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Abstract

A Truth Maintenance System (TMS) maintains a consistent state of belief given a set J of justifications, i.e. arguments for belief. To resolve contradictions dependency-directed backtracking is performed.

In this paper we introduce *Cautious Backtracking* as a method that can be used to track all dependency-directed backtracking methods simultaneously. This has been previously done by adding the contrapositions of justifications to J. We will show that contrapositions are not mandatory, namely that the same result can be reached by adding one disjunctive justification (the consequent is not one atom, but can be a disjunction of atoms). This Cautious Backtracking is a minimal backtracking in that the revised beliefs are the beliefs agreed upon by every backtracking method.

The Well-Founded model for disjunctive logic programs, as described by Ross in [Ross 89b], gives a semantics to Cautious Backtracking.

Furthermore, we will give an alternative proof for the correctness of Witteveen's method [Witteveen 90] for computing the Well-Founded model.

1 Introduction

Informally, a TMS is a program that manages a database of *nodes* and *justifications*. Each node stands for a piece of information. Each justification says that one node should be believed if some others are respectively believed and disbelieved. There are two kinds of truth maintenance systems. One kind is the justification-based TMS (JTMS), the other is the assumption-based TMS (ATMS).

A JTMS maintains a single context of belief (a model) at a time and supports nonmonotonic justifications. In everyday life, conclusions have to be drawn in the absence of complete information. This type of common sense reasoning unavoidably depends on default assumptions, so it is intrinsically nonmonotonic: new information can cause old conclusions to loose their validity.

An ATMS maintains multiple models simultaneously, but does not support nonmonotonicity. This makes it suitable for applications where several partial solutions are to be considered and compared in the process of developing complete solutions. Contrary to JTMSs in which the switching of context to compare solutions is possible but inefficient.

In the sequel we focus on the JTMS, the nonmonotonicity supporting system.

A TMS can be seen as a system working together with a problem solver. Whenever the problem solver performs an inference step, it sends a justification to the TMS. The task of the TMS is to maintain a consistent state of belief. The TMS determines this state from the statements of belief, the reasons for belief, and the known inconsistent states of belief (constraints) which have been given to the TMS by the problem solver.

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Among the normal justifications, constraints may exist. Constraints say that a conflict or contradiction arises, if the nodes of some set are believed and the nodes of another set are disbelieved.

If a conflict occurs, some sort of backtracking must be performed to reach a consistent state of belief. In the process of backtracking new beliefs are formed by reasoning from the conflict: if belief in n nodes causes a conflict, then we can reason in the contrapositive direction: Belief in $n-1$ of the nodes indicates that we should disbelieve the last node. This implies that there are n such contrapositive reasonings possible for this one conflict.

Giving a semantics for belief revision one can produce a consistent state of belief for every possible contrapositive or one can give one consistent state of belief which satisfies every contraposition.

The problem is that one has to produce every contraposition, and their number is linear in the number of beliefs causing the conflict. Furthermore, two kinds of negation get mixed. In TMSs negation as failure is used, meaning that we only disbelieve a node if there is no reason to believe it. However, reasoning in the contrapositive direction can tell us to believe in the falsehood of a node (so to disbelieve it) if others are respectively believed of disbelieved. This belief in falsehood is by definition not a negation as failure. As only negation as failure is allowed in justifications, new nodes are created to stand for the negated node. Then extra justifications must be added to ensure that the new node and the original node get opposite truth-values.

In this paper we propose a way to reason from the contradiction in which no contrapositions and no extra nodes are needed. Instead, only one, possibly disjunctive, justification is added. Furthermore, we do not have the problem of new negations.

This new approach to conflict resolution is called Cautious Backtracking (CB), which makes no choices during backtracking. We show that a consistent model of belief exists for this approach, which corresponds to the proposal of Witteveen ([Witteveen 91]). This proves that contrapositions are not necessary to perform dependency-directed backtracking. Simplifying the justification added in the CB approach creates intrinsic extensions of Witteveen's proposal.

In the next section preliminaries for the rest of the paper and the terminology used in this paper will be described.

In section 3 the Stable and Well-Founded semantics for logic programs will be discussed in detail, because these semantics are directly applicable to TMSs and will be referred to frequently. Here the alternative proof for the correctness of Witteveen's method for computing the Well-Founded model will be given.

Section 4 explains when conflicts arise and how these conflicts depend on the truth-values of the nodes. Then the use of contrapositions is explained and dependency-directed backtracking is described in short. The last part of this section describes two semantics for conflict resolution based on contrapositions.

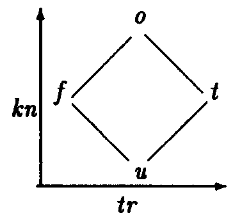
In section 5 we introduce a new approach to dependency-directed backtracking based on reasoning from a conflict, but not on contrapositions. Two variations of this approach will be explained in detail and compared with the semantics of the previous section.

2 Preliminaries

A TMS is a pair $D = (N, J)$, where N is a finite set of propositional atoms, called nodes, and J is a finite set of directed propositional formulas j, called justifications, of the form $\alpha \wedge \beta \rightarrow c$, where c is an atom, called the conclusion or head of j ($hd(j) = c$), α is the conjunction of positive literals (i.e. atoms) and β the conjunction of negative literals (i.e. negated atoms). $\beta(j)$ denotes the nonmonotonic part of justification j. The body of a justification $j = \alpha \wedge \beta \rightarrow c$ is $body(j) = \alpha \wedge \beta$. A justification with an empty body is called a premise. Justifications are called directed because c is distinguished from β in the sense that $\sim b \rightarrow c$ is not the same as $\sim c \rightarrow b$. $At(\alpha)$ ($At(\beta)$) denotes the set of atoms in α (β). $Lit(\phi)$ is the set of literals used in wff ϕ. The intended meaning of such a justification is the following: "Belief in the elements of $At(\alpha)$ and disbelief in the elements

of $At(\beta)$ justify the belief in c". A premise can now be seen as a justification with an empty antecedent. A constraint is a justification $\alpha \wedge \beta \rightarrow \perp$ where $\perp$ is seen as the node indicating a contradiction if it has truth-value *true*. If $\alpha \wedge \beta$ is believed, $\perp$ gets truth-value *true* as well.
In this paper we will use 3-valued interpretations. In 3-valued interpretations a third value called *undefined*, written u, is used for the nodes with no reason for belief or disbelief. An interpretation is a tuple $I = (I_t, I_f)$. I_t (I_f) is the set of atoms considered true (false) in the interpretation.

In this paper two orderings are used. One to compare truth-values in an interpretation and one to compare interpretations. These orderings are the orderings of Belnap's four valued logic, called $\mathcal{FOUR}$. In $\mathcal{FOUR}$ one can think of truth-values as sets of truth values in the ordinary sense, namely $\{true\}$, which we will write as t, $\{false\}$, written as f, $\{\}$, which we will write as u and read as underdetermined or as undefined; and $\{true, false\}$, written as o and read as overdetermined.
The four truth-values can be given a simple natural mathematical structure. We give $\mathcal{FOUR}$ two partial orderings namely the *knowledge ordering*, denoted $\leq_{kn}$, and the *truth ordering*, denoted $\leq_{tr}$ as depicted adjacent.
Thus $a \leq_{kn} b$ if there is a way up from a to b, where the intuition is "b gives more information than a" so $\leq_{kn}$ represents an increase in knowledge.
And $a \leq_{tr} b$ if there is a path from left to right from a to b. The intuition is that $\leq_{tr}$ represents an increase in "truth" (or decrease in "falseness").



Given an interpretation I propositions are evaluated as follows:

- $val_I(\sim a) = \sim val_I(a)$, where $\sim t = f$ and $\sim f = t$ and $\sim u = u$
- $val_I(\phi \wedge \psi) = min\{val_I(\phi), val_I(\psi)\}$
- $val_I(\phi \rightarrow \psi) = t$ if $val_I(\phi) \leq val_I(\psi)$ otherwise f.

Instead of val_I we will write I. Note that $\phi \rightarrow \psi \neq \sim \phi \vee \psi$ as it is classically.

Definition 2.1 *Let M and M' be interpretations of a* JTMS *D.*

- *M is a* model *of D if it satisfies every justification in D.*
- *$M \sqcup M' = (M_t \cup M'_t, M_f \cup M'_f)$ is called the* union *of M and M'.*
- *M and M' are* consistent *if $M \sqcup M'(\perp) \neq t$.*
- *M' is an* extension *of M if M and M' are consistent and $M_f \subseteq M'_f$ and $M_t \subseteq M'_t$.*

Notice that with this definition a 3-valued interpretation can be a 3-valued model while not having a 2-valued extension which is a 2-valued model. An example can be found in example 4.1.

When considering a semantics for a *TMS* a choice is to be made between a minimal, unique, model and a maximal, not unique, model. A minimal will in general be 3-valued, some choices have not been made. A maximal model is 2-valued, but several choices have been made, for which there is no grounded proof. Which of these models to prefer, depends on the application. But some nodes can only have one truth-value (t or f), because the other choice would lead to a contradiction ($\perp$). Such a node has an *intrinsic* truth-value.

