Termination of term rewriting: from many-sorted to one-sorted

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Abstract

A property of many-sorted term rewriting systems is called persistent if it is not affected by removing the corresponding typing restriction. Persistency turns out to be a generalization of direct sum modularity. We show that strong normalization is persistent for the class of term rewriting systems for which not both duplicating rules and collapsing rules occur, generalizing a similar result of Rusinowitch for modularity.

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1 Introduction

Usually term rewriting systems are one-sorted: all terms and subterms are of the same type. The notion of term rewriting systems extends in a natural way to many-sorted terms. In this case a set of sorts, a set of operation symbols and a set of variable symbols is given. Each operation symbol has a sort (one of the sorts) and an arity. This arity is not simply a number, but a sequence of sorts. Each variable symbol has a sort. For every sort the set of terms of that sort is defined inductively in a straightforward way. This definition of terms is standard in the theory of algebraic specifications ([3]).

A many-sorted term rewriting system (TRS) is a set of pairs $l \to r$, where l and r are terms of the same sort. As usual l is not allowed to be a single variable and all variables in r also occur in l. Contexts, substitutions and the reduction relation are defined as expected, inducing definitions of normal forms, weak and strong normalization and confluency.

This is a very natural definition. One important application of TRS theory is in algebraic specifications, and the nature of algebraic specifications is many-sorted. The above notion of a many-sorted TRS is exactly what is needed for automatic implementation of many-sorted algebraic specifications and for applying Knuth-Bendix completion for the many-sorted case. Further, many variations of the λ-calculus can be described as many-sorted TRS's, like the $\lambda\tau$-calculus.

Any many-sorted TRS is trivially mapped to a one-sorted TRS by removing all sort information and keeping the same reduction rules. We call a property of many-sorted TRS's *persistent* if a many-sorted TRS has the property if and only if its adjoined one-sorted TRS has the property.

In this paper we show that every persistent property of the reduction relation of a TRS is also a modular property. In our view persistency is more basic than modularity; modularity can be considered as a particular case of persistency. Since strong normalization is not modular we conclude that it is neither persistent. We prove that restricting to TRS's in which not both duplicating rules and collapsing rules occur, strong normalization is persistent, generalizing Rusinowitch' result ([6]) stating that strong normalization is modular for the same class of TRS's.

2 Many-sorted term rewriting

First we introduce some standard terminology. Let S be a finite set representing the set of types or sorts. An S-*sorted set* X is defined to be a family of sets $(X_s)_{s \in S}$. If S with $\#S > 1$ is not specified we speak about *many-sorted* instead of S-sorted. By *one-sorted* we mean S-sorted with $\#S = 1$; by n-*sorted* we mean S-sorted with $\#S = n$. For S-sorted sets X and Y an S-*sorted map* $\phi : X \to Y$ is defined to be a family of maps $(\phi_s : X_s \to Y_s)_{s \in S}$. For an S-sorted sets X an S-*sorted relation* r on X is defined to be a family of relations $(r_s \subseteq X_s \times X_s)_{s \in S}$.

By S^* we denote the set of finite sequences of elements of S, including the empty sequence. Let $\mathcal{F}$ be a set of symbols, called *operation symbols*. For every operation symbol an *arity* and a *sort* is given, described by functions

$$ar : \mathcal{F} \to S^* \quad \text{and} \quad st : \mathcal{F} \to S.$$

The combination of $\mathcal{F}$ and S is called an S-sorted *signature*. Operation symbols of which the arity is the empty sequence are called *constants*. Let $\mathcal{X}$ be an S-sorted set of symbols, called *variables*. We define the S-sorted set $\mathcal{T}(\mathcal{F}, \mathcal{X})$ of *terms* inductively by

- $\mathcal{X}_s \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})_s$ for $s \in S$;

- $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{X})_s$ for $f \in \mathcal{F}$ with $ar(f) = (s_1, \ldots, s_n)$ and $st(f) = s$, and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})_{s_i}$ for $i = 1, \ldots, n$.

In this description we do not allow overloading: all operation symbols and variable symbols are assumed to be distinct.

A *substitution* σ is defined to be an S-sorted map $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. It is extended to an S-sorted map $\bar{\sigma} : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ by defining inductively

- $\bar{\sigma}_s(x) = \sigma(x)$ for $x \in \mathcal{X}_s$ and $s \in S$;

- $\bar{\sigma}_s(f(t_1, \ldots, t_n)) = f(\bar{\sigma}_{s_1}(t_1), \ldots, \bar{\sigma}_{s_n}(t_n))$ for $f \in \mathcal{F}$ with $ar(f) = (s_1, \ldots, s_n)$ and $st(f) = s$, and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})_{s_i}$ for $i = 1, \ldots, n$.

For any $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})_s$ we write t^σ instead of $\bar{\sigma}_s(t)$.

