The power of parallel projection

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Abstract

It is shown that from d projections of a k-flat into k-dimensional linear subspaces one can still reconstruct the k-flat. Furthermore, we show that a k-flat and a j-flat intersect in d-space if and only if they intersect in $\binom{d}{k+j+1}$ linear subspaces of dimension $(k+j+1)$ (and which are independent of the k-flat and the j-flat). Applications of these projection results are k-dimensional simplex searching for a set S of n points in d-dimensional space with a structure of size $O(n^{k+1+\epsilon})$ and $O(\log n)$ query time, for arbitrarily small positive ϵ. A second application is ray shooting in axis-parallel boxes in d-dimensional space, with a structure of size $O(n^{2+\epsilon})$ and $O(\log n)$ query time. Thirdly, we obtain a structure for k-flat intersection searching in a set of j-flats.

1 Introduction

Well-studied is the simplex query problem: preprocess a set $\mathcal{P}$ of n points in d-dimensional space into a data structure, such that for any query simplex s, all points of $\mathcal{P}$ that lie inside s can be counted or reported efficiently. The research of a long list of authors has lead to data structures of size $O(n^{d+\epsilon})$ and query times $O(\log n)$ or $O(\log n + K)$, where K is the number of answers, see e.g. [8, 16, 18]. (In this paper, ϵ denotes an arbitrarily small positive constant that should be chosen beforehand.) Other data structures have size $O(n)$ and query time $O(n^{1-1/d})$, plus $O(K)$ in the reporting case [9, 17, 18]. In a certain restricted model of computation, this is close to the best one could hope for [4]. These data structures are also the best known for searching with a hyperplane in a set of points, to find all points that intersect it, and it is easy to see that searching with a k-flat yields the same results (a k-flat is a k-dimensional affine subspace of d-space, see [12]). But can we do better with a k-flat? In this paper we answer this question to the affirmative. With the use of a basic lemma on parallel projections, which has other applications as

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well, we show that k-flat searching can be reduced to a 'combination' of d searches in $(k+1)$-dimensional space. This search is performed with a k-flats, thus hyperplanes. Furthermore, we present two more basic lemmas on projections, which we explain below.

Let d-dimensional space (or d-space for short) be equipped with a set of d base vectors $\vec{x}_1, \ldots, \vec{x}_d$. Let a k-dimensional (linear) subspace be standard if it is spanned by k of the base vectors, originating at the origin. Thus d-space has $\binom{d}{k}$ standard k-subspaces. We use parallel projections, denoted $\mathcal{PJ}_I$, which simply 'remove' one or more of the base vectors. The subscript I denotes the set of base vectors that span the image space, and are maintained. The inverse mapping $\mathcal{PJ}_I^{-1}$, applied to an object in the image space of $\mathcal{PJ}_I$, yields the region of d-space that maps onto the object. For a k-flat f in d-space, define F_I to be $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$. The first projection lemma that we prove states that if $\mathcal{PJ}_I$ ranges over all possibilities of projecting into standard $(k+1)$-space, then the intersection of the F_I is precisely f again. In fact, rather than taking all $\binom{d}{k+1}$ possibilities, only d of the F_I suffice to obtain precisely f. The second projection lemma considers the intersection of a k-flat f and a j-flat g in d-space. It shows that f and g intersect if and only if $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect for $\mathcal{PJ}_I$ ranging over all possibilities of projecting into standard $(k+j+1)$-space. The third projection lemma states that a k-flat f lies above a j-flat g if and only if f and g do not intersect, they do intersect in the projection on the hyperplane $x_d = 0$, and for at least one projection into standard $(k+1)$-space (including the x_d-coordinate), the projection of f lies above the projection of g.

The dimensions of the spaces into which we project are sharp in the sense that the lemmas generally do not hold when we map to a lower dimensional space.

We also show that if one of the flats is axis-parallel, then we can do better. In that case, the dimension of the axis-parallel flat plays no role in the dimension of the standard subspaces in which the intersection can be regarded.

