A Note on Semi-Adjunctions

R. Hoofman

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Utrecht University
Department of Computer Science
Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

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R. Hoofman
Department of Computer Science, Utrecht University
P.O. Box 80.089, 3508 TB Utrecht, the Netherlands

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Abstract

We consider two methods to generalise categorical notions involving functors to *semi*-functors. It turns out that the two methods give in general the same results. In particular, the notion of *semi-adjunction* is studied. Semi-adjunctions in our sense prove to be exactly the *normal* semi-adjunctions of Hayashi [2].

1 Introduction

In [2] the notion of *semi*-functor is introduced. A *semi*-functor is a functor except that it need not preserve identities. Various categorical notions involving functors can be generalised to semi-functors. For example, [2] defines the notion of a *semi*-adjunction.

In this paper we consider two methods to generalise in a systematic way categorical notions. The first method uses the *Karoubi envelope* construction to translate ordinary notions to semi-notions. The Karoubi envelope construction is a canonical way to transform semi-functors into functors. The second method uses the fact that the category $\mathbf{Cat}_s$ of categories and semi-functors is a 2-category. Various categorical notions involving functors and natural transformations can be generalised to arbitrary 2-categories, and in particular to $\mathbf{Cat}_s$.

It turns out that, in general, the two methods give the same results. In particular, the two methods give the same kind of generalisation of the notion of adjunction. Semi-adjunctions in our sense correspond to the *normal* semi-adjunctions of [2].

The rest of this paper is organised as follows. In section 2 the notion of semi-functor and the Karoubi envelope construction are reviewed. In section 3 semi-adjunctions are defined by means of the first method. In section 4 the relation between our semi-adjunctions and the semi-adjunctions of [2] is studied. In section 5 semi natural transformations are introduced, and in section 6 we show that they give $\mathbf{Cat}_s$ the structure of a 2-category. Finally, in section 7 semi-adjunctions are defined by means of the second method, and we show that they coincide with the semi-adjunctions of the third section.

2 Semi-functors

Let C, D be categories. A *semi-functor* $F : \mathsf{C} \to \mathsf{D}$ is defined just as a functor, except that it need not preserve identities [2].

Example 1 *Let* Set *be the category of sets and functions. The semi-functor* $F : \mathsf{Set} \to \mathsf{Set}$ *is defined on objects by* $F(A) = A \times A$ *and on functions* $f : A \to B$ *by* $F(f)(< a, a' >) = < f(a), f(a) >$*. If* $a, a' \in A$ *and* $a \neq a'$*, then* $F(id)(< a, a' >) = < a, a > \neq < a, a' >$*.*

Let Cat be the category of categories and functors, and let Cat_s be the category of categories and semi-functors. Because each functor is also a semi-functor, there is an inclusion $I : \mathsf{Cat} \to \mathsf{Cat}_s$. The *Karoubi envelope* construction provides a functor in the other direction.

Definition 2 *The* Karoubi envelope $\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}$ *is the functor defined by*

objects *If* C *is a category, then* $\mathcal{K}(\mathsf{C})$ *is the category with as objects the* idempotent *arrows* $f \in \mathsf{C}(A, A)$ *(i.e.* $f \circ f = f$*). An arrow* $\phi \in \mathcal{K}(\mathsf{C})(f : A \to A, g : B \to B)$ *is an arrow* $\phi \in \mathsf{C}(A, B)$ *such that* $g \circ \phi \circ f = \phi$*. Composition is as in* C*, and* $id_f = f$*.*

arrows *If* $F : \mathsf{C} \to \mathsf{D}$ *is a semi-functor, then* $\mathcal{K}(F) : \mathcal{K}(\mathsf{C}) \to \mathcal{K}(\mathsf{D})$ *is the functor defined on objects* f *by* $\mathcal{K}(F)(f) = F(f)$*, and on arrows* ϕ *by* $\mathcal{K}(F)(\phi) = F(\phi)$*.*

The Karoubi envelope $\mathcal{K}(F)$ of a semi-functor F preserves identities because

$$\mathcal{K}(F)(id_f) = F(f) = id_{F(f)}$$

In fact, the Karoubi envelope construction is a *canonical* way to transform semi-functors into functors.