An S-sorted *term rewriting system* (TRS) is defined to be an S-sorted set R with $R_s \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})_s \times \mathcal{T}(\mathcal{F}, \mathcal{X})_s$ for $s \in S$. Elements (l, r) of R_s are called *rules* of sort s and are often written as $l \to r$ if there is no confusion with the many other meanings of the symbol $\to$. The *reduction relation* of an S-sorted TRS R is the S-sorted relation $\to_R$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ inductively defined by

- $l^\sigma \to_{R,s} r^\sigma$ for every $(l, r) \in R_s$ and every substitution σ;

- $f(t_1, \ldots, t_n) \to_{R,s} f(t_1, \ldots, t'_k, \ldots, t_n)$ (only t_k replaced by t'_k) for every $f \in \mathcal{F}$ with $ar(f) = (s_1, \ldots, s_n)$ and $st(f) = s$, and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})_{s_i}$ for $i = 1, \ldots, n$, $t'_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})_{s_k}$ and $t_k \to_{R,s_k} t'_k$.

The usual definition of TRS as in [4, 1] corresponds to the one-sorted case, i.e., $\#S = 1$.

An S-sorted TRS is called terminating or *strongly normalizing* if for every $s \in S$ there exist no infinite reductions of the reduction relation $\to_{R,s}$. Notions like weak normalization and confluence are similarly generalized from one-sorted to many-sorted TRS's.

3 Persistency

By removing all sort information every many-sorted term can be mapped to a one-sorted term as follows. Let $\mathcal{F}'$ be the set of symbols obtained by adding a prime (') to every symbol of $\mathcal{F}$. For $f \in \mathcal{F}$ with $ar(f) = (s_1, \ldots, s_n)$ we define the arity of $f' \in \mathcal{F}'$ to be n. In this way $\mathcal{F}'$ defines a one-sorted signature. Since there is only one sort there is no need for an explicit notation for the sort. We choose $\mathcal{X}' = \bigcup_{s \in S} \mathcal{X}_s$ to be the set of one-sorted variable symbols; recall that the sets $\mathcal{X}_s$ are assumed to be disjoint.

Every term over $\mathcal{F}$ of any sort can be mapped to a term over $\mathcal{F}'$ by adding prime symbols to all operation symbols. This map

$$\Theta : \bigcup_{s \in S} \mathcal{T}(\mathcal{F}, \mathcal{X})_s \rightarrow \mathcal{T}(\mathcal{F}', \mathcal{X}'),$$

is inductively defined by

- $\Theta(x) = x$ for every $x \in \mathcal{X}_s$ for every $s \in S$;

- $\Theta(f(t_1, \ldots, t_n)) = f'(\Theta(t_1), \ldots, \Theta(t_n))$ for all $f \in \mathcal{F}$ and terms $t_1, \ldots, t_n$ of the right sort.

The map Θ is defined on TRS's in an obvious way: for any many-sorted TRS R the one-sorted TRS $\Theta(R)$ is defined to consist of the rules $\Theta(l) \rightarrow \Theta(r)$ for the rules $l \rightarrow r$ from R.i On easily observes that for $t, t' \in \mathcal{T}(\mathcal{F}, \mathcal{X})_s$:

$$t \rightarrow_{R,s} t' \iff \Theta(t) \rightarrow_{\Theta(R)} \Theta(t').$$

A property of (many-sorted) TRS's is called a *reduction property* if it only depends on the reduction relation defined by the TRS, and not on the shape of the rules of the TRS. For example, confluency, and weak and strong normalization are reduction properties, left-linearity and orthogonality are not.

A reduction property is called *persistent* if for every many-sorted TRS R the property holds for R if and only if it holds for $\Theta(R)$.

The notion of persistency is closely related to the notion of *modularity* as it is discussed in [4, 5]. Modularity has been extensively studied in [6, 7, 8, 5]. A reduction property of one-sorted TRS's is called *modular* if for every pair of (one-sorted) TRS's R_1 and R_2 with disjoint sets of operation symbols the property holds for both R_1 and R_2 if and only if it holds for $R_1 \oplus R_2$. Here $R_1 \oplus R_2$ denotes the union of both TRS's; it is a one-sorted TRS over the disjoint union of the sets of operation symbols.

