the notion of *semiadjunction.* By using semifunctors rather than functors in the definition of a Cartesian closed category (CCC) we get *semi Cartesian closed categories (semi-CCC's).* Of course each CCC is a semi-CCC, just like each functor is a semifunctor. We shall repeat the algebraic description of semi-CCC's of [3].

Definition 52 *A* semi Cartesian closed category (semi-CCC) C *is a category* C *with the following data:*

- *An object $T \in \mathsf{C}$, and for each object $A \in \mathsf{C}$ an arrow $!_A \in \mathsf{C}(A,T)$.*
- *For each pair of objects $A, B \in \mathsf{C}$ an object $A \times B \in \mathsf{C}$, and arrows $\pi_{A,B} \in \mathsf{C}(A \times B, A)$ and $\pi'_{A,B} \in \mathsf{C}(A \times B, B)$. Furthermore, for each pair of arrows f, g, with $f \in \mathsf{C}(D, A)$ and $g \in \mathsf{C}(D, B)$, an arrow $< f, g > \in \mathsf{C}(D, A \times B)$.*
- *For each pair of objects $A, B \in \mathsf{C}$ an object $B^A \in \mathsf{C}$, and an arrow $\varepsilon_{A,B} \in \mathsf{C}(B^A \times A, B)$. Furthermore, for each arrow $f \in \mathsf{C}(D \times A, B)$ an arrow $\Lambda(f) \in \mathsf{C}(D, B^A)$.*

satisfying the following equations (omitting subscripts):

1. $! \circ f = !$
2. $\pi \circ < f, g > = f$
3. $\pi' \circ < f, g > = g$
4. $< f, g > \circ h = < f \circ h, g \circ h >$
5. $\varepsilon \circ < \Lambda(f) \circ g, h > = f \circ < g, h >$
6. $\Lambda(f \circ < g \circ \pi, \pi' >) = \Lambda(f) \circ g$
7. $\varepsilon \circ < \pi, \pi' > = \varepsilon$

A semi-CCC C has *finite products* iff it satisfies $< \pi, \pi' > = id$ and $!_T = id_T$. This means that $A \times B$ is the categorical product in C, and that T is a terminal object. If C also satisfies $\Lambda(\varepsilon) = id$, then C is a CCC. ([7]).
The Karoubi-envelope transforms various semi notions to corresponding (normal) notions. For example, if we apply the Karoubi-envelope construction to semifunctors and semiadjunctions then we get functors, resp. adjunctions.

Theorem 53 *If* C *is a semi-CCC, then $K(\mathsf{C})$ is a CCC.*

2. If $X \subseteq x$ and $a \ll \bigvee X$, then $a \in x$.

3. If $a \in x$, then there is a finite $X \subseteq x$ such that $a \ll \bigvee X$.

Theorem 36 *Let D be a bounded complete continuous dcpo. The continuous function $\delta_D : Pt(Rep(D)) \to D : x \mapsto \bigvee x$ has inverse $\epsilon_D : D \to Pt(Rep(D)) : d \mapsto B_D(d)$ and is natural in D.*

Proof:

- δ_D is a continuous function.
  Define for an arbitrary element x the set $S_x = \{\bigvee X | X \subseteq x, \text{ and } X \text{ is finite }\}$. The set S_x is directed, hence $\bigvee S_x$ exists, and $\bigvee S_x = \bigvee x$, hence $\bigvee x$ exists. It follows that δ_D is well-defined. It is easy to check that δ_D is continuous.

- ϵ_D is a continuous function.
  We check that $\epsilon_D(d)$ is an element of $Pt(Rep(D))$.

  1. If $X \subseteq \epsilon_D(d)$, then $\forall d' \in X(d' \ll d)$, hence $\forall d' \in X(d' \leq d)$, hence X is bounded by d.
  2. If $X \subseteq \epsilon_D(d)$ and $a \ll \bigvee X$, then $a \ll \bigvee X \ll d$, hence $a \ll d$, and $a \in \epsilon_D(d)$.
  3. If $a \in \epsilon_D(d)$, then $a \ll d$, hence by the weak interpolation theorem there exists an a' such that $a \ll a' \ll d$.