Theorem 3 *[1]* $\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}$ *is a right-adjoint of* $I : \mathsf{Cat} \to \mathsf{Cat}_s$*.*

Proof: We have to show that there is a natural isomorphism

$$\mathsf{Cat}_s(\mathsf{C}, \mathsf{D}) \cong \mathsf{Cat}(\mathsf{C}, \mathcal{K}(\mathsf{D}))$$

Given a semi-functor $F : \mathsf{C} \to \mathsf{D}$ define a functor $F' : \mathsf{C} \to \mathcal{K}(\mathsf{D})$ by $F'(A) = F(id_A)$ on objects A, and $F'(f) = F(f)$ on arrows f. The other way round, if $G : \mathsf{C} \to \mathcal{K}(\mathsf{D})$ is a functor, then the semi-functor $G' : \mathsf{C} \to \mathsf{D}$ is defined by $G'(A) = Dom_{G(A)}$ and $G'(f) = G(f)$.
In this way we get a natural isomorphism between the Hom-sets. ■

It follows that Cat is a *coreflective* subcategory of Cat_s.

Example 4 *The counit $S = S_{\mathsf{C}} : \mathcal{K}(\mathsf{C}) \to \mathsf{C}$ of the adjunction $I \dashv \mathcal{K}$ provides a nice example of a semi-functor. It is defined on objects $f : A \to A$ by $S(f) = A$ and on arrows ϕ by $S(\phi) = \phi$.*

3 Semi-adjunctions

We would like to generalise categorical definitions involving functors to *semi*-functors. One way to do this is by transferring notions from **Cat** to **Cat$_s$** by means of the Karoubi envelope.
For example, how should the notion of *adjunction between semi-functors* be defined? This might be done by saying that a pair of semi-functors F, G form a *semi-adjunction* iff $\mathcal{K}(F), \mathcal{K}(G)$ form an adjunction. We shall give a more simple definition, and show that it coincides with the one above.
First, let $F : \mathsf{C} \to \mathsf{D}$ be a semi-functor. There is a natural transformation $F(id) : F \to F$ with components $F(id_A) : FA \to FA$. We write $\mathsf{D}(FA, B)_s$ for the set of arrows $f \in \mathsf{D}(FA, B)$ which satisfy

$$f \circ F(id_A) = f$$

Note that $\mathsf{D}(FA, B)_s = \mathcal{K}(\mathsf{D})(F(id_A), id_B)$. If F happens to be a functor, then $\mathsf{D}(FA, B)_s = \mathsf{D}(FA, B)$. Analogously, the set $\mathsf{D}(B, FA)_s$ is defined.

Definition 5 *A* semi-adjunction *is a tuple $< F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}, \mu >$, where F, G are semi-functors and μ is a natural isomorphism*

$$\mu_{A,B} : \mathsf{D}(F(A), B)_s \cong \mathsf{C}(A, G(B))_s$$

Notation: we write $F \dashv_s G$ iff F, G are part of one semi-adjunction as defined.
If F, G are functors, then $F \dashv_s G \Leftrightarrow F \dashv G$.

Example 6 *Let $R = R_{\mathsf{C}} : \mathsf{C} \to \mathcal{K}(\mathsf{C})$ and $S = S_{\mathsf{C}} : \mathcal{K}(\mathsf{C}) \to \mathsf{C}$ be unit and counit of the adjunction $I \dashv \mathcal{K}$. Then $S \dashv_s R$ and $R \dashv_s S$, because*

$$\mathsf{C}(S(f), B)_s = \mathcal{K}(\mathsf{C})(f, R(B))$$

and

$$\mathcal{K}(\mathsf{C})(R(A), g) = \mathsf{C}(A, S(g))_s$$

There is a semi-adjunction between two semi-functors iff their Karoubi envelope form an adjunction.

Theorem 7 *$F \dashv_s G$ iff $\mathcal{K}(F) \dashv \mathcal{K}(G)$.*

Proof:

- ($\Leftarrow$): Suppose $\mathcal{K}(F) \dashv \mathcal{K}(G)$, then

$$\mathcal{K}(\mathsf{D})(\mathcal{K}(F)(f), g) \cong \mathcal{K}(\mathsf{C})(f, \mathcal{K}(G)(g))$$

Hence in particular

$$\mathcal{K}(\mathsf{D})(F(id_A), id_B) \cong \mathcal{K}(\mathsf{C})(id_A, G(id_B))$$

Because $\mathcal{K}(\mathsf{D})(F(id_A), id_B) = \mathsf{D}(F(A), B)_s$ and $\mathcal{K}(\mathsf{C})(id_A, G(id_B)) = \mathsf{C}(A, G(B))_s$ it follows that

$$\mathsf{D}(F(A), B)_s \cong \mathsf{C}(A, G(B))_s$$

It is easy to see that this isomorphism is natural in A, B.