Theorem 1 *Every persistent reduction property is modular.*

Proof: Let p any persistent reduction property of many-sorted TRS's. Let R_1 and R_2 be one-sorted TRS's with disjoint sets of operation symbols. We define a two-sorted TRS R as follows. The sorts are denoted by s_1 and s_2, the operation symbols are the operation symbols from both R_1 and R_2. The arity of an operation symbol from R_1 of arity n is defined to be $(s_1, s_1, \ldots, s_1)$, and its sort is defined to be s_1. Similarly the arities and sorts of the operation symbols from R_2 are defined to consist solely from s_2. Now the terms of sort s_1 of R correspond one-to-one to the terms of R_1. Further the reduction relation at sort s_1 of R corresponds one-to-one to the reduction relation of R_1. The same holds for '1' replaced by '2', we conclude that

$$p(R) \quad \Longleftrightarrow \quad p(R_1) \wedge p(R_2).$$

On the other hand the terms of $\Theta(R)$ correspond one-to-one to the terms of $R_1 \oplus R_2$ and the reduction relation of $\Theta(R)$ corresponds one-to-one to the reduction relation of $R_1 \oplus R_2$. So

$$p(\Theta(R)) \quad \Longleftrightarrow \quad p(R_1 \oplus R_2).$$

Since p is persistent $p(R)$ and $p(\Theta(R))$ are equivalent; combining this with the above results gives

$$p(R_1 \oplus R_2) \quad \Longleftrightarrow \quad p(R_1) \wedge p(R_2),$$

which we had to prove. $\square$

4 Termination

Since strong normalization is not a modular property we conclude from theorem 1 that strong normalization is neither a persistent property. The basic counterexample in [8] and the proof of theorem 1 lead to the following counterexample. Let $S = \{s_1, s_2\}$; the following variables and operation symbols are defined:

- x is a variable of sort s_1;

- y, z are variables of sort s_2;

- $0, 1$ are constants of sort s_1;

- f is an operation symbol of sort s_1 and arity (s_1, s_1, s_1);

- g is an operation symbol of sort s_2 and arity (s_2, s_2).

Let the S-sorted TRS R consist of the following rules:

$$
\begin{aligned}
f(0, 1, x) &\quad\rightarrow\quad f(x, x, x) \\
g(y, z) &\quad\rightarrow\quad y \\
g(y, z) &\quad\rightarrow\quad z.
\end{aligned}
$$

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One easily shows that the S-sorted TRS R is strongly normalizing, while

$$f(g(0,1),g(0,1),g(0,1)) \to f(0,g(0,1),g(0,1)) \to f(0,1,g(0,1)) \to$$

$$f(g(0,1),g(0,1),g(0,1)) \to \cdots$$

is an infinite reduction in $\Theta(R)$. This implies that strong normalization is not a persistent property.

In this paper we show that strong normalization is persistent for a particular class of many-sorted TRS's. For defining that class we need some definitions. A reduction rule is called a *collapsing* rule if its right hand side is a single variable. A reduction rule is called a *duplicating* rule if for some variable the number of occurrences in the right hand side is greater than the number of occurrences in the left hand side. For example, in the above example the first rule is duplicating and the second and the third are collapsing rules.

In [6] it is shown that strong normalization is modular in the class of one-sorted TRS's without collapsing rules and also in the class of one-sorted TRS's without duplicating rules. In this paper we generalize this result: we show that strong normalization is persistent for the class of many-sorted TRS's not containing both collapsing rules and duplicating rules.

Any infinite reduction of R is trivially translated to an infinite reduction of $\Theta(R)$. As a consequence, strong normalization of $\Theta(R)$ implies strong normalization of R. The difficult part is the converse: assume strong normalization of R and derive strong normalization of $\Theta(R)$. Without loss of generality we assume that $\mathcal{F}'$ contains some symbol of empty arity; so the existence of an infinite reduction of $\Theta(R)$ implies the existence of an infinite ground reduction of $\Theta(R)$.

5 No collapsing rules

We shall define a well-founded partial order on ground terms in such a way that the value of a term decreases by applying any non-collapsing rule of $\Theta(R)$. This will prove strong normalization of $\Theta(R)$ if R contains no collapsing rules.

Let R be a strongly normalizing S-sorted TRS over function symbols $\mathcal{F}$. Choose one variable symbol y_s for every sort s, let $\mathcal{Y}$ be defined by $\mathcal{Y}_s = \{y_s\}$. Let $T = \bigcup_{s \in S} \mathcal{T}(\mathcal{F}, \mathcal{Y})_s$ and let $>_T$ be the relation on T defined by

$$t >_T t' \iff t, t' \text{ have the same sort } s \text{ and } t \to^+_{R,s} t'.$$

Here $\to^+_{R,s}$ denotes the transitive closure of $\to_{R,s}$. Since R is strongly normalizing $>_T$ is a well-founded partial order on T. Let $>_l$ be the lexicographic order on $\mathbb{N} \times T$ defined by

$$(k,t) >_l (k',t') \iff k > k' \vee (k = k' \wedge t >_T t');$$

it is also well-founded. Let $M(\mathbb{N} \times T)$ be the set of finite multisets over $\mathbb{N} \times T$ and let $>_M$ be the corresponding multiset order induced by $>_l$. It is well-founded again,

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see [2]. The empty multiset is denoted by [ ], a one element multiset by $[(k,t)]$ and multiset union by $\bigsqcup$.