The applications of the results on projections are obtained by using multi-layer data structures (see e.g. [1, 11, 13, 15, 25]). Multi-layer structures allow to express more complicated query problems as a combination of several more simple query problems. In our case, it is possible to express certain d-dimensional query problems as the combination of several lower dimensional query problems. This results in a gain in efficiency. In the first application, we study searching with a k-flat or a k-simplex in a set of points in d-space. By the first projection lemma, this is equivalent to a combination of $\binom{d}{k+1}$ searches with a k-flat or k-simplex in a set of points in $(k+1)$-space. Therefore, multi-layer structures solve the problem using $O(n^{k+1+\epsilon})$ storage rather than $O(n^{d+\epsilon})$ storage. The second application is a data structure for ray shooting in axis-parallel boxes in d-space. In this case, we can reduce the problem to a combination of 2-dimensional line intersection queries and ray shooting in axis-parallel line segments, to obtain an $O(n^{2+\epsilon})$ size data structure which supports ray shooting queries in $O(\log n)$ time. This result is a generalization of a result of de Berg et al. [3], who considered the 3-dimensional problem. We close with a data structure that stores a set of j-flats in d-space for k-flat intersection queries. The structure is a generalization of the results on lines in 3-space [2, 5, 24], and uses the Plücker transform and random sampling.

2 Projections

In this section we discuss aspects of the mapping called projection. A projection reduces the dimension of the space. We describe the *standard parallel projection* $\mathcal{PJ}$, which is most useful for our purposes. The projection $\mathcal{PJ}$ removes a subset of $d-k$ coordinates of any point in d-space, thereby giving a point in k-space. The linear subspace that is the image space of the projection $\mathcal{PJ}$ is spanned by k of the base vectors of d-space, and is called a *standard k-dimensional subspace* or *standard k-space.* Let $p = (p_1, \ldots, p_d)$ be any point in d-space, $d \geq 2$. We define the projection $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(p)$, where $1 \leq i_1 < \cdots < i_k \leq d$ is a subset of the coordinates, to be the point in k-space of which the i_j-coordinates are maintained and the other coordinates are omitted. For any geometric object o, consisting of a collection of points in d-space, we define $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(o)$ to be the set of points $\{(z_1, \ldots, z_k) \mid (z_1, \ldots, z_k) = \mathcal{PJ}_{\{i_1,\ldots,i_k\}}(p) \text{ where } p \in o\}$ in k-space. Projections have the following property:

Lemma 1 *Let o_1 and o_2 be two objects in d-space. If o_1 and o_2 intersect, then $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(o_1)$ and $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(o_2)$ intersect in standard k-space, for any $1 \leq i_1 < \cdots < i_k \leq d$.*

The inverse of Lemma 1 is not necessarily true. In any (non-trivial) projection, certain information is lost. However, there are situations in which it is possible to obtain information about the intersection of two objects from their intersections in a number of projections. In particular, this is the case for objects in higher dimensional space that do not use all the dimensions. Such objects are k-flats. In the following subsections we consider general k-flats first, and then we study orthogonal k-flats.

2.1 General k-flats

For a set $I \subseteq \{1, \ldots, d\}$, we define $\mathcal{PJ}_I^{-1}$ to be the inverse mapping of the projection $\mathcal{PJ}_I$ in the following way: $\mathcal{PJ}_I^{-1}(p)$ is the collection of points q in d-space for which $\mathcal{PJ}_I(q) = p$. The mapping $\mathcal{PJ}_I^{-1}$ *adds the base vectors* $\vec{x}_j$, where $j \notin I$, to a flat f' that lies in the image space of $\mathcal{PJ}_I$. The result is the subspace spanned by the flat f' and the vectors $\vec{x}_j$ which are translated to originate in f'. For a k-flat f and a set I, we define F_I to be $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$, or in words, the flat that is obtained if f is projected with $\mathcal{PJ}_I$ into a standard subspace, and then mapped back into d-space with $\mathcal{PJ}_I^{-1}$. Obviously, f is contained in F_I.