Definition 2.2 *A model M of J is* intrinsic *if M is consistent with every other model of J.*

Definition 2.3 *A model M of J is* maximal intrinsic *if M is a intrinsic model of J, which extends every other intrinsic model of J.*

Let M be an intrinsic model of J, then $a \in M_t$ ($a \in M_f$) implies that $M' = (M_t - \{a\}, M_f \cup \{a\})$ ($M' = (M_t \cup \{a\}, M_f - \{a\})$) is not a model of J. For example, let $J = \{\sim a \rightarrow b, \sim b \rightarrow a, a \rightarrow c, b \rightarrow c\}$, then M is a model of J if and only if $M = (\emptyset, \emptyset)$ or $c \in M_t$. But only $(\emptyset, \emptyset)$ and $(\{c\}, \emptyset)$ are intrinsic models of J.

Definition 2.4 *[vGRS 88]*

- *For a set of literals S we denote the set formed by taking the complement of each literal in S by $\circ S$.*
- *Literal q is* inconsistent *with S if $q \in \circ S$.*
- *Sets of literals R and S are* inconsistent *if some literal in R is inconsistent with S (or vice versa), i.e., if $R \cap \circ S \neq \emptyset$*
- *A set of literals is* inconsistent *if it is inconsistent with itself; otherwise it is* consistent.
- *Sets of literals R and S are* disjoint *if no literal in R has the same atom as a literal in S (or vice versa), i.e., if $R \cap S = R \cap \circ S = \emptyset$*

Definition 2.5 *Let $D = (N, J), j \in J$, we define*

- $C(j, I) := \{hd(j) \mid I(body(j)) = t\}$
- $C(D, I) := \bigcup\{C(j, I) \mid j \in J\} \cup I$
- $Cl(D, I) := (\lambda I'.C(D, I'))^{\omega}(I)$

Informally, *Cl* is the transitive closure of the premises.

3 Stable and Well-Founded Semantics

In this section we will summarize the definition of the Stable and Well-Founded semantics, because they are key notions in this paper.

The importance of the Stable semantics follows from its relation with the Well-Founded semantics.

The Well-Founded model is the knowledge-minimal 3-valued stable model. Therefore the Well-Founded model is unique and, like all stable models, it has the useful property of reproducing itself under a certain operation. This operation will be explained in the following. In the last part of this section we will prove Witteveen's method for computing the Well-Founded model correct. This proof hinges on the fact that $Gus(D, I)$ can be found in an efficient way, using a transitive closure operation.

3.1 Stable model

The Stable model semantics has been introduced by Gelfond en Lifschitz in [GL 88]. The semantics is 2-valued and is based on Moore's Autoepistemic Logic. In this semantics every choice made is grounded, i.e. does not depend on a circular chain of beliefs. Informally a model is a stable model if it reproduces itself under a certain operation. This operation, is defined as follows:

Definition 3.1 *(2-valued stable) Let $D = (N, J)$ be a* TMS. *Then M is a stable model of D iff M is the unique minimal model of $D' = (N, J(M))$, where $J(M) = \{\alpha \to c \mid \alpha \wedge \beta \to c \in J, M \models \beta\}$.*

The stable model semantics has a few drawbacks as it is not universal and not unique. However its reproductive property is attractive as is the easy method of checking whether or not a model is stable. In order to overcome the problem of non-universality, several people proposed (3-valued) generalizations of stable models.

Giordano and Martelli [GiMar1] proposed generalized stable models to handle conflict resolution in Truth Maintenance. More details of their proposal are given in section 4.2.

Przymusinska and Przymusinski gave a definition of a 3-valued stable model of Logic Programs in [PaPi 90b], which Witteveen [Witteveen 90] translated to a definition for JTMSs.

Definition 3.2 *(3-valued stable) Let $M = (M_t, M_f)$ be an interpretation of $D = (N, J)$ and let $J(M,t) = \{\alpha \to c \mid \alpha \wedge \beta \to c \in J$ and $M(\beta) = t\}$ and $J(M,u) = \{\alpha \to c \mid \alpha \wedge \beta \to c \in J$ and $M(\beta) \geq_{tr} u\}$. M is a* 3-valued stable *model of D iff*

> *(a) M_t is the truth-minimal 2-valued model of $D(M,t) = (N, J(M,t))$ in short: $M_t = Cl((N, J(M,t)))$*
>
> *(b) $M_t \cup M_u$ is truth-minimal 2-valued model of $D(M,u) = (N, J(M,u))$ in short: $M_t \cup M_u = Cl((N, J(M,u)))$.*

3.2 Well-Founded model

In 1988 Van Gelder, Ross and Schlipf [vGRS 88] proposed a new definition of canonical model, which they called the Well-Founded model. This semantics is unique and universal, in that it always produces a 3-valued model and exactly one. In this subsection we will only describe their original formulation for the propositional case and the formulation by Witteveen [Witteveen 90] for JTMSs. Other general formulations have been proposed in [Van Gelder 89] and [Przymusinski 89].

The key idea in the original formulation by Van Gelder, Ross and Schlipf is the concept of an "unfounded" set. This concept is used to capture negation by failure.

Definition 3.3 *Given a* JTMS *$D = (N, J)$ and a 3-valued interpretation I. We say that $A \subseteq N$ is an* unfounded set of D with respect to I *(denoted $unf(A, D, I)$) if*

$$\forall c \in A\ \forall(\alpha \wedge \beta \to c) \in J : \begin{cases} I(\alpha \wedge \beta) = f \\ or \\ At(\alpha) \cap A \neq \emptyset \end{cases}$$

Equivalent with this definition is

Definition 3.4 *Let $D = (N, J)$ be a* JTMS *and I a 3-valued interpretation.*

- *$Poss(D, X) = \{C(J, Y) \mid X$ and Y consistent$\}$*
- *$unf(A, D, I)$ if $\circ A$ is consistent with $I \cup Poss(D, I \cup \circ A)$*

Informally, the Well-Founded semantics uses the condition above to draw negative conclusions. Justifications that satisfy $I(\alpha \wedge \beta) = f$ are not usable for further derivations. Condition $At(\alpha) \cap A \neq \emptyset$ is the unfoundedness condition: of all the justifications that still might be usable to derive something in A, each requires an atom in A to be true. Note that not necessarily $A \cap I = \emptyset$ (however in the following this gives no problems as one starts with $I = \emptyset$). It is immediately clear that, with respect to any interpretation I, the union of arbitrary unfounded sets is an unfounded set.

Definition 3.5 *The* greatest unfounded set with respect to I *($Gus(D, I)$) is the union of all sets that are unfounded with respect to I, $Gus(D, I) = \bigcup\{A \mid unf(A, D, I)\}$.*

3.2.1 Well-Founded 3-valued Models

The Well-Founded semantics of a JTMS D is defined as the least fixpoint of an operator, called V, on interpretations.

Definition 3.6 *[vGRS 88]*

- *$U(D, I)$ is defined by: $U(D, I) = \circ Gus(D, I)$.*
- *$V(D, I) = C(D, I) \cup U(D, I)$.*

Lemma 3.7 *[vGRS 88] $C(D, I)$, $U(D, I)$ and $V(D, I)$ are monotonic transformations.*

Definition 3.8 *[vGRS 88] For all countable ordinals κ the set V^κ, whose elements are literals in $N \cup \circ(N)$, is defined recursively by:*

1. *For limit ordinal κ, $V^\kappa = \bigcup_{\mu<\kappa} V^\mu$*
2. *For successor ordinal $\kappa+1$, $V^{\kappa+1} = V(D, V^\kappa)$*

Note that 0 is a limit ordinal, and $V^0 = \emptyset$.

Lemma 3.9 *[vGRS 88] $(V^\kappa)_{\kappa \geq 0}$ as defined in definition 3.8 is a monotonic sequence of 3-valued interpretations (i.e, I_κ is consistent for all κ).*

The set N is countable, so the sequence of definition 3.8 reaches a fixed point V^* after some countable ordinal. Now the Well-Founded semantics can be defined:

Definition 3.10 *The Well-Founded semantics of a* JTMS *D is the "meaning" represented by the limit V^* described above.*

Examples can be found in section 4. The complexity of this semantics is obviously very high, so Van Gelder [Van Gelder 89] and (independently) Przymusinski [Przymusinski 89] have proposed faster construction methods for this semantics, but with these the complexity of computing the Well-Founded model is still very high. Fortunately Witteveen proved a faster method when the program is restricted to the propositional case (as in JTMSs) based on work by Goodwin ([Goodwin 82] and [Goodwin 87]).

3.2.2 A faster method for finding the Well-Founded model for a JTMS.

In [Witteveen 90] a faster method for computing the Well-Founded model is described. This method seems not use the notion of unfounded sets, as it is computed solely using transitive closure operations.