- ($\Rightarrow$): Suppose $F \dashv_s G$, then there is a natural isomorphism

$$\mu_{A,B} : \mathsf{D}(F(A), B)_s \cong \mathsf{C}(A, G(B))_s$$

We have to define a natural isomorphism

$$\nu_{f,g} : \mathcal{K}(\mathsf{C})(\mathcal{K}(F)(f), g) \cong \mathcal{K}(\mathsf{C})(f, \mathcal{K}(G)(g))$$

for idempotents $f : A \to A, g : B \to B$. We show that we can take $\nu_{f,g} = \mu_{A,B}$. First $\nu_{f,g}(\phi)$ is defined for $\phi : F(f) \to g$:

$$\begin{aligned} \phi \circ F(id) &= g \circ \phi \circ F(f) \circ F(id) \\ &= g \circ \phi \circ F(f) \\ &= \phi \end{aligned}$$

Second $\nu_{f,g}(\phi)$ is well-defined:

$$\begin{aligned} G(g) \circ \nu_{f,g}(\phi) \circ f &= G(g) \circ \mu_{A,B}(\phi) \circ f \\ &= \mu_{A,B}(g \circ \phi \circ F(f)) \\ &= \mu_{A,B}(\phi) \\ &= \nu_{f,g}(\phi) \end{aligned}$$

It is easy to check that $\nu_{f,g}$ is a natural isomorphism.

■

4 Hayashi semi-adjunctions

In [2] Hayashi gave a definition of semi-adjunction.

Definition 8 *A* Hayashi semi-adjunction *is a tuple* $< F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}, \alpha, \beta >$*, where* F, G *are semi-functors and* α, β *are collections of arrows* $\{\alpha_{A,B}, \beta_{A,B}\}_{A \in \mathsf{C}, B \in \mathsf{D}}$ *such that the four squares in the following diagram commute.*

$$
\begin{array}{ccccccc}
B & & \mathsf{D}(FA,B) & \underset{\beta_{A,B}}{\overset{\alpha_{A,B}}{\rightleftarrows}} & \mathsf{C}(A,GB) & & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \circ \_ \circ Fg} & & \downarrow{\scriptstyle Gf \circ \_ \circ g} & & \uparrow{\scriptstyle g} \\
B' & & \mathsf{D}(FA',B') & \underset{\beta_{A',B'}}{\overset{\alpha_{A',B'}}{\rightleftarrows}} & \mathsf{C}(A',GB') & & A'
\end{array}
$$

Notation: we write $F \dashv_{Hs} G$ iff F, G are part of a Hayashi semi-adjunction.
We shall give another characterisation of Hayashi semi-adjunctions, which resembles more closely our definition of semi-adjunction.

Theorem 9 *The tuple* $< F, G, \alpha, \beta >$ *is a Hayashi semi-adjunction iff* α *and* β *are families of functions*

$$\mathsf{D}(FA,B) \underset{\beta_{A,B}}{\overset{\alpha_{A,B}}{\rightleftarrows}} \mathsf{C}(A,GB)$$

natural in A, B*, which reduce to an isomorphism*

$$\mathsf{D}(FA,B)_s \cong \mathsf{C}(A,GB)_s$$

Proof:

- ($\Rightarrow$): Suppose $< F, G, \alpha, \beta >$ is a Hayashi semi-adjunction. It follows that α, β are natural. We show that they reduce to an isomorphism on the restricted Hom-sets. Take $h \in \mathsf{D}(F(A), B)_s$, then

$$\begin{aligned} \beta\alpha(h) &= \beta\alpha(h \circ F(id)) \\ &= \beta(G(id) \circ \alpha(h)) \\ &= h \circ F(id) \\ &= h \end{aligned}$$

  Analogously, $\alpha\beta(h) = h$ for $h \in \mathsf{C}(A, G(B))_s$.