The first projection lemma that we prove states that if $\mathcal{PJ}_I$ ranges over all possibilities of projecting into standard $(k+1)$-space, then the intersection of the F_I

is precisely f again. In fact, rather than taking all $\binom{d}{k+1}$ possibilities, d projections and inverse mappings suffice.

As a concrete example of the first projection lemma, let ℓ be any line in 3-space. Project ℓ into the x_1x_2-plane, the x_1x_3-plane and the x_2x_3-plane, and we add to the projected line the base vectors $\vec{x}_3$, $\vec{x}_2$ and $\vec{x}_1$, respectively, then the intersection of these three flats (usually planes) is the original line ℓ again. See Figure 1 for an example.

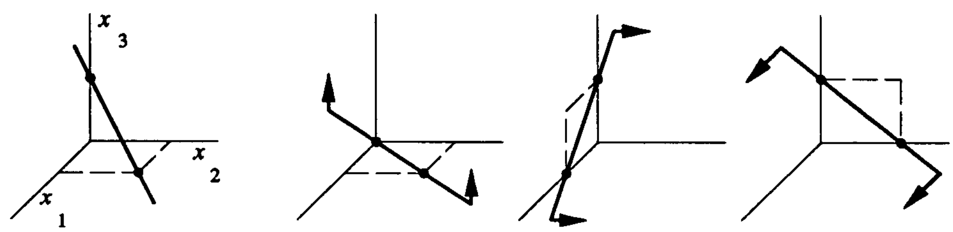

Figure 1: Left: a line in 3-space which intersects the x_3-axis in the one fat point and the x_1x_2-plane in the other fat point. The other three figures show the line projected in the x_1x_2-plane, the x_1x_3-plane and the x_2x_3-plane. The arrows show how the projected line is mapped back into 3-space by the inverse mappings.

Lemma 2 *For $0 \le k \le d-1$, let $\mathcal{I}$ be the collection of $\binom{d}{k+1}$ subsets of $\{1,\ldots,d\}$ of size $k+1$. For a k-flat f in d-space, we have*

$$\bigcap_{I \in \mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f,$$

where $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$ is the maximal region F_I for which $\mathcal{PJ}_I(F_I) = \mathcal{PJ}_I(f)$.

Proof: Since $f \subseteq F_I$ for every I, we have that $f \subseteq \bigcap_{I\in\mathcal{I}} F_I$. Because f is a k-flat, it is sufficient to prove that the dimensionality $\dim(\bigcap_{I\in\mathcal{I}} F_I) \le k$. We use induction on d, and there are two base cases.

Base case (i): If $k = d-1$, then $\mathcal{I} = \{1,\ldots,d\}$. Hence, $\mathcal{PJ}_I$ and $\mathcal{PJ}_I^{-1}$ are the identity transforms, and the statement is trivial.

Base case (ii): If $k = 0$, then f is a point $(p_1,\ldots,p_d)$, and the flats F_I are d hyperplanes with mutually perpendicular normal vectors, which intersect in one point, namely, f.

Assume that $0 < k < d-1$. Split $\mathcal{I}$ into $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_1$ contains all subsets of $\mathcal{I}$ that exclude d, and $\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1$. Notice that $\mathcal{I}_1$ and $\mathcal{I}_2$ are non-empty.

First, assume that f does not contain a line parallel to the x_d-axis. Let $\tilde{f}$ be the projection of f onto the hyperplane $x_d = 0$. Then $\tilde{f}$ is a k-flat in $(d-1)$-space, and

by induction, $\dim(\bigcap_{I\in\tilde{\mathcal{I}}_1} \tilde{F}_I) \leq k$, where $\tilde{\mathcal{I}}_1 = \{ \{a_1,\ldots,a_k\} \mid \{a_1,\ldots,a_k,d\} \in \mathcal{I}_1\}$ and $\tilde{F}_I = \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(\tilde{f}))$. Consequently, $\dim(\bigcap_{I\in\mathcal{I}_1} F_I) \leq k+1$, because the d-th base vector is added.
Claim: There is a subset $J \in \mathcal{I}_2$ of indices for which F_J does not contain a line parallel to the x_d-axis.
Proof: Take the projection $\tilde{f}$ of f onto the hyperplane $x_d = 0$. Choose any set $\mathcal{K}$ of k linearly independent vectors in $\tilde{f}$. Choose a set $\mathcal{J}$ of $(d-1)-k$ base vectors such that $\mathcal{J} \cup \mathcal{K}$ spans the hyperplane $x_d = 0$ (which is $(d-1)$-space). The set J of indices that corresponds to the base vectors that are not in the set $\mathcal{J}$ is the required set of the claim.