Definition 3.11 *Let I and I' be 3-valued interpretations for* JTMS *$D = (N, J)$.*

- *$Ker(I) = I_t \cup I_f$.*
- *$I' - I = (I'_t - I_t, I'_f - I_f)$.*
- *$Un(\alpha, I)$ $(Un(\beta, I))$ is the conjunction of atoms in α (β) evaluating to u under I.*
- *$N^{-I} = N - Ker(I)$*
- *$J^{-I} = \{Un(\alpha) \wedge Un(\beta) \rightarrow c \mid \alpha \wedge \beta \rightarrow c \in J \& c \notin Ker(I) \& I(\alpha \wedge \beta) >_{tr} f\}$.*
- *$D^{-I} = (N^{-I}, J^{-I})$.*
- *$Cl^+(D, I) := Cl((D^{-I})_+, \emptyset)$*
- *$J_+ = \{\alpha \rightarrow c \mid \alpha \wedge \beta \rightarrow c \in J\}$.*
- *$D_+ = (N, J_+)$*
- *$W(D, I) = I \sqcup (Cl(D, I), N^{-I} - Cl^+(D, I))$.*

Theorem 3.12 *[Witteveen 90] The least fixpoint W^* of W is the Well-Founded model of D.*

A proof of this theorem is given in [Witteveen 90], but in the next section we give an alternative proof, which shows that this method does use the notion of the Greatest Unfounded Set.
The time-complexity of W^* is $\mathcal{O}(\mid N \mid \times \mid D \mid)$, because Dowling and Gallier [DG 84] and Minoux [Minoux 88] gave linear-time algorithms to find a least truth-assignment for a set of Horn clauses. Notice that a set of justifications without negation is a set of Horn clauses.

3.2.3 Some new facts and new proofs of old facts about the Well-Founded semantics.

In this section we will prove that $N^{-I} - Cl^{+}(D,I)$ is equal to $Gus(D,I)$, i.e. that there is a constructive way to find $Gus(D,I)$. In the following, we will think of an interpretation I as a set of literals $I_t \cup \circ I_f$.

Lemma 3.13 $Gus(D,I) \subseteq N - (Cl^{+}(D,I) \cup I)$.

Proof: $Cl^{+}(D,I) \cup (I \cap N) = Cl((D^{-I})_{+}, \emptyset) \cup (I \cap N) = (\lambda X.C((D^{-I})_{+}, X))^{\omega}(\emptyset) \cup (I \cap N) = \bigcup_{i \in \omega} I_i \cup (I \cap N)$ where $I_0 = \emptyset$ and

$$\begin{aligned} I_{n+1} &= C((D^{-I})_{+}, I_n) = \\ &= \{p \mid \alpha\prime \rightarrow p \in (J^{-I})_{+}, I_n \models \alpha'\} \\ (*) \quad &= \{p \mid (Lit(\alpha \wedge \beta) - I) \cap N \subseteq I_n, \alpha \wedge \beta \rightarrow p \in J, p \notin I, consistent(I, Lit(\alpha \wedge \beta))\} \end{aligned}$$

which implies that $\neg\exists A(p \in A \wedge unf(A,D,I))$.

Suppose $p \in A$ and $unf(A,D,I)$, then $\circ A$ is consistent with $I \cup Poss(D, I \cup \circ A)$. This can be reformulated to $A \cap I = \emptyset$ and $A \cap Poss(D, I \cup \circ A) = \emptyset$. $A \cap Poss(D, I \cup \circ A) = \emptyset$ is the same as $A \cap \bigcup\{C(D,X) \mid I \cup \circ A$ and X consistent$\} = \emptyset$. So $\neg\exists X(p \in C(D,X)$ and consistent $(I \cup \circ A, X))$ which is $\forall X($consistent$(I \cup \circ A, X) \rightarrow \forall(Y \rightarrow p) \in J : Y \not\subseteq X)))$.

(**) $\forall \tilde{I}((I \cup \circ A) \cap \circ \tilde{I} \neq \emptyset \vee \forall(\tilde{I}' \rightarrow p) \in J : \tilde{I}' \not\subseteq \tilde{I})$

But from (*): $\exists(\alpha \wedge \beta \rightarrow p) \in J$, where I and $Lit(\alpha \wedge \beta)$ are consistent, i.e. $I \cap \circ Lit(\alpha \wedge \beta) = \emptyset$
$(Lit(\alpha \wedge \beta) - I) \cap N \subseteq I_n$ (Ind. hyp.: $I_n \cap Gus(D,I) = \emptyset$).
From (**) with $X := Lit(\alpha \wedge \beta)$ and $\tilde{I}' := Lit(\alpha \wedge \beta)$ we can derive
$(I \cup \circ A) \cap \circ Lit(\alpha \wedge \beta) \neq \emptyset$.
To summarize:

1. $I \cap \circ Lit(\alpha \wedge \beta) = \emptyset \Leftrightarrow \circ I \cap Lit(\alpha \wedge \beta) = \emptyset$
2. $(I \cup \circ A) \cap \circ Lit(\alpha \wedge \beta) \neq \emptyset \Leftrightarrow (\circ I \cup A) \cap Lit(\alpha \wedge \beta) \neq \emptyset$
3. $(Lit(\alpha \wedge \beta) - I) \cap N \cap A = \emptyset$

From (2) we get $(\circ I \cap Lit(\alpha \wedge \beta)) \cup (A \cap Lit(\alpha \wedge \beta)) \neq \emptyset$, we combine this with (1) to $A \cap Lit(\alpha \wedge \beta) \neq \emptyset$. Furthermore, from (3) combined with $A \subseteq N$ we can conclude $A \cap Lit(\alpha \wedge \beta) - I = \emptyset$. We already had $A \cap I = \emptyset$, so $A \cap Lit(\alpha \wedge \beta) - I = A \cap Lit(\alpha \wedge \beta) = \emptyset$, contradiction.

□

Lemma 3.14 $N - (Cl^{+}(D,I) \cup I) \subseteq Gus(D,I)$.

Proof: Let $X = N - (Cl^{+}(D,I) \cup I)$. We have to show that X is an unfounded set of D with respect to I, i.e. $unf(X,D,I)$. This is true if and only if $\circ X$ is consistent with $I \cup Poss(D, I \cup \circ X)$. So $unf(X,D,I)$ if and only if ($X \cap I = \emptyset$ and $X \cap Poss(D, I \cup \circ X) = \emptyset$). If and only if $X \cap Poss(D, I \cup \circ X) = \emptyset$. If and only if

$\odot$ $Poss(D, I \cup \circ X) \subseteq N - X$

Now $p \in Poss(D, I \cup \circ X)$ iff $\exists Y \exists \alpha \wedge \beta \rightarrow p \in J$ such that Y is consistent with $I \cup \circ X)$ and $Y(\alpha \wedge \beta) = t$. So $p \in Poss(D, I \cup \circ X)$ iff ($\exists \alpha \wedge \beta \rightarrow p \in J$ such that $Lit(\alpha \wedge \beta) \cap (\circ I \cup X) = \emptyset$) iff ($\exists(\alpha \wedge \beta \rightarrow p) \in J$ such that $Lit(\alpha \wedge \beta) \cup \circ I = \emptyset$ and $Lit(\alpha \wedge \beta) \cap X = \emptyset$)
iff

$\oplus$ $(\alpha \wedge \beta \rightarrow p) \in J$ and $Lit(\alpha \wedge \beta) \cap \circ I = \emptyset$ and $Lit(\alpha \wedge \beta) \subseteq N - X = (N \cap I) \cup Cl^{+}(D,I)$

We have to prove that $p \in (I \cap N) \cup Cl^{+}(D,I)$ in order to prove $\odot$. So we have to show $p \in I$ or $p \in Cl^{+}(D,I)$. If $p \in I$, we are done, so assume $p \notin I$. $p \in Cl^{+}(D,I)$ if $\exists j \in J$ with $hd(j) = p$

and $p \notin I$ and I consistent with $Lit(body(j))$, such that $(Lit(body(j)) - I) \cap N \subseteq Cl^+(D, I)$. Take $j = \alpha \wedge \beta \rightarrow p$, then $(Lit(\alpha \wedge \beta) - I) \cap N \subseteq Cl^+(D, I)$ because of $\oplus$.

□

Corollary 3.15 $N^{-I} - Cl^+(D, I) = Gus(D, I)$.

Theorem 3.16 *Let K and L be monotonic operators on sets. Let $A(X) = K(X) \cup L(X)$ and $B(X) = K^\omega(X) \cup L(X)$ for any set X. If K is increasing and continuous, then $A^\omega(X) = B^\omega(X)$.*

Proof: Note that

1. $K^\omega(B^\omega(X)) = B^\omega(X)$
2. $L(B^\omega(X)) \subseteq B^\omega(X)$
3. $K(A^\omega(X)) = A^\omega(X)$
4. $L(A^\omega(X)) \subseteq A^\omega(X)$
5. $K^\omega(A^\omega(X)) = A^\omega(X)$.
6. $K(B^\omega(X)) = B^\omega(X)$

Properties (1), (3), (5) and (6) hold because $A^\omega(X)$ and $B^\omega(X)$ are fixpoints of K (and therefore of K^ω). Properties (2) and (4) hold because of $B(B^\omega(X)) = K^\omega(B^\omega(X)) \cup L(B^\omega(X)) = B^\omega(X)$ and $A(A^\omega(X)) = K(A^\omega(X)) \cup L(A^\omega(X)) = A^\omega(X)$.
First we prove that $A^\omega(X) \subseteq B^\omega(X)$, then we prove that $B^\omega(X) \subseteq A^\omega(X)$.