We define four functions on one-sorted ground terms:

$$\mathsf{sort} : T(\mathcal{F}') \to S, \quad \mathsf{rank} : T(\mathcal{F}') \to \mathbb{N},$$

$$\mathsf{top} : T(\mathcal{F}') \to T, \quad \mathsf{mult} : T(\mathcal{F}') \to M(\mathbb{N} \times T).$$

For any operation symbol f of arity $(s_1, \ldots, s_n)$ and sort s, and any $u_1, \ldots, u_n \in T(\mathcal{F}')$ we define

$$\mathsf{sort}(f'(u_1, \ldots, u_n)) = s.$$

We define a partition of $\{1, \ldots, n\}$ into two subsets A and B by

$$A = \{i \in \{1, \ldots, n\} \mid \mathsf{sort}(u_i) = s_i\},$$

$$B = \{i \in \{1, \ldots, n\} \mid \mathsf{sort}(u_i) \neq s_i\}.$$

Now $\mathsf{rank}, \mathsf{top}$ and mult are defined inductively by

$$
\begin{aligned}
\mathsf{rank}(f'(u_1, \ldots, u_n)) &= \max(\max_{i \in A} \mathsf{rank}(u_i), \max_{i \in B}(\mathsf{rank}(u_i) + 1)), \\
\mathsf{top}(f'(u_1, \ldots, u_n)) &= f(t_1, \ldots, t_n), \\
&\quad \text{where} \quad t_i = \mathsf{top}(u_i) \quad \text{if} \quad i \in A, \\
&\qquad\qquad\quad\ t_i = y_{s_i} \qquad\ \text{if} \quad i \in B, \\
\mathsf{mult}(f'(u_1, \ldots, u_n)) &= (\bigsqcup_{i=1}^{n} \mathsf{mult}(u_i)) \bigsqcup (\bigsqcup_{i \in B} [(\mathsf{rank}(u_i), \mathsf{top}(u_i))]).
\end{aligned}
$$

The base of this inductive definition is in the operation symbols of empty arity; the maximum of an empty set of natural numbers is defined to be 0, the multiset union of an empty set of multisets is defined to be [ ].

The idea of this technical definition is the following. Each one-sorted term can uniquely be split up into maximal many-sorted parts. These many-sorted building blocks are organized in a tree structure. Now sort denotes the sort of the root of the one-sorted term, rank the height of the corresponding tree structure, top the building block containing the root, and mult the multiset of all building blocks, labelled with their heights, except for the building block containing the root.

We define the weight function $W : T(\mathcal{F}') \to M(\mathbb{N} \times T)$ by

$$W(u) = \mathsf{mult}(u) \bigsqcup [(\mathsf{rank}(u), \mathsf{top}(u))],$$

so the weight is the multiset of all building blocks labelled with their heights, including the building block containing the root. We shall prove that $W(u) >_M W(v)$ for every ground reduction step $u \to v$ in $\Theta(R)$ if R contains no collapsing rules. Before doing so we need some lemmas.

Lemma 2 *Let* $u \in T(\mathcal{F}')$. *Then* $k < \mathsf{rank}(u)$ *for every* $(k,t) \in \mathsf{mult}(u)$.

Proof: By induction on the structure of u. Let $u = f'(u_1, \ldots, u_n)$ and let $(k, t) \in \text{mult}(u)$. Then either

$$(k, t) \in \text{mult}(u_i) \text{ for some } i, \text{ giving } k < \text{rank}(u_i) \leq \text{rank}(u),$$

or

$$k = \text{rank}(u_i) \text{ for some } i \in B, \text{ giving } k < \text{rank}(u_i) + 1 \leq \text{rank}(u).$$

□

For any one-sorted ground substitution $\sigma : \mathcal{X}' \to \mathcal{T}(\mathcal{F}')$ we define a corresponding S-sorted substitution $\tilde{\sigma}$ for which $\tilde{\sigma}_s : \mathcal{X}_s \to \mathcal{T}(\mathcal{F}, \mathcal{Y})_s$ is defined by

$$\tilde{\sigma}_s(x) \;=\; \begin{cases} y_s & \text{if } \text{sort}(\sigma(x)) \neq s \\ \text{top}(\sigma(x)) & \text{if } \text{sort}(\sigma(x)) = s. \end{cases}$$

Lemma 3 *Let* $\sigma : \mathcal{X}' \to \mathcal{T}(\mathcal{F}')$ *be any one-sorted ground substitution and let* $t \in \bigcup_{s \in S} \mathcal{T}(\mathcal{F}, \mathcal{X})_s$ *be any term with* $t \notin \bigcup_{s \in S} \mathcal{X}_s$. *Then*

$$\text{top}(\Theta(t)^\sigma) \;=\; t^{\tilde{\sigma}}.$$

Proof: By induction on the structure of t. Let $t = f(t_1, \ldots, t_n)$ for some f of sort s and arity $(s_1, \ldots, s_n)$. Define for $i = 1, \ldots, n$:

$$\tilde{t}_i \;=\; \begin{cases} y_{s_i} & \text{if } \text{sort}(\Theta(t_i)^\sigma) \neq s_i \\ \text{top}(\Theta(t_i)^\sigma) & \text{if } \text{sort}(\Theta(t_i)^\sigma) = s_i. \end{cases}$$