- ($\Leftarrow$): Suppose α, β are natural, then two of the four squares in the diagram above commute. To see that the other two squares commute, we calculate

$$\begin{aligned} \beta(G(g) \circ \alpha(h) \circ f) &= \beta\alpha(g \circ h \circ F(f)) \\ &= g \circ h \circ F(f). \end{aligned}$$

Analogously, the other square commutes.

∎

Now the relation between semi-adjunctions and Hayashi semi-adjunctions can be stated.

Theorem 10 *$F \dashv_{Hs} G$ iff $F \dashv_s G$.*

Proof:

- $(\Rightarrow)$: Trivial.
- $(\Leftarrow)$: Suppose $< F, G, \mu >$ is a semi-adjunction. Define $\alpha(h) = \mu(h \circ F(id))$ and $\beta(h) = \mu^{-1}(G(id) \circ h)$, then α, β are natural. For example,

$$\begin{aligned} \alpha(g \circ h \circ F(f)) &= \mu(g \circ h \circ F(f) \circ F(id)) \\ &= \mu(g \circ h \circ F(id) \circ F(f)) \\ &= G(g) \circ \mu(h \circ F(id)) \circ f \\ &= G(g) \circ \alpha(h) \circ f \end{aligned}$$

∎

However, it looks like we give slightly *too much* information in a Hayashi semi-adjunction: we also define $\alpha(h)$ for $h \notin \mathsf{D}(F(A), B)_s$. Given a Hayashi semi-adjunction $< F, G, \alpha, \beta >$ we may restrict α, β to get a semi-adjunction $< F, G, \mu >$. Extending μ as in the proof of theorem 10 gives us α', β' again. However, because in general $\alpha(h \circ F(id)) = \alpha(h)$ does not hold, it is not true that $\alpha = \alpha'$. In the same way we do not have $\beta = \beta'$ in general.

Definition 11 *[2] A Hayashi semi-adjunction $< F, G, \alpha, \beta >$ is* normal *iff*

- $G(id) \circ \alpha(h) = \alpha(h)$
- $\beta(h) \circ F(id) = \beta(h)$

Hence, a Hayashi semi-adjunction is normal iff the values of α, β are fully determined by their values on the restricted Hom-sets.

Theorem 12 *If $< F, G, \alpha, \beta >$ is a Hayashi semi-adjunction with F or G a functor, then it is normal.*

Proof: Suppose F is a functor, then $G(id) \circ \alpha(h) = \alpha(h \circ F(id)) = \alpha(h)$ and $\beta(h) \circ F(id) = \beta(h)$. Analogously if G is a functor. ∎

Theorem 13 *The class of semi-adjunctions is isomorphic to the class of normal Hayashi semi-adjunctions.*

Proof: Trivial. ■

5 Semi natural transformations

We would like to transform the *class* of semi-functors between two categories D, E into a *category* $\mathsf{D} \Rightarrow \mathsf{E}$. As a first attempt, we define $\mathsf{D} \Rightarrow \mathsf{E}$ as the category with as objects semi-functors $F : \mathsf{D} \to \mathsf{E}$ and as arrows natural transformations $\alpha : F \to G$. The composition $\beta \cdot \alpha : F \to G$ of natural transformations $\alpha, \beta : F \to G$ is defined by $(\beta \cdot \alpha)_D = \beta_D \circ \alpha_D$ for $D \in \mathsf{D}$, and the identity natural transformation $1_F : F \to F$ has components $(1_F)_D = id_{F(D)}$ for $D \in \mathsf{D}$.

Theorem 14 $\mathbf{Cat}_s$ *with* $(-) \Rightarrow (-)$ *is a weak Cartesian closed category*[1].

Proof: Finite products in $\mathbf{Cat}_s$ are the same as in $\mathbf{Cat}$. The evaluation semi-functor $E : (\mathsf{D} \Rightarrow \mathsf{E}) \times \mathsf{D} \to \mathsf{E}$ is defined on objects $< H, D >$ by $E(< H, D >) = H(D)$, and on arrows $< \alpha, f >:< H, D > \to < H', D' >$ by $E(< \alpha, f >) = \alpha_{D'} \circ H(f)$. If $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is a semi-functor, then $F^* : \mathsf{C} \to (\mathsf{D} \Rightarrow \mathsf{E})$ is the semi-functor defined as follows: on objects $F^*(C) = F(C, -) : \mathsf{D} \to \mathsf{E}$ and on arrows $F^*(f : C \to C')$ is the natural transformation with components $F^*(f)_D = F(f, id_D)$. ■