  It is easy to check that ϵ_D is continuous.

- $\delta_D \circ \epsilon_D = id_D$
  We have:
  $\delta_D(\epsilon_D(d)) =$
  $\bigvee B_D(d) =$
  d.

- $\epsilon_D \circ \delta_D = id_{Pt(Rep(D))}$
  We have:
  $\epsilon_D(\delta_D(x)) =$
  $\epsilon_D(\bigvee x) =$
  $B_D(\bigvee x)$
  We have to show that this set is equal to x. Suppose $a \in x$, then there exists a finite $X \subseteq x$ such that $a \ll \bigvee X$. Hence $a \ll \bigvee X \ll \bigvee x$, and $a \ll \bigvee x$. For the reverse inclusion suppose $a \ll \bigvee x$. Because $\bigvee x \leq \bigvee S_x$ there is a finite $X \subseteq x$ such that $a \leq \bigvee X$. But for each $b \in X$ there exists an $Y_b \subseteq x$ such that $b \ll \bigvee Y_b$. Take $Y = \bigcup\{Y_b | b \in X\}$, then $b \ll \bigvee Y_b \leq \bigvee Y$ for each $b \in X$, hence $b \ll \bigvee Y$, hence $\bigvee X \ll \bigvee Y$. It follows that $a \ll \bigvee Y$ with $Y \subseteq x$, hence $a \in x$.

hypothesis If Y has size n and $X \vdash Y$, then $X \cup Y \in Con_A$.

step Suppose the size of Y is $n+1$ and $X \vdash Y$. Because $Y \neq \emptyset$ there is a $b' \in Y$. Take $Y' = Y - \{b'\}$, then $X \vdash Y'$ and by induction hypothesis $X \cup Y' \in Con_A$. Because $X \vdash b'$ we have by clause 5 $X \cup Y' \vdash b'$, hence by entailment consistency $X \cup Y = X \cup Y' \cup \{b'\} \in Con_A$.

Finally $Y \subseteq X \cup Y \in Con_A$, hence by clause 2 it follows that $Y \in Con_A$. ∎

Because for a bounded complete continuous dcpo $Rep(D)$ is entailment consistent the proof in the previous subsection of equivalence between clS and BCCont can be repeated for clS_{ec}. So clS $\simeq$ BCCont $\simeq$ clS_{ec}, and we get clS $\simeq$ clS_{ec}.
We could have taken entailment consistent cis as our continuous information systems (by replacing clause 4 in the definition of a cis by the axiom of entailment consistency). However in section 3 we shall be rewarded for having the more general definition of a continuous information system. There we will prove a theorem which is immediate with our notion of a cis.

3.4 Reflexivity

The category of algebraic information systems is a subcategory of clS.

Definition 40 *A cis A is* reflexive *iff it satisfies*

$$\forall X \in Con_A \forall a \in Dom_A (a \in X \Rightarrow X \vdash a)$$

Theorem 41 *A is a reflexive cis iff it is an algebraic information system.*

Proof: Let A be a reflexive cis. We only have to prove that A is entailment consistent. Suppose $X \vdash_A a$, then by reflexivity $X \vdash_A X \cup \{a\}$, hence by clause 4 in the definition of a cis it follows that $X \cup \{a\} \in Con_A$.
Let A be an algebraic information system. It is trivial that A satisfies the first three clauses in the definition of a cis. For clause 5 suppose $X \subseteq Y$ and $X \vdash a$, then $Y \vdash X \vdash a$, hence $Y \vdash a$. Clause 4 follows then by theorem 39 because A is entailment consistent. For clause 6 from right to left take $Y = X$. ∎

Continuous approximable mappings between reflexive cis are the same as algebraic approximable mappings.