If t_i is not a variable then $\text{sort}(\Theta(t_i)^\sigma) = s_i$. From the induction hypothesis we conclude $t_i^{\tilde{\sigma}} = \tilde{t}_i$. If t_i is a variable, then $t_i^{\tilde{\sigma}} = \tilde{t}_i$ according to the definition of $\tilde{\sigma}$. So for all $i = 1, \ldots, n$ we have $t_i^{\tilde{\sigma}} = \tilde{t}_i$. We conclude

$$\begin{aligned} \text{top}(\Theta(t)^\sigma) &= \text{top}(\Theta(f(t_1, \ldots, t_n))^\sigma) \\ &= \text{top}(f'(\Theta(t_1)^\sigma, \ldots, \Theta(t_n)^\sigma)) \\ &= f(\tilde{t}_1, \ldots, \tilde{t}_n) \\ &= f(t_1^{\tilde{\sigma}}, \ldots, t_n^{\tilde{\sigma}}) \\ &= t^{\tilde{\sigma}}. \end{aligned}$$

□

Lemma 4 *Let* $l \to r$ *be any rule of an* S-*sorted TRS. Let* $\sigma : \mathcal{X}' \to \mathcal{T}(\mathcal{F}')$ *be any one-sorted ground substitution for which* $\text{sort}(l^\sigma) = \text{sort}(r^\sigma)$. *Then*

$$\text{top}(\Theta(l)^\sigma) >_T \text{top}(\Theta(r)^\sigma).$$

Proof: From lemma 3 we obtain $\text{top}(\Theta(l)^\sigma) = l^{\tilde{\sigma}}$. If r is no variable we obtain from lemma 3 that $\text{top}(\Theta(r)^\sigma) = r^{\tilde{\sigma}}$; if r is a variable we obtain the same from the definition of $\tilde{\sigma}$. In both cases we derive from the definition of $>_T$:

$$\text{top}(\Theta(l)^\sigma) = l^{\tilde{\sigma}} >_T r^{\tilde{\sigma}} = \text{top}(\Theta(r)^\sigma).$$

$\square$

Lemma 5 *Let $l \to r$ be any rule of an S-sorted TRS. Let $\sigma : \mathcal{X}' \to \mathcal{T}(\mathcal{F}')$ be any one-sorted ground substitution. Then*

$$\text{rank}(\Theta(l)^\sigma) \geq \text{rank}(\Theta(r)^\sigma).$$

Proof: For any term t in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ define

$$A(t) = \bigcup_{s \in S} \{x \in X_s \mid x \text{ occurs in } t \text{ and } \text{sort}(x^\sigma) = s\},$$

and

$$B(t) = \bigcup_{s \in S} \{x \in X_s \mid x \text{ occurs in } t \text{ and } \text{sort}(x^\sigma) \neq s\}.$$

Using the definition of rank one easily proves by induction on the structure of t that

$$\text{rank}(\Theta(t)^\sigma) = \max(\max_{x \in A(t)} \text{rank}(x^\sigma), \max_{x \in B(t)} (\text{rank}(x^\sigma) + 1))$$

for every non-variable term t. Since all variables of r occur in l we have $A(r) \subseteq A(l)$ and $B(r) \subseteq B(l)$, and the lemma follows, also if r is a variable. $\square$

Lemma 6 *Let $l \to r$ be any rule of an S-sorted TRS. Let $\sigma : \mathcal{X}' \to \mathcal{T}(\mathcal{F}')$ be any one-sorted ground substitution for which $\text{sort}(l^\sigma) = \text{sort}(r^\sigma)$. Then*

$$W(\Theta(l)^\sigma) >_M W(\Theta(r)^\sigma).$$

Proof: For any term t in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ define

$$p(t) = (\text{rank}(\Theta(t)^\sigma), \text{top}(\Theta(t)^\sigma));$$

by definition $p(t)$ is an element of $W(t)$. Lemmas 4 and 5 imply

$$p(\Theta(l)^\sigma) >_l p(\Theta(r)^\sigma).$$

Lemma 2 implies that all other elements of $W(\Theta(r)^\sigma)$ are smaller than $p(\Theta(r)^\sigma)$ with respect to $>_l$. We conclude that all elements of $W(\Theta(r)^\sigma)$ are strictly smaller than the element $p(\Theta(l)^\sigma)$ of $W(\Theta(l)^\sigma)$ with respect to $>_l$. This proves the lemma. $\square$