• $A^\omega(X) \subseteq B^\omega(X)$:
We prove this by induction. The basis $A^0(X) = X = B^0(X) \subseteq B^\omega(X)$ follows from the fact that $Y \subseteq K(Y)$ for any set Y. For the induction step suppose that $Y \subseteq B^\omega(X)$ for a set Y. Then $K(Y) \subseteq K(B^\omega(X))$ because of the monotonicity of K. And $K(B^\omega(X)) \subseteq B^\omega(X)$ because of property (6). Property (2) and the monotonicity of L prove that $L(Y) \subseteq B^\omega(X)$. So we can conclude that $A(Y) \subseteq B^\omega(X)$, which concludes this proof by induction.

• $B^\omega(X) \subseteq A^\omega(X)$:
We prove this by induction. The basis $B^0(X) = X = A^0(X) \subseteq A^\omega(X)$ follows from the fact that $Y \subseteq K(Y)$ for any set Y. For the induction step suppose that $Y \subseteq A^\omega(X)$ for a set Y. Then $K^\omega(Y) \subseteq K^\omega(A^\omega(X))$ because of the monotonicity of K, which implies that K^ω is monotonic. And $K^\omega(A^\omega(X)) \subseteq A^\omega(X)$ because of property (5). Property (4) and the monotonicity of L prove that $L(Y) \subseteq A^\omega(X)$. So we can conclude that $B(Y) \subseteq A^\omega(X)$, which concludes this proof by induction.

□

Corollary 3.17 *Witteveen's method for constructing the Well-Founded model is correct; i.e. $W^* = V^*$, conform theorem 3.12.*

Proof: Recall $W^* = (\lambda I.Cl(D, I) \cup \circ Gus(D, I))^\omega(\emptyset)$. And $V^* = (\lambda I.C(J, I) \cup \circ Gus(D, I))^\omega(\emptyset)$. Theorem 3.16 is applicable, when taking $K := C$ (and $K^\omega = Cl$) and $L := Gus$, $A^\omega := V^*$ and $B^\omega := W^*$. C and Gus are monotonic, continuous operators, and C is increasing.

□

Comparing Well-Founded models with intrinsic models, the following holds:
The Well-Founded model need not be intrinsic, for example, let $D = (\{a, b, c, d\}, J)$, where $J = \{\sim a \rightarrow b, \sim c \rightarrow d, \sim c \rightarrow a, b \wedge d \rightarrow \perp\}$, then the Well-Founded model is $(\{a, d\}, \{b, c\})$, but $(\{b, c\}, \{a, d\})$ is also a model of J and these two models are not consistent.

Note that $Cl(D, \emptyset)$ is always an intrinsic model of J, and it is the least (i.e. knowledge-minimal) intrinsic model of D.

4 Conflict resolution using contrapositions

In this section, we will discuss several methods to resolve conflicts. All of these methods are based on the use of justifications in their contrapositive directions. To solve a conflict, some choices need to be revised, i.e. a form of belief revision must be performed. When creating a model, a path of choices will be followed, leading to a contradiction if one or more choices are wrong.

The method attempts to resolve this contradiction by backtracking along our path of choices and revising some. As we want to revise our model as little as possible, we don't want to revise choices that didn't cause the contradiction; the choices to be revised are defined by the dependencies of the contradiction on these choices. Therefore, the backtracking performed is dependency-directed. Also, we only want to perform backtracking when it is mandatory, i.e. if there is no (3-valued) stable model for D. As the Well-Founded model is the knowledge-minimal 3-valued stable model, backtracking is necessary only if the Well-Founded model is inconsistent. That is why we concentrate on the Well-Founded semantics.

In section 4.1, we will first show how conflicts can arise. Subsequently, we will show the relation between belief revision and reasoning from the contradiction using justifications in their contrapositive direction. The idea of using contrapositions can be found in Dependency-directed backtracking, which will be described in short.

In sections 4.2 and 4.3, we will discuss the semantics for DDB in Justification-based Truth Maintenance Systems, which were presented by [GiMar1] and [Witteveen 91]. These different semantics depend strongly on the use of contrapositions.

4.1 Introduction

Now remember that the task of a JTMS is to maintain a model. Given a set of justifications J, we let the JTMS calculate the Well-Founded model. One of the problems arising in the presence of constraints is that it is possible that no 2-valued extension of the Well-Founded model is a 2-valued model. For example:

Example 4.1

Let $D = (N, J)$ with $N = \{a\}$ and let J be:

$$\begin{array}{ll} \sim a & \rightarrow a \\ \sim a & \rightarrow \bot \\ a & \rightarrow \bot \end{array}$$

For D the Well-Founded model is $(\emptyset, \emptyset)$, but D is inconsistent and therefore has no 2-valued model at all. Then one can question the JTMS, for example "$WF \models p$?", or "does a model exist for D such that p (or $\sim p$) evaluates to true?". In most of these cases an alternative model must be found for D to satisfy the user, or a negative answer was provided. In most cases the question will not be trivial; the Well-Founded model will not be the required model. This implies that the set of justifications does not give enough information to render the model sought after by the user (or problem solver). On the other hand, if the Well-Founded model contains a contradiction, then it could be the case that the set of justifications gives too much information. This is shown in the following.

Example 4.2

Let $D_1 = (N_1, J_1)$ with $N_1 = \{a, b\}$ and let J_1 be:

$$\begin{array}{rl} & \rightarrow a \\ & \rightarrow b \\ a \wedge b & \rightarrow \perp \end{array}$$

For D_1 the Well-Founded model is $(\{a, b, \perp\}, \emptyset)$, which contains a contradiction. In this case J_1 contains too much information. The problem solver (or user) will have to retract one of the justifications before a 3-valued model can be found.

Example 4.3

Let $D_2 = (N_2, J_2)$, with $N_2 = \{a\}$ and $J_2 = \{\sim a \rightarrow \perp\}$.

The Well-Founded model is $(\{\perp\}, \{a\})$ in which disbelief in a leads to belief in $\perp$. The JTMS has to perform backtracking to find a model for D_2, adding justification $\rightarrow a$ and calculating the Well-Founded model now gives $(\{a\}, \{\perp\})$. So a model does exist for J_2, but J_2 contains not enough information to find it without backtracking.

In the next example we will try to illustrate the dependencies between several 3-valued interpretations when constructing the Well-Founded model.

Example 4.4

Consider $D = (N, J)$ with $N = \{a, b, c, d\}$ and let J be:

$$\begin{array}{ll} \sim a & \rightarrow b \\ \sim c & \rightarrow d \\ b \wedge d & \rightarrow \perp \\ \sim c & \rightarrow a \end{array}$$

$WF = (\emptyset, \emptyset) \sqcup (\emptyset, \{c\}) \sqcup (\{a, d\}, \emptyset) \sqcup (\emptyset, \{b, \perp\}) \sqcup (\emptyset, \emptyset)$. Here the truth of a and d depends only on the falseness of c, and the falseness of b and $\perp$ depends only on the truth of a.

The idea to use the contrapositives of the justifications requires some remarks. The justification $a \wedge \sim b \rightarrow c$ allows c to be concluded, given that a is believed (that is, provable) and b is disbelieved (that is, not provable). Contrapositioning gives two new clauses: $a \wedge \neg c \rightarrow b$ and $\sim b \wedge \neg c \rightarrow \neg a$. where $\neg$ is the classical negation and $\sim$ a negation by default. The first new clause says that b can be concluded when a is provable and $\neg c$ is provable, while the second new clause says that it is possible to conclude $\neg a$ when b is not provable (disbelieved) and $\neg c$ is provable. Accordingly, when one wants to use a justification in its contrapositive direction, it is not sufficient to have its consequent disbelieved, instead the negation of the consequent must be provable (or the consequent cannot be consistently assumed). As this is only possible in the presence of constraints, contrapositives can only be used if constraints occur in the set of justifications.

A form of contraposition is used in dependency-directed backtracking (DDB).

Dependency-directed backtracking has been explained more elaborately by Doyle in [Doyle 79], here we only give a glossary of his work.

We need some extra definitions. Informally, an *assumption* is a node which truth-value relies on one or several nodes, which are *false* as a result of negation as failure.

Definition 4.5 *Let D be a* JTMS.