Theorem 42 *Let A, B be reflexive cis, then $f : A \longrightarrow B$ is a cam iff it is an algebraic approximable mapping.*

Proof: Straightforward. ∎

- $\nu_A \circ \mu_A = I_{Rep(Pt(A))}$
  We have:
  $\alpha \nu_A \circ \mu_A [X] \Leftrightarrow$
  $\exists Y(\alpha \mu_A Y \& Y \nu_A [X]) \Leftrightarrow$
  $\exists Y(Y \subseteq \bigvee \alpha \& [X] \ll [Y]) \Leftrightarrow$
  $[X] \ll \bigvee \alpha \Leftrightarrow$
  $\alpha I_{Rep(Pt(A))}[X]$

- $\mu_A \circ \nu_A = I_A$
  We have:
  $Y \mu_A \circ \nu_A a \Leftrightarrow$
  $\exists \alpha(Y \mu_A \alpha \& \alpha \nu_A a) \Leftrightarrow$
  $\exists \alpha(\bigvee \alpha \ll [Y] \& a \in \bigvee \alpha) \Leftrightarrow$
  $a \in [Y] \Leftrightarrow$
  $Y \vdash a \Leftrightarrow$
  $Y I_A a$

- μ_A is natural in A.
  Let $f : A \to B$ be a cam. We have to show that $\mu_B \circ Rep(Pt(f)) = f \circ \mu_A$.
  First we consider $Rep(Pt(f))$.
  $\alpha Rep(Pt(f))[Y] \Leftrightarrow$
  $[Y] \ll Pt(f)(\bigvee \alpha) \Leftrightarrow$
  $\exists Z([Y] \subseteq [Z] \& Z \subseteq Pt(f)(\bigvee \alpha)) \Leftrightarrow$
  $\exists Z([Y] \subseteq [Z] \& Z \subseteq \{b | \exists X \subseteq \bigvee \alpha(X f b)\})$
  We have:
  $\alpha(\mu_B \circ Rep(Pt(f)))b' \Leftrightarrow$
  $\exists \beta(\alpha Rep(Pt(f))\beta \& \beta \mu_B b') \Leftrightarrow$
  $\exists \beta, Z(Z \subseteq \{b | \exists X \subseteq \bigvee \alpha(X f b)\} \& \bigvee \beta \subseteq [Z] \& b' \in \bigvee \beta) \Leftrightarrow$
  (($\Rightarrow$): $\forall b \in Z \exists X_b \subseteq \bigvee \alpha(X_b f b)$. Take $X = \bigcup\{X_b | b \in Z\}$, then $X \subseteq \bigvee \alpha$, hence X is consistent. $X_b \subseteq X$ and $X_b f b$ for each $b \in Z$, hence $X f b$. It follows that $X f Z$. Furthermore, $b' \in \bigvee \beta \subseteq [Z]$, hence $Z \vdash b'$, hence $X f b'$.
  ($\Leftarrow$): Suppose $\exists X(X \subseteq \bigvee \alpha \& X f b')$. There is an Y such that $X f Y \vdash b'$, and hence there is an Z such that $X f Z \vdash Y \vdash b'$. Take $\beta = \{[Y]\}$.)
  $\exists X(X \subseteq \bigvee \alpha \& X f b') \Leftrightarrow$
  $\exists X(\alpha \mu_A X \& X f b') \Leftrightarrow$
  $\alpha(f \circ \mu_A)b'$.

■

We consider $Pt(Rep(D))$, with D a bounded complete continuous dcpo. A point $x \in Pt(Rep(D))$ is an element of $Rep(D)$. Hence x is a subset of the basis of D satisfying the following:

1. If $X \subseteq x$ and X finite, then X is bounded.

- δ_D is natural in D.
Suppose $f : D \to E$ is a continuous function. We have to show that $\delta_E \circ Pt(Rep(f)) = f \circ \delta_D$.
$\delta_E(Pt(Rep(f)(x)) =$
$\delta_E(\{e | \exists X \subseteq x(X Rep(f) e)\} =$
$\delta_E(\{e | \exists X \subseteq x(e \ll f(\bigvee X))\}) =$
$\bigvee(\{e | \exists X \subseteq x(e \ll f(\bigvee X))\}) =$
$\bigvee\{f(\bigvee X) | X \subseteq x\} =$
$f(\bigvee S_x) =$
$f(\bigvee x) =$
$f(\delta_D(x))$.