The category $\mathbf{Cat}_s$ with $(-) \Rightarrow (-)$ is only *weak* Cartesian closed, because for semi-functors $F : \mathsf{C} \to (\mathsf{D} \Rightarrow \mathsf{E})$ the identity $(E(F \times Id))^* = F$ does not hold. More specific, if $f : C \to C'$ is an arrow in C, then $(E(F \times Id))^*(f)$ is the natural transformation with components $(E(F \times Id))^*(f)_D = F(f)_D \circ F(C)(id_D)$. In general, this is not equal to the natural transformation $F(f)$, which has components $F(f)_D$. The equality does hold, however, if $F(f)$ is a *semi* natural transformation.

Definition 15 *A* semi natural transformation *is a natural transformation* $\alpha : F \to G : \mathsf{D} \to \mathsf{E}$ *which satisfies*

$$\alpha_D \circ F(id_D) = \alpha_D$$

for every $D \in \mathsf{D}$.

Note that in general a semi-functor is "*less*" than a functor, whereas a semi natural transformation is "*more*" than a natural transformation.

Theorem 16 *If* $\alpha : F \to G : \mathsf{D} \to \mathsf{E}$ *is a natural transformation, and* F *or* G *is a functor, then* α *is a semi natural transformation.*

[1] A *weak* Cartesian closed category is defined as a Cartesian closed category, except that the exponential type constructor $\Rightarrow$ need only be a *semi-functor*.

Proof: If G is a functor, then

$$\begin{aligned}\alpha_D \circ F(id_D) &= G(id_D) \circ \alpha_D \\ &= id_{G(D)} \circ \alpha_D \\ &= \alpha_D\end{aligned}$$

■

Let $\mathsf{D} \Rightarrow_s \mathsf{E}$ be the subcategory of $\mathsf{D} \Rightarrow \mathsf{E}$ with the same objects, but with *semi* natural transformations as arrows. The composition in $\mathsf{D} \Rightarrow_s \mathsf{E}$ is the same as in $\mathsf{D} \Rightarrow \mathsf{E}$, but the identity $1_F : F \to F$ has components $(1_F)_D = F(id_D)$.

Theorem 17 $\mathbf{Cat}_s$ *with* $(-) \Rightarrow_s (-)$ *is a Cartesian closed category.*

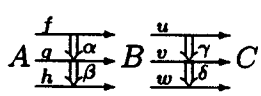
Proof: The evaluation semi-functor E and $(-)^*$ are defined as in the proof of theorem 14. ■

6 Cat_s as 2-category

In this section we show that $\mathbf{Cat}_s$ is a 2-category, and that $\mathcal{K}$ is a 2-functor. Intuitively, a 2-category C is a category C in which each Hom-set $\mathsf{C}(A, B)$ is a category again.

Definition 18 *[3] A* 2-category C *has objects or 0-cells* A*, arrows or 1-cells* $f : A \to B$*, and 2-cells* $\alpha : f \to g : A \to B$ *such that*

- *The objects and arrows form a category* C_0 *with identities* $id_A : A \to A$ *and composition* gf.
- *For fixed* A, B *the arrows* $A \to B$ *and the 2-cells between them form a category* $\mathsf{C}(A, B)$ *with identities* $1_f : f \to f : A \to B$ *and composition* $\beta \cdot \alpha$.
- *The 2-cells form a category: if* $\alpha : f \to g : A \to B$ *and* $\beta : u \to v : B \to C$ *are 2-cells, then there is a 2-cell* $\beta * \alpha : uf \to vg : A \to C$*. The identity of these composition is the 2-cell* $1_{id_A} : id_A \to id_A : A \to A$.
- *In the situation*



we have $(\delta * \beta) \cdot (\gamma * \alpha) = (\delta \cdot \gamma) * (\beta \cdot \alpha)$*. Furthermore, if* $f : A \to B, g : B \to C$ *are 1-cells, then* $1_u * 1_f = 1_{uf}$.