- *Atom a is a* fact *if $a \in Cl(D, \emptyset)$.*
- *Literal a is an* assumption *if a is not a fact.*

- *The* maximal foundations *of a node c are the literals of the bodies of any justification having c as its conclusion.*

When the TMS makes $\bot$ *in* (truth-value: *true*), it invokes DDB (dependency-directed backtracking) to find and remove *at least one* of the current assumptions in order to make the contradiction node *out* (truth-value: *false*). Suppose $\bot$ is *in* because of justification j, we show this relation by writing $\bot_j$.

DDB consists of several steps:

1. *Find the maximal assumptions*

   Let S be the set of maximal assumptions underlying $\bot_j$. Node $a \in S$ iff $a \in Lit(body(j))$ and a is an assumption and no node b exists such that $b \in Lit(body(j))$ with a an element of the maximal foundations of b. Let $S = \{a_1, \ldots, a_n\}$.

2. *Summarize the cause of the inconsistency.*

   If no previous backtracking attempt on j discovered S to be the set of maximal assumptions, then create a new justification: $\bigwedge_{A \in S} \rightarrow \bot$, otherwise this justification would already have been created.

3. *Select and reject a culprit.*

   Select some a_i, the culprit, from S. Let $\alpha \wedge \beta \rightarrow a_i$ be a_i's supporting justification. Select $d \in At(\beta)$, d is called the *denial*, and justify it with $a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_n \wedge \beta^{-\{\sim d\}} \rightarrow d$. If the backtracker erred in choosing the culprit or denial, presumably a future contradiction will involve d and the remaining assumptions in β. However, if $\beta^{-\{\sim d\}} \neq \emptyset$, d will be an assumption, of higher level than the remaining assumptions, and so will be the first to be denied.

4. *Repeat if necessary.*

   If the TMS finds other arguments so that $\bot_j$ remains *in* after the addition of the new justification for d, repeat DDB. (Presumably a_i will no longer be an assumption.)

   Finally, if $\bot_j$ becomes *out* then halt, or if no assumptions can be found in $\bot_j$'s maximal foundations, notify the problem solver of an unanalyzable contradiction, then halt.

Note that in step (3) no selection criteria have been given. Several criteria are possible and give rise to the different DDB-strategies.

4.2 Multiple states of belief for constraint satisfaction

Giordano and Martelli present a logical semantics for justification-based truth maintenance systems which is able to capture the idea of dependency-directed backtracking. This semantics is based on a generalization of the Stable model semantics and an active use of constraints. Of overall importance is the use of contrapositions of justifications.

In the presence of constraints, conflicts can arise. The conflict resolution process is shown mainly to rely on the intuitive idea of a contrapositive use of justifications to resolve inconsistencies. To provide a logic characterization of this contradiction resolution process, they propose a generalization of stable models.

They claim that it is reasonable to require a unique canonical model when dealing with the problem of defining a semantics for negation as failure in logic programs. But they say: "On the contrary, when reasoning on the possible states of beliefs supported by a given set of justifications, multiple states of beliefs make perfectly sense and, thus, also sets of justifications with multiple stable models."

We do agree with their opinion in part, but not when dealing with *justification*-based truth maintenance, which has to deal with frequent non-monotonic reasoning. Then as said before it is too costly to maintain several possible states of beliefs (models). Giordano and Martelli note that it is possible to adapt the use of stable models to the situation when constraints are present in the set of justifications, by simply computing the stable models of the set of general clauses in the set of justifications, those that are no constraint, and then, later on, eliminating the stable models that do not satisfy the constraints (that is, the inconsistent stable models). In this way, constraints are simply used to cut out some inconsistent labelings and no belief revision is performed. Problems arise when for a set of justifications J no stable model exists which satisfies all constraints, then backtracking must be performed. However, since the Well-Founded model of the set of justifications J is the least 3-valued stable model of J, computing the Well-Founded model, directly tells whether or not backtracking is required.

In order to give a logical characterization for the TMS performing dependency-directed backtracking, Giordano and Martelli define a generalization of the notion of *stable model*, with this generalization they want to use constraints actively to perform belief revision. In this definition they take care of the fact that they need to use justifications also in their contrapositive form.

Let M be an interpretation in the classical propositional calculus. M can be regarded as a subset of the propositions occurring in J, those true in the interpretation. Let $\mathcal{J}'_M$ be the set of justifications obtained from J by deleting from the body of each justification all the literals $\sim b$ for which $b \notin M$, i.e. $\mathcal{J}'_M = \{\alpha \wedge \beta_1 \rightarrow c \mid \alpha \wedge \beta \rightarrow c \in J$ and $\beta = \beta_1 \wedge \beta_2$ and $M \models \beta_2\}$. Now write each justification $a_1 \wedge \ldots \wedge a_n \wedge \sim b_1 \wedge \ldots \wedge \sim b_h \rightarrow c$ in $\mathcal{J}'_M$ as the disjunction of literals $\neg a_1 \vee \ldots \neg a_n \vee b_1 \vee \ldots \vee b_h \vee c$. Let $\mathcal{J}_M$ be the set justifications in $\mathcal{J}'_M$ having one and only one literal true in M.

Definition 4.6 *[GiMar1] Let M be a set of propositions occurring in J. M is a* generalized stable model *of J if M is a model of J and $M = \{a \mid a$ proposition in J and $\mathcal{J}_M \models a\}$.*

Notice that, while negation in the initial set J, $\sim$, is a default negation, negation occurring in $\mathcal{J}_M$ is the classical one and $\models$ is the logical consequence in classical logic.

As the clauses in $\mathcal{J}_M$ have only one literal true in M, each of these clauses can be regarded as an implication having as its consequent that single literal which is true in model M. Accordingly, all the literals occurring in its body will also be true in M. This implies that $\mathcal{J}_M$ will not contain constraints, since each constraint containing a unique literal true in M occurs in $\mathcal{J}_M$ as an implication with that literal as its head, while other constraints are deleted. Thus negation in the head or body of the clauses of $\mathcal{J}_M$ is always interpreted as *classical negation*.

Notice that a set J of justifications can have several generalized stable models. On the other hand in the case that J is inconsistent there should not be a generalized stable model. J can be inconsistent if J contains constraints, however, since $\mathcal{J}_M$ is always consistent, it can happen that there is an interpretation M satisfying $M = \{a \mid a$ is a proposition in J and $\mathcal{J}_M \models a\}$, even if J is inconsistent. This is the reason that M is also required to be a model of J.

If there is a unique generalized stable model of J, then it precisely corresponds to the labeling computed by the TMS on backtracking. However sets of justifications can be found for which several generalized stable models exist, examples can be found in [GiMar1], so using this method, the TMS seems to lose its merit of finding only the preferred solution. However Doyle, in his original paper [Doyle 79], describes the behavior of the TMS *incrementally*, this incremental process can obtain solutions different from those expected in the static approach. This is due to the different methods for backtracking, the generalized stable models correspond to all situations which can be obtained by giving to the TMS the justifications in any order.

4.3 Single state of Belief for all possible belief revisions

Using contrapositions, classical negation appears in the head and in the body of justifications. Both occurrences are troublesome, as classical negation must be proved and negation is not allowed in the head of justifications. Therefore, the classical negation must be circumvented by adding a new atom a^- to stand for $\neg a$. To preserve the meaning of such negated atoms, some justifications and constraints must be added:

$$a \wedge a^- \rightarrow \bot$$

This ensures that not both a and a^- can be true. But we also have to ensure that not both a and a^- can be false. This can be done by adding:

$$\sim a \rightarrow a^- \text{ and } \sim a^- \rightarrow a$$

Now define the set $Back(J)$ of backward justifications as follows:

Definition 4.7 *Let D be a* JTMS.

- $E(a^-) = \{a \wedge a^- \rightarrow \bot, \sim a \rightarrow a^-, \sim a^- \rightarrow a\}$.
- $Back(j, \alpha) = \{\alpha' \wedge c^- \wedge \beta \wedge \sim a \rightarrow a^- \mid \alpha' = \alpha^{-\{a\}}, a \in \alpha\}$.
- $Back(j, \beta) = \{\alpha \wedge c^- \wedge \beta' \wedge \sim b^- \rightarrow b \mid \beta' = \beta^{-\{\sim b\}}, b \in At(\beta)\}$.
- $N^-(j) = \{a^- \mid a^- \in At(Back(j, \alpha) \cup Back(j, \beta))\}$.
- *Let $Anc(c)$ be the set of* direct ancestors *of $c \in N$: $Anc(c) = \{a \mid \exists j \in J : a \in body(j)$ and $hd(j) = c\}$.*
- *Let $Anc^*(c)$ be the transitive closure of $Anc(c)$.*
- $N^- = \bigcup\{N^-(j) \mid j \in J \text{ and } hd(j) \in Anc^*(\bot)\}$
- $Back^-(j) = \bigcup\{E(a^-) \mid a^- \in N^-\}$.
- $Back(j) = Back(j, \alpha) \cup Back(j, \beta) \cup Back^-(j)$.
- $B(J) = \bigcup\{Back(j) \mid j \in J \text{ and } hd(j) \in Anc^*(\bot)\}$.