■

3.3 Entailment consistency

There is a subcategory of clS which is equivalent to the whole category clS.

Definition 37 *A cis A is* entailment consistent *iff it satisfies*

$$\forall X \in Con_A \forall a \in Dom_A (X \vdash_A a \Rightarrow X \cup \{a\} \in Con_A)$$

Define clS_{ec} as the full subcategory of clS with as objects the entailment consistent cis. clS_{ec} is a subcategory of clS.

Example 38 *Let A be the following cis:*

- $Dom_A = [0, 1]$
- $Con_A = \{X | X \subseteq [0, 1] \& 1 \notin X \& X \text{ finite }\} \cup \{\{1\}\}$
- $X \vdash_A a = a < \bigvee X$

The cis A is not entailment consistent. For example $\{1\} \vdash_A 0$, *but* $\{0, 1\}$ *is not consistent.*

For a entailment consistent cis clause 4 in the definition of a cis is unnecessary.

Theorem 39 *Clause 4 in the definition of a cis is the result of clause 2,5 and entailment consistency.*

Proof: Suppose $X \vdash Y$, then we prove by induction to the size n of Y that $X \cup Y \in Con_A$ using only clause 5 and entailment consistency.

basis If $n = 0$, then $X \cup Y = X \cup \emptyset = X \in Con_A$.

An algebraic approximable mapping is fully determined by giving the pairs of the form $Xf\{b\}$. Hence we can give an alternative definition of an algebraic approximable mapping as a relation between $Con_A \times Dom_B$.

Definition 17 *An* algebraic approximable mapping (aam) f *between algebraic information systems* A *and* B *is a relation* $f \subseteq Con_A \times Dom_B$ *which satisfies:*

1. $XfY \Rightarrow Y \in Con_B$
2. $\exists X, Y(X' \vdash_A X \& XfY \& Y \vdash_B b) \Rightarrow X'fb$

where we abbreviate $\forall b \in Y(Xfb)$ *as* XfY.

The identity aam $I_A : A \to A$ is defined as $XI_Aa \Leftrightarrow X \vdash_A a$. Let $f : A \to B$ and $g : B \to C$ be aam, then their composition $g \circ f : A \to C$ is defined as $X(g \circ f)c := \exists Y(XfY \& Ygc)$. aIS is the category with as objects algebraic information systems and as arrows algebraic approximable mappings.
We shall not go into the precise manner in which an algebraic information system presents a Scott domain, and in which we can transform an arbitrary Scott domain into an algebraic information system (see [8, 11]). In the next section a more general situation will be considered.

Theorem 18 aIS $\simeq$ BCAlg

3 Continuous information systems

In this section we define the category of continuous information systems and approximable mappings. Continuous information systems are concrete representations of bounded complete continuous dcpo's. This correspondence is similar to that between algebraic information systems and Scott domains.

3.1 The category cIS

Definition 19 *A* continuous information system (cis) A *is a tuple* $< Dom_A, Con_A, \vdash_A >$ *where*

- Dom_A *is a set, the set of* tokens,
- $Con_A \subseteq \mathcal{P}_f(Dom_A)$, *the set of* consistent sets,
- $\vdash_A \subseteq Con_A \times Dom_A$, *the* entailment relation,

satisfying the following clauses $(X, Y \in \mathcal{P}_f(Dom_A))$:

1. $\emptyset \in Con_A$
2. $X \subseteq Y \in Con_A \Rightarrow X \in Con_A$

Continuous Information Systems

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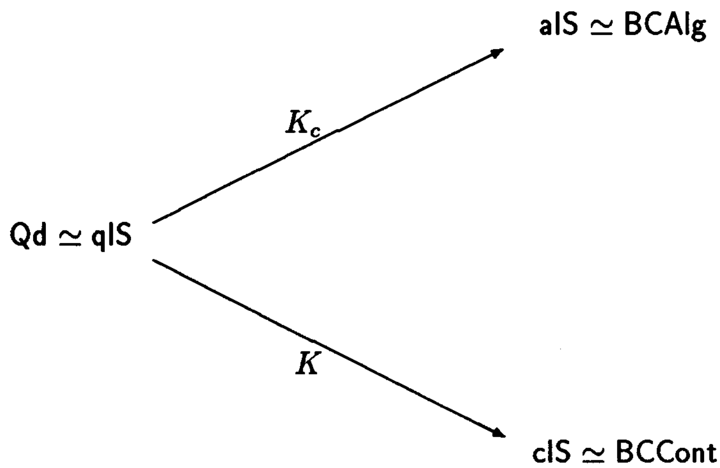
Abstract

In this paper we generalise the notion of *algebraic information system* ([8],[11]) to that of *continuous information system.* Just as algebraic information systems are concrete representations of bounded complete algebraic dcpo's (Scott domains), continuous information systems are concrete representations of bounded complete continuous dcpo's.
A certain subclass of information systems, consisting of the so-called qualitative information systems, which corresponds to the class of qualitative domains ([1]), is basic in the sense that all other information systems are generated by this class. It follows that the category of bounded complete continuous dcpo's and continuous functions is equivalent to the Karoubi-envelope of the category of qualitative domains and continuous functions.
Furthermore, we show how certain constructions on qualitative information systems (such as product and function space) can be "translated" to constructions on continuous information systems. Among other things, it is proven that the category of qualitative domains and continuous functions is a semi Cartesian closed category ([3]). Finally, two *universal* information systems are defined.

1 Introduction

Scott domains (i.e. bounded complete algebraic dcpo's) are a special kind of posets which are used in the mathematical semantics of programming languages. In general, Scott domains are rather *abstract* structures. In [11] an alternative type of structures was defined, called *information systems.* These information systems are much more intuitive than Scott domains.
However, it was proven that information systems are concrete representations of Scott domains. Each Scott domain can be presented as an information system, and the other way round each information system can be converted into a Scott domain. Technically, there is an equivalence of categories between the category BCAlg of Scott

domains and continuous functions and the category aIS of information systems ([8]). Some important data types can not be represented as Scott domains. For example, the set of real numbers with the usual ordering is not a Scott domain. Therefore the class of Scott domains is widened to the class of *continuous* Scott domains (i.e. bounded complete continuous dcpo's [2]). In this paper we shall define *continuous* information systems, which are concrete presentations of continuous Scott domains. To avoid confusion the usual information systems will be called *algebraic* information systems. Just as in the algebraic case, we are able to prove that the category BCCont of continuous Scott domains and continuous functions and the category cIS of continuous information systems are equivalent.
The category aIS is a full subcategory of cIS, corresponding to the inclusion between BCAlg and BCCont. In fact, algebraic information systems are exactly the *reflexive* continuous information systems.
Another important subcategory of cIS (and aIS) is the category qIS of *qualitative* information systems. These structures are more simple than general information systems. It is easy to show that qualitative information systems correspond to *qualitative domains* ([1]), i.e. there is an equivalence of categories between the category Qd of qualitative domains and continuous functions and qIS.



The category qIS is basic in a special sense: both cIS and aIS can be constructed out of qIS in a very natural way. Technically, we shall prove that the *Karoubi envelope* K(qIS) is equivalent to cIS, and that the *Closure Karoubi envelope* K_c(qIS) is equivalent to aIS. Among other things, this implies that the Karoubi envelope of the category of qualitative domains Qd is equivalent to the category of Scott domains BCAlg. Hence qualitative domains *underlie* Scott domains in a certain sense.
In a way this seems strange, because the category Qd is for example not even Cartesian closed. For that reason in [1] the Cartesian closed subcategory of Qd was considered with the same objects as Qd, but with *stable* continuous functions as arrows. However, we shall see that Qd comes very close to being a Cartesian closed

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