The paradigmatic example of a 2-category is **Cat**, which has categories as objects, functors as arrows, and natural transformations as 2-cells. Note that if $\alpha : F \to G : \mathsf{C} \to \mathsf{D}$ and $\beta : U \to V : \mathsf{D} \to \mathsf{E}$ are natural transformations, then the natural transformation $\beta * \alpha : UF \to VG : \mathsf{C} \to \mathsf{E}$ has components $(\beta * \alpha)_C = \beta_{GC} \circ U(\alpha_C)$ for $C \in \mathsf{C}$.

Theorem 19 Cat_s *with categories as objects, semi-functors as arrows, and semi natural transformations as 2-cells is a 2-category.*

Proof: The composition $*$ of semi natural transformations is defined as in Cat. We check that $1_U * 1_F = 1_{UF}$:

$$\begin{aligned}(1_U * 1_F)_C &= (1_U)_{F(C)} \circ U((1_F)_C) \\ &= U(id_{F(C)}) \circ UF(id_C) \\ &= UF(id_C) \\ &= (1_{UF})_C\end{aligned}$$

It is left to the reader to check the remaining details. ■

A *2-functor* $F : \mathsf{C} \to \mathsf{D}$ between 2-categories C, D sends i-cells in C to i-cells in D for $i = 0, 1, 2$, preserving domains and codomains and all types of composition and identities.

Theorem 20 $\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}$ *can be extended to a 2-functor.*

Proof: We define $\mathcal{K}$ on semi natural transformations. Suppose $\alpha : F \to G : \mathsf{C} \to \mathsf{D}$ is a semi natural transformation, then $\mathcal{K}(\alpha) : \mathcal{K}(F) \to \mathcal{K}(G) : \mathcal{K}(\mathsf{C}) \to \mathcal{K}(\mathsf{D})$ is the natural transformation with components $\mathcal{K}(\alpha)_f = \alpha_A \circ F(f)$ for $f : A \to A$ an idempotent. $\mathcal{K}$ is well-defined because

$$\begin{aligned}G(f) \circ \mathcal{K}(\alpha)_f \circ F(f) &= G(f) \circ \alpha_A \circ F(f) \circ F(f) \\ &= \alpha_A \circ F(f) \\ &= \mathcal{K}(\alpha)_f\end{aligned}$$

and it is easy to see that $\mathcal{K}(\alpha)$ is natural.
$\mathcal{K}$ preserves 1_F:

$$\begin{aligned}\mathcal{K}(1_F)_f &= (1_F)_A \circ F(f) \\ &= F(id_A) \circ F(f) \\ &= F(f) \\ &= (1_F)_f\end{aligned}$$

$\mathcal{K}$ preserves $\cdot$-composition: Suppose $\alpha : F \to G$ and $\beta : G \to H$, then

$$\begin{aligned}\mathcal{K}(\beta \cdot \alpha)_f &= (\beta \cdot \alpha)_A \circ F(f) \\ &= \beta_A \circ \alpha_A \circ F(f) \circ F(f) \\ &= \beta_A \circ G(f) \circ \alpha_A \circ F(f) \\ &= \mathcal{K}(\beta)_f \circ \mathcal{K}(\alpha)_f\end{aligned}$$

$\mathcal{K}$ preserves $*$-composition: Suppose $\alpha : F \to G : \mathsf{C} \to \mathsf{D}$ and $\beta : U \to V : \mathsf{D} \to \mathsf{E}$, then

$$\begin{aligned}
\mathcal{K}(\beta * \alpha)_f &= (\beta * \alpha)_A \circ UF(f) \\
&= \beta_{G(A)} \circ U(\alpha_A) \circ UF(f) \\
&= \beta_{G(A)} \circ U(\alpha_A \circ F(f)) \\
&= \beta_{G(A)} \circ U(G(f) \circ \alpha_A) \\
&= \beta_{G(A)} \circ U(G(f)) \circ U(\alpha_A) \\
&= \mathcal{K}(\beta)_{\mathcal{K}(G)(f)} \circ \mathcal{K}(U)(\mathcal{K}(\alpha)_f) \\
&= (\mathcal{K}(\beta) * \mathcal{K}(\alpha))_f
\end{aligned}$$

■

7 2-Adjunctions in $\mathbf{Cat}_s$

In section 2 we have seen that we can generalise categorical notions to *semi*-notions by means of the Karoubi envelope construction. Intuitively, we say that N is a semi-notion iff $\mathcal{K}(N)$ is the corresponding ordinary notion.
The fact that $\mathbf{Cat}_s$ is a 2-category gives us yet another way to define semi-notions. Various categorical definitions involving functors and natural transformations can be generalised to arbitrary 2-categories. In particular, we might consider the 2-category $\mathbf{Cat}_s$. As an example, we shall consider adjunctions in 2-categories.