Note that backward justifications will be created in $B(J)$ only for those justifications whose (in)direct consequences are $\bot$. Remarkable is that in $Back(j, \alpha)$ and analogously in $Back(j, \beta)$ the contrapositioned atom a (which appears as a^- as the head of the new justification) still appears (though negatively) in the body of the new justification. This ensures that in the Well-Founded model of $B(J) \cup J$, a (and a^-) will get truth-value u. The Revision-strategies defined below are therefore not equivalent to DDB-strategies as described in the introduction of this section.

Definition 4.8 *R is a* Revision-*strategy if it adds to J a subset of $B(J)$.*

As Witteveen shows in [Witteveen 91], $B(J)$ might enlarge the complexity of a JTMS D to $\mathcal{O}(|D| \times |N|)$ and because computing the Well-Founded model takes $\mathcal{O}(|N'| \times |D'|)$ for a JTMS $D' = (N', J')$, here $|N'| \leq 2|N|$ and $|D'| \leq |D| \times |N|$, the overall complexity of his algorithm is $\mathcal{O}(|N|^2 \times |D|)$.

Theorem 4.9 *[Witteveen 91] $WF(B(J) \cup J)$ is the knowledge-minimal 3-valued stable model for the class of all* Revision-*strategies.*

5 Conflict Resolution using one disjunctive justification (Cautious Backtracking)

We present two new backtracking methods which avoid the use of contrapositions. The first gives the same model as proposed by [Witteveen 91] in a more efficient and natural way. The second is an extension of the first in which more information is obtained from the contradiction.

The techniques used are variations of a method that will be called Cautious Backtracking (CB). In the following we will use WF_D to denote the Well-Founded model for JTMS D.

5.1 Introduction

There is another way to make use of the dependencies among beliefs. Given a model, some of the justifications denote the actual dependencies between truth values of propositions. These are the justifications whose body is evaluated true under the given model. In the Well-Founded model, these dependencies can be seen in the layered structure of the model.

Consider a JTMS D with constraints, for D we calculate the Well-Founded model according to Witteveen. But now we are not only interested in the resulting 3-valued model, but also in the several 3-valued interpretations from which this model has been comprised. These interpretations contain some of the information regarding the dependencies between the elements of N.

Definition 5.1 *Let $I_D^0 = (\emptyset, \emptyset)$, $I_D^{i+1} = W(D, I_D^i)$ and let k be such that I_D^k is the least fixpoint of W.*

So the Well-founded model is given by:

$$WF_D = (\bigcup_{0 \leq i \leq k} I_{D,t}^i, \bigcup_{0 \leq i \leq k} I_{D,f}^i)$$

From now on we will discard the subscript D, unless there can be doubt about which TMS is considered.

Definition 5.2 *Let $\tilde{I}_i = \bigcup_{l \leq i} I^l$ for all $i \geq 0$.*

Note that I_t^1 is the least fixpoint of the immediate consequence operator applied on the positive part of D, i.e. the justifications without negations. I_f^1 contains the atoms from the greatest unfounded set (Gus) with respect to I^0. Analogously, I_t^{i+1} is the least fixpoint of the immediate consequence operator applied on the positive part of $D^{-\tilde{I}^i}$ and I_f^{i+1} contains the atoms from the Gus with respect to I^i. Given the nature of the immediate consequence operator, each of the elements of I_t^{i+1} were justified by a justification $\alpha \wedge \beta \rightarrow c$ which had this element as its head and for which at least one literal in β was justified. In the same way the elements of I_f^{i+1} depend directly on I_t^i and I_f^i and maybe indirectly on several other I^k $(k < i)$.

Lemma 5.3 *If $I_f^i = \emptyset$ then $I_t^{i+1} = \emptyset$.*

Proof: Let $J' = J^{-\tilde{I}_i}$. Suppose $I_f^i = \emptyset$ but $c \in I_t^{i+1}$. Without loss of generality assume that c is a fact in J'. Known is that $c \notin I_t^i$, which implies that the body of all justifications with conclusion c in J' (of which there is at least one) are evaluated to u under interpretation I^i $(I^i(\alpha \wedge \beta) = u)$ while the positive part of this body (α) is evaluated to t under this interpretation $(I^i(\alpha) = t)$. (Remember that $At(\beta) \neq \emptyset$ otherwise $c \in I_t^i$ and that $I_f^i = \emptyset$.) But $\rightarrow c \in J'$ implies that $At(\beta) \in I_f^i$.

□

How can the knowledge of these dependencies be used to efficiently compute a new Well-Founded model for an altered J? Suppose that an atom $p \in N$ is in WF_f for a certain set of justifications J and the problem solver would rather have $p \in WF_t$, it can add a justification to J like $\rightarrow p$, then the JTMS will have to adjust the Well-Founded model to encompass this new justification. The only way p can be made false is by negation by failure, unless a constraint is present in J with p in its body. So in most cases adding $\rightarrow p$ will help. Suppose the JTMS knows (by keeping track of the list of interpretations I^i) that $p \in I_f^i$, then the only nodes that could be affected by p's change of truth-value are those not in $\tilde{I}_{i-1}$. So we can recompute the Well-Founded model starting at I^i. The only problem that could occur is that one of the constraints could be violated by this Well-Founded model, then the JTMS must perform backtracking. This will be possible unless the body of this constraint consists entirely of facts (positive literals all of which are element of I_t^1). More will be said about backtracking later on.

On the other hand suppose that $p \in N$ is in WF_t and the problem solver would rather have $p \in WF_f$, then it is not allowed to add $\rightarrow \neg p$ to J, but it can add a constraint $p \rightarrow \perp$. However there are other possibilities, by looking at the interdependencies of the atoms, it is sometimes possible to recalculate a part of the Well-Founded model. First the JTMS has to check whether or not $p \in WF_f$ would contradict $p \in I_t^1$, if this is not the case, then (if no constraints are present) it is possible to recalculate the Well-Founded model from I^i, where $p \in I_t^{i+1}$. Because p directly depends on the atoms in I_f^i. The JTMS acts as if one of the elements of I_f^i occurring in a justification with head p were added as fact to J and continues calculating the Well-Founded model at I^i. Picking only on of the elements of I_f^i as a fact to be added, corresponds with a specific dependency-directed backtracking method.

Making no selection corresponds to the CB methods.

5.2 Cautious Backtracking

We can avoid using contrapositions by noting that only *assumptions* can be responsible for the contradiction (otherwise D would not have any model). Assumptions are created by the use of negation as failure and apparently a number of atoms was not correctly considered *false*, leading to the contradiction. This set of atoms can be characterized as follows:

Definition 5.4 *Atom b belongs to the foundations of c in interpretation M ($b \in Gr(c, M)$) if $\exists \alpha \wedge \beta \rightarrow c \in J$ such that $M(\alpha \wedge \beta) = t$ and at least one of the following holds:*

1. $\sim b \in Lit(\beta)$.
2. $a \in Lit(\alpha)$ *and* $b \in Gr(a, M)$.

Instead of reasoning in the contrapositive direction, one can also search for those negative atoms, that belong to the foundations of $\perp$, i.e. $Gr(\perp, M)$. Note that in the Well-Founded interpretation the following holds: if $\perp \in I^i$ then $Gr(\perp, M) \subseteq (\tilde{I}_{i-1})_f$. Therefore $Gr(\perp, M)$ can be easily found in WF, by comparing $(\tilde{I}_{i-1})_f$ to J.

The approach described below actually comprises two different methods, CB_1 and CB_2. For CB_1 we prove that it corresponds with $B(J)$ as defined in section 4.3. In CB_2 the added justification can be a fact, leading to a more intrinsic model. In every step of the method described below we will denote how the proposed methods differ.

CB-method

Suppose WF is an inconsistent interpretation for D and suppose $\perp$ is not a fact (there exists a model for D). Let constraint δ (consequence of δ is $\perp$) be the justification which caused WF to be inconsistent.

1. *Determine* $Gr(\perp_\delta, WF)$.
   This set can easily be found in WF, by looking at the dependencies.
2. For CB_1 add to J the following (disjunctive) justification j:

$$\bigwedge \circ Gr(\perp_\delta, WF) \rightarrow \bigvee Gr(\perp_\delta, WF)$$

3. For CB_2 add to J the following (disjunctive) justification j:

$$\rightarrow \bigvee Gr(\perp_\delta, WF)$$

The more intrinsic method (CB_2) renders a set of justifications for which the Disjunctive Well-Founded model, as defined by Ross [Ross 89b], can be computed, which extends the skeptical model for CB_1. De difference between (1) and (2) is that $\bigwedge \circ Gr(\bot_\delta, WF)$ ensures that all atoms in the head of the negation will neither become false as a result of negation as failure, nor true. As can easily be seen, the complexity of D will never be enlarged by more the complexity of the added justification.