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Lemma 7 *Let $u, v \in \mathcal{T}(\mathcal{F}')$ for which $\mathsf{rank}(u) > \mathsf{rank}(v)$. Then*

$$\mathsf{mult}(u) >_M \mathsf{mult}(v).$$

Proof: Note that $\mathsf{rank}(u) > 0$. From the definitions of rank and mult easily follows that $\mathsf{mult}(u)$ contains an element (k_0, t_0) with $k_0 = \mathsf{rank}(u) - 1$. From lemma 2 we conclude that $k < \mathsf{rank}(v) \leq k_0$ for every $(k, t) \in \mathsf{mult}(v)$. So all elements of $\mathsf{mult}(v)$ are strictly smaller with respect to $>_l$ than $(k_0, t_0) \in \mathsf{mult}(u)$. Hence $\mathsf{mult}(u) >_M \mathsf{mult}(v)$. $\square$

Lemma 8 *Let $f' \in \mathcal{F}'$ of arity $n > 0$ and let $u_1, \dots, u_n, u'_k \in \mathcal{T}(\mathcal{F}')$ for which $\mathsf{sort}(u_k) = \mathsf{sort}(u'_k)$ and $W(u_k) >_M W(u'_k)$ and $\mathsf{top}(u_k) \geq_T \mathsf{top}(u'_k)$. Let*

$$u = f'(u_1, \dots, u_n) \quad and \quad u' = f'(u_1, \dots, u'_k, \dots, u_n))$$

(only u_k replaced by u'_k). Then

$$W(u) >_M W(u') \quad and \quad \mathsf{top}(u) \geq_T \mathsf{top}(u').$$

Proof: By definition we have either $\mathsf{top}(u) = \mathsf{top}(u')$ or

$$\mathsf{top}(u) = f(\dots, \mathsf{top}(u_k), \dots) \quad \text{and} \quad \mathsf{top}(u') = f(\dots, \mathsf{top}(u'_k), \dots).$$

Since $\mathsf{top}(u_k) \geq_T \mathsf{top}(u'_k)$ and $>_T$ is closed under contexts we conclude in both cases that $\mathsf{top}(u) \geq_T \mathsf{top}(u')$.

From lemma 2 and $W(u_k) >_M W(u'_k)$ we conclude that $\mathsf{rank}(u_k) \geq \mathsf{rank}(u'_k)$. From the definition of rank follows $\mathsf{rank}(u) \geq \mathsf{rank}(u')$.

Let $(s_1, \dots, s_n) = ar(f)$ and $s = st(f)$. As before let

$$A = \{i \in \{1, \dots, n\} \mid \mathsf{sort}(u_i) = s_i\},$$

$$B = \{i \in \{1, \dots, n\} \mid \mathsf{sort}(u_i) \neq s_i\}.$$

Now by definition

$$W(u) \;=\; (\bigsqcup_{i=1}^{n} \mathsf{mult}(u_i)) \;\bigsqcup\; (\bigsqcup_{i \in B}[(\mathsf{rank}(u_i), \mathsf{top}(u_i))]) \;\bigsqcup\; [(\mathsf{rank}(u), \mathsf{top}(u))];$$

we have to prove that $W(u) >_M W(u')$.

We distinguish two cases: $k \in B$ and $k \in A$. First assume $k \in B$. Combining the inequalities $W(u_k) >_M W(u'_k)$ and $(\mathsf{rank}(u), \mathsf{top}(u)) \geq_l (\mathsf{rank}(u'), \mathsf{top}(u'))$ gives

$$\mathsf{mult}(u_k) \bigsqcup [(\mathsf{rank}(u_k), \mathsf{top}(u_k))] \bigsqcup [(\mathsf{rank}(u), \mathsf{top}(u))] \;>_M$$

$$\mathsf{mult}(u'_k) \bigsqcup [(\mathsf{rank}(u'_k), \mathsf{top}(u'_k))] \bigsqcup [(\mathsf{rank}(u'), \mathsf{top}(u'))].$$

Add to both the left hand side and the right hand side of this inequality

$$(\bigsqcup_{i=1,...,n,i\neq k} \mathsf{mult}(u_i)) \bigsqcup (\bigsqcup_{i\in B\setminus\{k\}} [(\mathsf{rank}(u_i),\mathsf{top}(u_i))]).$$

Then the left hand side is equal to $W(u)$ and the right hand side is equal to $W(u')$, proving $W(u) >_M W(u')$.