Because $\bigcap_{I\in\mathcal{I}_1} F_I$ is a $(k+1)$-flat which contains a line parallel to the x_d-axis, and F_J is a j-flat with $j \leq d-1$ which does not contain a line parallel to the x_d-axis, we have $\dim(\bigcap_{I\in\mathcal{I}_1} F_I \cap F_J) \leq k$. Thus also $\dim(\bigcap_{I\in\mathcal{I}} F_I) \leq k$, which proves the induction step.

Second, assume that f contains a line parallel to the x_d-axis. Let $\tilde{f}$ be the projection of f onto the hyperplane $x_d = 0$. Then $\tilde{f}$ is a $(k-1)$-flat in $(d-1)$-space, and by induction we have $\dim(\bigcap_{I\in\tilde{\mathcal{I}}_1} \tilde{F}_I) \leq k-1$. Consequently, $\dim(\bigcap_{I\in\mathcal{I}_1} F_I) \leq k$, because the d-th base vector is added. Thus also $\dim(\bigcap_{I\in\mathcal{I}} F_I) \leq k$, which proves the induction step. □

Corollary 1 *With the notation of the above lemma, there exists a subcollection $\mathcal{I}'$ of $\mathcal{I}$ of size d such that*

$$\bigcap_{I\in\mathcal{I}'} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f.$$

Proof: Follows immediately from the proof of the above lemma. □

Corollary 2 *With the notation of the above lemma,*

$$\bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(g)) = g$$

holds when g is a j-flat and $j \leq k$.

Corollary 3 *With the notation of the above lemma, we have for a point p in d-space: $p \in f$ if and only if for all $I \in \mathcal{I}$: $\mathcal{PJ}_I(p) \in \mathcal{PJ}_I(f)$.*

Proof: The 'only if' follows from Lemma 1. Next, suppose that for all $I \in \mathcal{I}$: $\mathcal{PJ}_I(p) \in \mathcal{PJ}_I(f)$. Then $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(p)) \subseteq \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$ for all $I \in \mathcal{I}$, by definition of $\mathcal{PJ}_I^{-1}$. By the previous corollary, $\bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(p)) = p$, thus $p \in \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$ for all $I \in \mathcal{I}$, and $p \in \bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f$ by Lemma 2. □

The second projection lemma considers the intersection of a k-flat f and a j-flat g in d-space. It shows that f and g intersect if and only if $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect for all $\mathcal{PJ}_I$ ranging over all possibilities of projecting into standard $(k+j+1)$-space. As a concrete example, two lines in 4-space intersect if and only if they intersect in all four projections into standard 3-dimensional subspaces.

Lemma 3 *For $k \geq 0$, $j \geq 0$ and $j+k \leq d-1$, let $\mathcal{I}$ be the collection of $\binom{d}{k+j+1}$ subsets of $\{1,\ldots,d\}$ of size $k+j+1$. A k-flat f and a j-flat g in d-space intersect if and only if for all $I \in \mathcal{I}$, we have that $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect (in $(k+j+1)$-space).*