5.2.1 Skeptical model of disjunctive logic programs

Here we define a model for disjunctive logic programs which contains less information than the Well-Founded models (Weak and Strong) for disjunctive logic programs as defined by Ross in [Ross 89b]. This skeptical model is related to the Well-Founded model for normal logic programs. The reason we define such a skeptical model (Sk) is that we can prove that for all elements $a \in N$ $Sk(\mathsf{CB}_1(J))(a) = WF(B(J) \cup J)(a)$, where $B(J)$ is the set of justifications defined in 4.3.

Definition 5.5

- *A disjunctive justification has the format: $\alpha \wedge \beta \rightarrow \delta$, where $\alpha \wedge \beta$ is as before and δ is a disjunction of atoms.*
- *Let* DJTMS *stand for a* JTMS *containing disjunctive justifications.*

Definition 5.6 *Let I be an interpretation and let D be a* DJTMS.

- *Normal(D) is the part of D not containing disjunctive justifications.*
- *The Transitive Closure DCl of D is the transitive closure of Normal(D), i.e. $DCl(D, I) = Cl(Normal(D), I)$.*

Note that if δ contains more than one atom, then $I(\delta) = t$ will not be used to derive extra information, since δ does not occur in the body of any justification. In the Well-Founded semantics as defined by Ross extra information can sometimes be derived. The definition of an *unfounded set* is only slightly adjusted:

Definition 5.7 *Given a* JTMS *D with disjunctive justifications and a 3-valued interpretation I. Then we say that $A \subseteq N$ is an* unfounded set of D with respect to I *(denoted $unf(A, D, I)$) if for each atom $c \in A$ the following holds: For each justification $j \in J$ in whose head p occurs ($p \in At(\delta)$), at least one of the following holds: $I(\alpha \wedge \beta) = f$ or $At(\alpha) \cap A \neq \emptyset$.*

Definition 5.8 *The Skeptical model $Sk(D)$ of a* DJTMS *D is the least fixpoint of $\lambda I.(DCl(D, I) \cup \circ DGus(D, I))$.*

The time-complexity of *Sk* equals the complexity of the Well-Founded model, namely $\mathcal{O}(|N| \times |D|)$.

Corollary 5.9 *The time-complexity of backtracking and finding a new model (i.e. $Sk(\mathsf{CB}_1(J))$) equals $\mathcal{O}(|N| \times |D|)$.*

We extend the Skeptical model to an interpretation for all atoms in $B(J)$ and prove the extension to be a stable model for $B(J) \cup J$.

Definition 5.10 *Let M be the interpretation defined by:*

$$M(x) = \begin{cases} Sk(\mathsf{CB}_1(J))(x) \text{ if } x \in N \\ \sim Sk(\mathsf{CB}_1(J))(y) \text{ if } x = y^- \in N^- \end{cases}$$

Lemma 5.11 *M is a model of $B(J) \cup J$.*

Proof: Let $x \in N \cup N^-$ and suppose there is a justification $j \in B(J) \cup J$ with head x such that $M(x) <_{tr} M(body(j))$. As M is a model of J, $j \notin J$. Hence $At(j) \cap N^- \neq \emptyset$. For x we have two possibilities: $x \in N$ or $x \in N^-$.
• $x \in N^-$ then $x = y^-$, $y \in N$, this implies that $\sim y \in Lit(body(j))$. So $M(body(j)) \leq M(\sim y) = \sim M(y) = M(y^-) = M(x)$. And M satisfies j.
• $x \in N$. Remember $j \notin J$ so there is a $y^- \in At(j)$. Apparently $j \in Back(\alpha \wedge \beta \wedge \rightarrow y, \beta)$. This implies that $\sim x^- \in Lit(body(j))$. We have to check each of the three possible truth-values for x in M. If $M(x) = t$, then always $M(body(j)) \leq_{tr} M(x)$, so M satisfies j. If $M(x) = u$, then $M(\sim x^-) = u$. As $M(body(j)) = min_{tr}\{M(l) \mid l \in Lit(body(j))\}$, $M(body(j)) \leq_{tr} M(\sim x^-) = u$. So m satisfies j. If $M(x) = f$, then $M(\sim x^-) = f$ as well. Therefore, $M(body(j)) = f$ and M satisfies j.

□

Lemma 5.12 *M is a 3-valued stable model of $B(J) \cup J$.*

Proof: Let Min_a be the truth-minimal 2-valued model of $(N \cup N^-, B(J) \cup J)(M, t)$. Let Min_b be the truth-minimal 2-valued model of $(N \cup N^-, B(J) \cup J)(M, u)$. We have to prove that (see definition 3.2)

(a) $M_t = Min_a$ and

(b) $M_t \cup M_u = Min_b$.

a: Suppose $M_t \neq Min_a$, then there is a $x \in N \cup N^-$ such that

i. $x \in M_t$ but $x \notin Min_a$ or

ii. $x \in Min_a$ but $x \notin M_t$.

i:
• If $x \in N$, then $M(x) = t$ implies that $x \notin Anc^*(\perp)$, therefore $B(J)$ contains no justification with conclusion x. So $Min_a(x) = t$ as well.

• If $x = y^-$, then $M(x) = t$ implies that $M(y) = f$. Therefore no subset δ of N exists such that $y \in \delta$ and such that $\delta = hd(j)$ with $j \in \mathrm{CB}_1(J) - J$. Apparently $y \notin Anc^*(\perp)$, but this contradicts $x = y^-$.

ii:
• Suppose $x \in N$, then $x \in Min_a$ implies that there exists a justification j' in $(N \cup N^-, B(J) \cup J)(M, t)$ such that $Min_a(body(j')) = t$ and such that $Hd(j') = x$. Associated with j' must be a justification $j \in B(J) \cup J$. This justification j is a backward justification or $j \in E(x^-)$ or $j \in J$. In the first two cases $\sim x^-$ occurs in the body of j. As $j' \in (N \cup N^-, B(J) \cup J)(M, t)$, we know that $M(\sim x^-) = t$. This implies that $M(x^-) = f$, but then $M(x)$ must be t.

So $j \in J$ and $At(\alpha(j)) \subseteq Min_a$. So $\forall a \in At(\alpha(j))$ a monotonic proof for a exists in $(N \cup N^-, B(J) \cup J)(M, t)$, which begins with facts $\rightarrow c$. This implies that $\forall a \in At(\alpha(j))$ a non-monotonic proof exists in J, which begins with justifications $\beta \rightarrow c$, for which $M(\beta) = t$. This implies that $M(body(j)) = t$ and $M(x) = t$.
• Suppose $x \in N^-$, then there must be a justification $j = \alpha \wedge \beta \rightarrow x \in B(J)$ such that $M(\beta) = t$ and $Min_a(\alpha) = t$. Now $j \notin J$, so $\sim y \in Lit(\beta)$. Therefore $M(\sim y) = t$ and $M(y^-) = t$. Conclusion $M(x) = t$.

So $M_t = Min_a$ cannot be contradicted.

b: Suppose $M_t \cup M_u \neq Min_b$, then there is a $x \in N \cup N^-$ such that

i. $x \in M_f$ & $x \in Min_b$ & $x \in N$, or

ii. $x \in M_f$ & $x \in Min_b$ & $x \in N^-$, or

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[Witteveen 91] CEES WITTEVEEN, *Skeptical Belief Revision is Tractable.*, TU Delft, February 1991.

iii. $x \notin M_f$ & $x \notin Min_b$ & $x \in N$, or
iv. $x \notin M_f$ & $x \notin Min_b$ & $x \in N^-$.

i: $x \in Min_b$ implies that there is a justification $j = \alpha \wedge \beta \rightarrow x \in B(J) \cup J$ such that $M(\beta) \neq f$ and $Min_b(\alpha) = t$. If $j \notin J$, then $\sim x^- \in Lit(\beta)$, so $M(\sim x^-) \neq f$. Therefore $M(x^-) \neq t$ and $M(x) \neq f$. Contradiction, so $j \in J$. Then $M(\alpha) = f$ and $M(\beta) >_{tr} f$. $Min_b(\alpha) = t$ if and only if $\forall a \in At(\alpha)$ a monotonic proof for a exists, which begins with facts $\rightarrow c$. If and only if $\forall a \in At(\alpha)$ a non-monotonic proof for a exists, beginning with justifications $\beta' \rightarrow c$, such that $M(\beta') >_{tr} f$ (and so $M(c) >_{tr} f$. Therefore $M(\alpha \wedge \beta) >_{tr} f$ and $M(x) >_{tr} f$.

ii: As in i $x \in Min_b$ implies that there is a justification $j = \alpha \wedge \beta \rightarrow x \in B(J) \cup J$ such that $M(\beta) \neq f$ and $Min_b(\alpha) = t$. But now $j \notin J$. Let $x = y^-$, then $\sim y \in Lit(\beta)$. So $M(\sim y) \neq f$, implying that $M(y^-) \neq f$.

iii: The only case not properly treated earlier is: $x \in M_u$ & $x \in Anc^*$. This means that $\forall j = \alpha \wedge \beta \rightarrow x \in B(J) \cup J$ holds that if $M(\beta) \neq f$ then $Min_b(\alpha) = f$. Consider $\sim x^- \rightarrow x$, this justification is in $B(J) \cup J$ and $M(\sim x^-) = \sim M(x^-) = M(x) = u$. So $\rightarrow x \in (N \cup N^-, B(J) \cup J)(M, u)$. Therefore $Min_b(x) = t$, contradiction.

iv: Let $x = y^-$. In this case $y \in Anc^*(\perp)$. Consider $\sim y \rightarrow y^-$ which is in $B(J)$. As $M(y^-) \neq f$, $M(\sim y) \neq f$ as well. Therefore $\rightarrow y^- \in (N \cup N^-, B(J) \cup J)(M, u)$. Conclusion: $Min_b(y^-) = Min_b(x) = t$.