For the remaining case $k \in A$ first assume that $\mathsf{top}(u_k) >_T \mathsf{top}(u_k')$. Since $>_T$ is closed under contexts we then have $\mathsf{top}(u) >_T \mathsf{top}(u')$. Since $\mathsf{rank}(u) \geq \mathsf{rank}(u')$ we obtain

$$(\mathsf{rank}(u),\mathsf{top}(u)) >_l (\mathsf{rank}(u'),\mathsf{top}(u')).$$

From lemma 2 we conclude that all elements of $W(u')$ are strictly smaller with respect to $>_l$ than $(\mathsf{rank}(u),\mathsf{top}(u)) \in W(u)$, proving $W(u) >_M W(u')$.

In the remaining case we have $k \in A$ and $\mathsf{top}(u_k) = \mathsf{top}(u_k')$. If $\mathsf{rank}(u_k) = \mathsf{rank}(u_k')$ we conclude from $W(u_k) >_M W(u_k')$ that $\mathsf{mult}(u_k) >_M \mathsf{mult}(u_k')$. On the other hand if $\mathsf{rank}(u_k) > \mathsf{rank}(u_k')$ then we conclude from lemma 7 that again $\mathsf{mult}(u_k) >_M \mathsf{mult}(u_k')$. Adding $(\mathsf{rank}(u),\mathsf{top}(u)) \geq_l (\mathsf{rank}(u'),\mathsf{top}(u'))$ gives

$$\mathsf{mult}(u_k) \bigsqcup [(\mathsf{rank}(u),\mathsf{top}(u))] >_M \mathsf{mult}(u_k') \bigsqcup [(\mathsf{rank}(u'),\mathsf{top}(u'))].$$

Add to both the left hand side and the right hand side of this inequality

$$(\bigsqcup_{i=1,...,n,i\neq k} \mathsf{mult}(u_i)) \bigsqcup (\bigsqcup_{i\in B} [(\mathsf{rank}(u_i),\mathsf{top}(u_i))]).$$

Now the left hand side equals $W(u)$ and the right hand side equals $W(u')$, so $W(u) >_M W(u')$, concluding the proof. $\square$

Lemma 9 *Let R be a strongly normalizing S-sorted TRS without collapsing rules. Let $u,v \in T(\mathcal{F}')$ for which $u \to_{\Theta(R)} v$. Then $W(u) >_M W(v)$ and $\mathsf{top}(u) \geq_T \mathsf{top}(v)$.*

Proof: Induction on the structure of the relation $\to_{\Theta(R)}$. The base of this induction follows from lemma 4 and lemma 6; the condition $\mathsf{sort}(l^\sigma) = \mathsf{sort}(r^\sigma)$ is trivially fulfilled since there are no collapsing rules. The induction step is exactly lemma 8. $\square$

Theorem 10 *Let R be an S-sorted TRS without collapsing rules. Then R is strongly normalizing if and only if $\Theta(R)$ is strongly normalizing.*

Proof: The 'if'-part is trivial. For the 'only if'-part let R be a strongly normalizing S-sorted TRS and assume that there is an infinite $\Theta(R)$-reduction. Ground substitution in this reduction leads to an infinite ground $\Theta(R)$-reduction. Applying the weight function W to this reduction leads to an infinite strictly descending chain in $M(\mathbb{N} \times T)$ with respect to $>_M$, according to lemma 9. This contradicts the well-foundedness of the order $>_M$ on $M(\mathbb{N} \times T)$. $\square$

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6 No duplicating rules

For proving persistency of strong normalization of TRS's without duplicating rules, we need an extra function size : $T(\mathcal{F}') \to \mathbb{N}$ defined inductively by

$$\text{size}(f'(u_1, \ldots, u_n)) = \#B + \sum_{i=1}^{n} \text{size}(u_i)$$

for any operation symbol f of arity $(s_1, \ldots, s_n)$ and sort s, and any $u_1, \ldots, u_n \in T(\mathcal{F}')$, where B is defined by

$$B = \{i \in \{1, \ldots, n\} \mid \text{sort}(u_i) \neq s_i\}.$$

So size is the number of many-sorted building blocks minus one.

We define a new weight function $W' : T(\mathcal{F}') \to \mathbb{N} \times M(\mathbb{N} \times T)$ by

$$W'(u) = (\text{size}(u), W(u)).$$

The lexicographic order $>_L$ on $\mathbb{N} \times M(\mathbb{N} \times T)$ is defined by

$$(n, m) >_L (n', m') \iff n > n' \vee (n = n' \wedge m >_M m'),$$

clearly this order is well-founded. We shall prove that $W'(u) >_L W'(v)$ for every ground reduction step $u \to v$ in $\Theta(R)$ if R contains no duplicating rules. First we derive some properties of size.