Proof: Let $\vec{u}_1,\ldots,\vec{u}_k$ be a set of base vectors for f, and let p be a point in f. Then $f = p+a_1\vec{u}_1+\cdots+a_k\vec{u}_k$ for reals $a_1,\ldots,a_k$. Similarly, let $g = q+b_1\vec{v}_1+\cdots+b_j\vec{v}_j$ for a point q in g, a base $\vec{v}_1,\ldots,\vec{v}_j$ of g, and reals $b_1,\ldots,b_j$. Define F as follows. Take a maximal set of independent vectors from $\vec{u}_1,\ldots,\vec{u}_k,\vec{v}_1,\ldots,\vec{v}_j$—let it be $\vec{w}_1,\ldots,\vec{w}_m$. Then $F = p+c_1\vec{w}_1+\cdots+c_m\vec{w}_m$ for reals $c_1,\ldots,c_m$. We have the following relations for f, g and F: (i) $f \subseteq F$, (ii) g and F are parallel, that is, all vectors of g also appear in F after the appropriate translation, and (iii) g and f intersect if and only if $g \subseteq F$.

The important observation is: for all $I \in \mathcal{I}$, we have either $\mathcal{PJ}_I(g) \cap \mathcal{PJ}_I(F) = \emptyset$, or $\mathcal{PJ}_I(g) \subseteq \mathcal{PJ}_I(F)$. (The observation is true by (ii).) The proof can be completed as in Corollary 3. □

A flat is vertical if and only if it contains a line parallel to the x_d-axis. We say that a non-vertical k-flat f lies above a non-vertical j-flat g if and only if f and g do not intersect, and there is a vertical line intersecting f and g for which the intersection point with f lies above the intersection point with g.

Lemma 4 *For $k \geq 0$, $j \geq 0$ and $j+k \leq d-1$, a non-vertical k-flat f lies above a non-vertical j-flat g if and only if (i) f and g do not intersect, (ii) $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$ do intersect, and (iii) for at least one subset I of $\{1,\ldots,d-1\}$ of size $k+j+1$, $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$.*

Proof: 'only if': Assume that f lies above g. Then (i) and (ii) are true by definition. Let ℓ be a vertical line that intersects f in a point p and g in a point q. Then $\mathcal{PJ}_I(p)$ and $\mathcal{PJ}_I(q) \in \mathcal{PJ}_I(\ell)$, and since $\mathcal{PJ}_I$ is a parallel projection perpendicular to the x_d-axis, $\mathcal{PJ}_I(p)$ lies above $\mathcal{PJ}_I(q)$. Consequently, for any subset I as in the lemma, $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$ or they intersect. Assume that for all subsets I they intersect. Because $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$ also intersect, we conclude by Lemma 3 that f and g intersect, a contradiction. Hence, there is at least one subset I for which $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$.

'if': Assume that (i), (ii) and (iii) of the lemma hold. Let ℓ be any vertical line that contains an intersection point of $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$, thus $\ell \subseteq \mathcal{PJ}_d^{-1}(\mathcal{PJ}_d(f) \cap \mathcal{PJ}_d(g))$. Let $p = \ell \cap f$ and let $q = \ell \cap g$. We have to show that p lies above q.

Let I' be the subset for which $f' = \mathcal{PJ}_{I'}(f)$ lies above $g' = \mathcal{PJ}_{I'}(g)$, as in (iii). By definition, f' and g' are non-vertical, and there is a vertical line ℓ' such that $p' = \ell' \cap f'$ lies above $q' = \ell' \cap g'$. We have that ℓ' is a vertical line in the image space of $\mathcal{PJ}_{I'}$. We now consider $\ell'' = \mathcal{PJ}_{I'}(\ell)$ in the same image space. Let $p'' = \mathcal{PJ}_{I'}(p)$ and $q'' = \mathcal{PJ}_{I'}(q)$. Let L be the 2-flat which contains ℓ'' and ℓ' in the image space of $\mathcal{PJ}_{I'}$ (see Figure 2). If p does not lie above q, then p'' does not lie above q'', and the line segments $\overline{p''p'}$ and $\overline{q''q'}$ intersect in L. Hence, f' and g' intersect (they contain these line segments), which contradicts the statement that f' lies above g'. Consequently, p lies above q. □

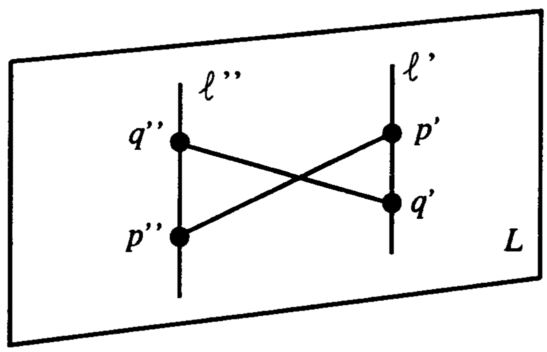

Figure 2: Situation inside L.