Conclusion: $M_t \cup M_u = Min_b$ cannot be contradicted. Therefore M is a stable model of $B(J) \cup J$. □

Corollary 5.13 $\forall a \in N\ Sk(\mathrm{CB}_1(J))(a) \geq_{kn} WF(B(J) \cup J)(a)$.

Lemma 5.14 *$WF(B(J) \cup J)$ is a model for* $\mathrm{CB}_1(J)$.

Proof: Suppose not, then there is a $\delta \subseteq N$ and a justification $j \in \mathrm{CB}_1(J)$ such that $hd(j) = \delta$ and $WF(B(J) \cup J)(\delta) \leq_{tr} WF(B(J) \cup J)(body(j))$. As $WF(B(J) \cup J)$ is a model of J, $j \notin J$. If δ consists of one atom c, then $c \in Anc^*(\perp)$, so $WF(B(J) \cup J)(c) \neq f$. Therefore $WF(B(J) \cup J)(\sim c) \neq t$, contradiction.

So δ is a disjunction of atoms and $j = \delta$. $WF(B(J) \cup J)$ is not a model of $\mathrm{CB}_1(J)$ implies that $WF(B(J) \cup J)(\delta) <_{tr} u$. $WF(B(J) \cup J)(\delta) <_{tr} u$ if and only if $\forall x \in At(\delta) : WF(B(J) \cup J)(x) = f$. If and only if $WF(B(J) \cup J)(\perp) = t$, contradiction. □

Lemma 5.15 $\forall a \in N\ Sk(\mathrm{CB}_1)(a) \leq_{kn} WF(B(J) \cup J)(a)$.

Proof: Here it is essential that $\rightarrow d$ was replaced by $\sim d \rightarrow d$. Suppose the lemma is not true, then an $a \in N$ exists such that $WF(B(J) \cup J)(a) <_{kn} Sk(\mathrm{CB}_1(J))(a)$. The only cases to be considered are, $Sk(\mathrm{CB}_1(J))(a) = t$ and $Sk(\mathrm{CB}_1(J))(a) = f$, while in both cases $WF(B(J) \cup J)(a) = u$.

- $WF(B(J) \cup J)(a) = u$ and $Sk(\mathrm{CB}_1(J))(a) = t$.

$WF(B(J) \cup J)(a) = u$ iff $\exists j \in B(J) \cup J$ with $hd(j) = a$ and $WF(B(J) \cup J)(body(j)) = u$, while $\forall j \in B(J) \cup J$ holds that $hd(j) = a$ implies $WF(B(J) \cup J)(body(j)) <_{tr} t$. Consider $Sk(\mathrm{CB}_1(J))$: $Sk(\mathrm{CB}_1(J))(a) = t$ iff $\exists j \in \mathrm{CB}_1(J) : hd(j) = a$ and $Sk(\mathrm{CB}_1(J))(body(j)) = t$. For this j there is only one possibility: $j \in J$, for $\mathrm{CB}_1(J) - J$ is not fact. Furthermore a must be an assumption, otherwise $WF(B(J) \cup J)(a) = t$ as well.

Apparently $\exists \sim b \in Anc^*(a)$ such that $Sk(\mathrm{CB}_1(J))(b) = f$ and $WF(B(J) \cup J)(b) >_{tr} f$. Iff $\exists j' \in B(J)$ with $hd(j') = b$ such that $WF(B(J) \cup J)(body(j')) >_{tr} f$. Iff $\sim b \in Anc^*(\perp)$. This implies that $b \in Gr(\perp, M)$, therefore $\exists j = \rightarrow \delta \in \mathrm{CB}_1(J) - J$ with $b \in At(\delta)$. This implies that $Sk(\mathrm{CB}_1(J))(b) = u$, contradiction.

- $WF(B(J) \cup J)(a) = u$ and $Sk(\mathrm{CB}_1(J))(a) = f$.

$Sk(\mathsf{CB}_1(J))(a) = f$ implies there is no $j \in \mathsf{CB}_1(J) - J$ with $a \in At(hd(j))$, but on the other hand $a \in Anc^*(\bot)$ (otherwise $WF(B(J) \cup J)(a) = Sk(\mathsf{CB}_1(J))(a))$. Furthermore $\forall j \in J$ holds that if $hd(j) = a$ then $Sk(\mathsf{CB}_1(J))(body(j)) = f$. So there must exist at least one $c \in At(hd(j))$, with $j \in \mathsf{CB}_1(J) - J$, such that $\sim c \in Anc^*(a)$ and such that $Sk(\mathsf{CB}_1(J))(c) = t$. But $\forall c \in At(hd(j))$ with $j \in \mathsf{CB}_1(J) - J$ holds that $Sk(\mathsf{CB}_1(J))(c) = u$, contradiction.

□

Theorem 5.16 *$Sk(\mathsf{CB}_1(D))$ is the knowledge minimal 3-valued Stable model for the class of all* Revision-*strategies.*

Proof: As can be seen from theorem 4.9, it is sufficient to prove: For all $a \in N$ $Sk(\mathsf{CB}_1(J))(a) = WF(B(J) \cup J)(a)$. Then the proof is direct from corollary 5.13 and lemma 5.15.

□

5.2.2 On the Semantics of CB_2

In the CB approach, we had to add $\sim c \rightarrow c$ instead of $\rightarrow c$ to the set of justifications, otherwise the Skeptical model would have contained more information than $WF(B(J) \cup J)$.
In CB_2 we add $\rightarrow c$ and not $\sim c \rightarrow c$, because disbelief in c was the only cause of the contradiction. Apparently the intrinsic value of c is t. This implies that the Skeptical model for CB_2 extends the Skeptical model for CB_1.
Furthermore we propose to use the Well-Founded model for disjunctive logic programs ([Ross 89b]) as the semantics of CB_2. The Well-Founded model for disjunctive logic programs is capable of making additional intrinsic choices.
Here we will give an example of the difference between CB_1 and CB_2, which concerns the inconsistency of D.

Example 5.17

Let $D = (N, J)$ with $N = \{a, b, c\}$ and let J be:

$$\begin{array}{ll} \sim c & \rightarrow b \\ \sim c & \rightarrow a \\ \sim b & \rightarrow a \\ a & \rightarrow \bot \end{array}$$

For D the Well-Founded model is $(\emptyset, \{c\}) \sqcup (\{b, a, \bot\}, \emptyset)$, the contradiction seems to depend on the falsity of c. So in the CB_2 method $\rightarrow c$ is add to J. Recomputing the Well-Founded model gives $(\{c\}, \emptyset) \sqcup (\emptyset, \{b\}) \sqcup (\{a, \bot\}, \emptyset)$. Again the contradiction can be traced to c, so J is inconsistent. The CB_1 method would add $\sim c \rightarrow c$ to J, for which the Well-Founded model is $(\emptyset, \emptyset)$. This is a 3-valued model, so the inconsistency of J will not be noticed.

In the previous section was mentioned that a Revision-strategy was not the same as a DDB-strategy. In general a DDB-strategy leads to more informative models than the corresponding Revision-strategy, because the truth-values will not always be shifted to u. Though we are still trying to prove it, we believe in the following:

Conjecture 5.18
Let DWF be the Disjunctive Well-Founded model. We conjecture that $DWF(\mathsf{CB}_2(D))$ is the knowledge-minimal 3-valued Stable model for the class of all DDB-*strategies.*

6 Conclusion

We presented a method for backtracking which avoids the use of contrapositions, but adds one (possibly disjunctive) justification to the JTMS. This method is called *Cautious* because, contrary to standard backtracking methods, it does not select a culprit, but leaves this choice open. Not only does the avoidance of contrapositions yield a more comprehensive method, but also the complexity of the algorithm is reduced by an order of magnitude. Finally, Cautious Backtracking avoids interchanging two different types of negation.

Two variants of Cautious Backtracking were described, the first (CB_1) has especially been designed to maintain all properties of Witteveen's proposal.

The second variant (CB_2) is an extension of the first in that more informative models can be produced, although the added justification is even less complex.

In some situations CB_2 is able to detect that no 2-valued model exists, whereas CB_1 renders a less informative 3-valued model. The choice between CB_1 and CB_2 therefore depends on the application.

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