Lemma 11 *Let $l \to r$ be any non-duplicating rule of an S-sorted TRS. Let $\sigma : \mathcal{X}' \to T(\mathcal{F}')$ be any one-sorted ground substitution. Then*

$$\text{size}(\Theta(l)^\sigma) \geq \text{size}(\Theta(r)^\sigma).$$

Proof: For any term t in $T(\mathcal{F}, \mathcal{X})$ and any $x \in X'$ let $n(t, x)$ be the number of occurrences of x in t, and let

$$B(t) = \bigcup_{s \in S} \{x \in X_s \mid x \text{ occurs in } t \text{ and } \text{sort}(x^\sigma) \neq s\}.$$

Using the definition of size one easily proves by induction on the structure of t that

$$\text{size}(\Theta(t)^\sigma) = \left(\sum_{x \in X'} n(t, x) * \text{size}(x^\sigma) \right) + \sum_{x \in B(t)} n(t, x).$$

for every non-variable term t. Since $n(l, x) \geq n(r, x)$ for all $x \in X'$ the lemma follows, also if r is a variable. $\square$

Lemma 12 *Let R be an S-sorted TRS. Let $u, v \in T(\mathcal{F}')$ for which $u \to_{\Theta(R)} v$ and $\text{sort}(u) \neq \text{sort}(v)$. Then $\text{size}(u) > \text{size}(v)$.*

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Proof: This is only possible if $u = \Theta(l)^\sigma$ and $v = x^\sigma$ for some $\sigma : \mathcal{X}' \to \mathcal{T}(\mathcal{F}')$ and some collapsing rule $l \to x$. Let $\Theta(l) = f'(u_1, \ldots, u_n)$. If $u_k = x$ for some k then the lemma follows from the definition. Otherwise x occurs in some non-variable u_k, and assuming $\text{size}(u_k^\sigma) > \text{size}(x^\sigma)$ as an induction hypothesis we obtain

$$\text{size}(u) = \text{size}(f'(\ldots, u_k, \ldots)^\sigma) \geq \text{size}(u_k^\sigma) > \text{size}(x^\sigma) = \text{size}(v).$$

$\square$

Lemma 13 *Let R be an S-sorted TRS without duplicating rules. Let $u, u' \in \mathcal{T}(\mathcal{F}')$ for which $u \to_{\Theta(R)} u'$. Then*

$$\text{size}(u) > \text{size}(u') \quad \vee \quad (\text{size}(u) = \text{size}(u') \wedge W(u) >_M W(u') \wedge \text{top}(u) \geq_T \text{top}(u')).$$

Proof: First assume that $u = \Theta(l)^\sigma$ and $u' = r^\sigma$ for some rule $l \to r$. If $\text{sort}(u) = \text{sort}(u')$ then the lemma follows from lemmas 11, 4 and 6; otherwise it follows from lemma 12.

Next let $u_k \to_{\Theta(R)} u_k'$ for

$$u = f'(u_1, \ldots, u_n) \quad \text{and} \quad u' = f'(u_1, \ldots, u_k', \ldots, u_n))$$

(only u_k replaced by u_k') for some operation symbol f of arity $(s_1, \ldots, s_n)$. As an induction hypothesis we assume that the lemma holds for (u_k, u_k'). If $\text{sort}(u_k) = \text{sort}(u_k')$ then from the definition of size follows that $\text{size}(u) > \text{size}(u') \Leftrightarrow \text{size}(u_k) > \text{size}(u_k')$ and $\text{size}(u) = \text{size}(u') \Leftrightarrow \text{size}(u_k) = \text{size}(u_k')$. Now the lemma follows from lemma 8.

In the remaining case we have $\text{sort}(u_k) \neq \text{sort}(u_k')$. From lemma 12 now follows $\text{size}(u_k) > \text{size}(u_k')$; from the definition of size we conclude

$$\text{size}(u) > \text{size}(u') \quad \vee \quad (\text{size}(u) = \text{size}(u') \wedge \text{sort}(u_k) = s_k).$$

If $\text{size}(u) > \text{size}(u')$ we are done, in the remaining case we have $\text{sort}(u_k) = s_k$ and we have to prove that $W(u) >_M W(u')$ and $\text{top}(u) \geq_T \text{top}(u')$. Since $\text{sort}(u_k) \neq \text{sort}(u_k')$ we conclude that $u_k = \Theta(l)^\sigma$ and $u_k' = x^\sigma$ for some collapsing rule $l \to x$. From the definition of top follows that $\text{top}(u) \to_R \text{top}(u')$, so $\text{top}(u) >_T \text{top}(u')$. Since $\text{rank}(u) \geq \text{rank}(u')$ we conclude $W(u) >_M W(u')$. $\square$

Theorem 14 *Let R be an S-sorted TRS without duplicating rules. Then R is strongly normalizing if and only if $\Theta(R)$ is strongly normalizing.*

Proof: The 'if'-part is trivial. The 'only if'-part follows from the fact that $W'(u) >_L W'(u')$ for every $u, u' \in \mathcal{T}(\mathcal{F}')$ for which $u \to_{\Theta(R)} u'$, which is immediate from lemma 13. $\square$

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