Definition 21 *[3] A* 2-adjunction $\eta, \epsilon : f \dashv g : A \to B$ *in a 2-category* C *consists of arrows* $f : A \to B$ *and* $g : B \to A$ *together with 2-cells* $\eta : id_A \to gf$ *and* $\epsilon : fg \to id_B$ *such that*

1. $(1_g * \epsilon) \cdot (\eta * 1_g) = 1_g$

2. $(\epsilon * 1_f) \cdot (1_f * \eta) = 1_f$

Note that a 2-adjunction in **Cat** is the same as an ordinary adjunction between functors. We can also consider 2-adjunctions in $\mathbf{Cat}_s$.

Definition 22 *A* 2-adjunction $\eta, \epsilon : F \dashv G : \mathsf{C} \to \mathsf{D}$ *in* $\mathbf{Cat}_s$ *consists of semi-functors* $F : \mathsf{C} \to \mathsf{D}$ *and* $G : \mathsf{D} \to \mathsf{C}$ *together with natural transformations* $\eta : Id_{\mathsf{C}} \to GF$ *and* $\epsilon : FG \to Id_{\mathsf{D}}$ *such that*

1. $G\epsilon \cdot \eta G = Gid$

2. $\epsilon F \cdot F\eta = Fid$

We show that, in general, 2-notions in $\mathbf{Cat}_s$ coincide with semi-notions defined by means of the Karoubi envelope construction.
First we need a lemma.

Lemma 23 *$\mathcal{K}$ reflects equalities, i.e.*

1. $\mathcal{K}(\mathsf{C}) = \mathcal{K}(\mathsf{D})$ *implies* $\mathsf{C} = \mathsf{D}$
2. $\mathcal{K}(F) = \mathcal{K}(G)$ *implies* $F = G$
3. $\mathcal{K}(\alpha) = \mathcal{K}(\beta)$ *implies* $\alpha = \beta$

Proof: Point 1,2 are trivial. Suppose $\mathcal{K}(\alpha) = \mathcal{K}(\beta)$, then $\mathcal{K}(\alpha)_f = \mathcal{K}(\beta)_f$ for all idempotent f. In particular $\mathcal{K}(\alpha)_{id_A} = \mathcal{K}(\beta)_{id_A}$ for all A, hence $\alpha_A \circ F(id_A) = \beta_A \circ F(id_A)$ for all A. Because α, β are semi natural transformations, it follows that $\alpha_A = \beta_A$ for all A, and hence that $\alpha = \beta$. ■

In general, a *notion* N consists of some *abstract* data, and some requirements on this data. In the following we shall only consider notions in which these requirements are equations. We shall write $N(d)$ iff an instance d of the abstract data associated with N satisfies the requirements of N. If N is a 2-categorical notion, then $N[\mathsf{C}]$ denotes N in the 2-category C.

Theorem 24 *Suppose N is a notion in* Cat. *Let M be a 2-categorical generalisation of N (i.e. $M[\mathsf{Cat}] = N$). Let the notion S in Cat_s be defined by: $S(d)$ iff $N(\mathcal{K}(d))$. Then*

$$M[\mathsf{Cat}_s] = S$$

Proof:

- $M[\mathsf{Cat}_s](d)$ implies $S(d)$.
  Because 2-functors preserve equalities, it follows from $M[\mathsf{Cat}_s](d)$ that $M[\mathsf{Cat}](\mathcal{K}(d))$, and hence $N(\mathcal{K}(d))$. By definition of S this implies $S(d)$.
- $S(d)$ implies $M[\mathsf{Cat}_s](d)$.
  By definition it follows from $S(d)$ that $N(\mathcal{K}(d))$, and hence $M[\mathsf{Cat}](\mathcal{K}(d))$. Because $\mathcal{K}$ reflects equalities (lemma 23), this implies $M[\mathsf{Cat}_s](d)$.

■

Corollary 25 *Semi-adjunctions are the same as 2-adjunctions in* Cat_s.

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