2.2 Orthogonal k-flats

Let an orthogonal k-flat be a k-dimensional affine subspace of d-space, spanned by the translation of k base vectors of d-space. In similar ways as above it is possible to obtain projection lemmas for orthogonal k-flats. For example, one can show that in order to test the intersection of an orthogonal k-flat and a general j-flat, one can consider a constant number of instances of the problem in $(j+1)$-space. We state the analogs of the Lemmas 2, 3 and 4 below. The proofs of these lemmas follow in the same way as in the general case.

Lemma 5 *For $0 \leq k \leq d-1$, let $\mathcal{I}$ be the collection of d subsets of $\{1, \ldots, d\}$ of size 1. For an orthogonal k-flat f in d-space, we have*

$$\bigcap_{I \in \mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f.$$

Lemma 6 *For $0 \leq k \leq d-1$ and $0 \leq j \leq d-1$, let $\mathcal{I}$ be the collection of $\binom{d}{j+1}$ subsets of $\{1, \ldots, d\}$ of size $j+1$. An orthogonal k-flat f and a general j-flat g* *in d-space intersect if and only if for all $I \in \mathcal{I}$, we have that $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect (in $(j+1)$-space).*

Lemma 7 *For $0 \leq k \leq d-1$ and $0 \leq j \leq d-1$, a non-vertical orthogonal k-flat f lies above a non-vertical general j-flat g if and only if (i) f and g do not intersect, (ii) $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$ do intersect, and (iii) for at least one subset I of $\{1, \ldots, d-1\}$ of size $j+1$, $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$.*

We remark that all given lemmas and corollaries also hold for geometric objects that lie inside the flats, rather than the whole flats themselves. For instance, a triangle t in d-space can be reconstructed from all $\binom{d}{3}$ projections into standard 3-space.

3 Applications

Before we give applications of the projection lemmas, we state a general result on the performance of multi-layer structures. Similar results can be found in [1, 11, 13, 15, 25]. If a data structure consists of a main tree that is used for simplex queries, and each node stores an associated structure for some query problem, then we say that a simplex composition has been applied to (the data structure for) that query problem. The main tree can be used to select all points that lie in a given query simplex in a small number of subsets that are associated to certain nodes. At these nodes, the associated structure is queried. The answers to the query are those objects that are an answer both in the main tree and in the associated structure. The performance of any data structure to which a simplex composition is applied, is as follows:

Theorem 1 *Let $\mathcal{P}$ be a set of n points in d-space ($d \geq 2$), and let $\mathcal{S}$ be a set of n objects. Let T' be a data structure on $\mathcal{S}$ of size $O(f(n))$ and with query time $O(g(n))$. For an arbitrarily small constant $\epsilon > 0$, the application of simplex composition on $\mathcal{P}$ to T' results in a data structure*

(i) *of size $O(n^\epsilon(n^d + f(n)))$ and query time $O(\log n + g(n))$,*

(ii) *of size $O(n + f(n))$ and query time $O(n^\epsilon(n^{1-1/d} + g(n)))$,*

(iii) *of size $O(m^\epsilon(m + f(n)))$ and query time $O(n^\epsilon(g(n) + n/m^{1/d}))$, for any $n \leq m \leq n^d$,*

assuming that $f(n)/n$ is non-decreasing and $g(n)/n$ is non-increasing.

3.1 k-Simplex intersection searching in sets of points

We stated in the introduction that efficient solutions to simplex query problem on a set $\mathcal{P}$ of n points in d-space exist. It can be solved with a data structure of size $O(n^{d+\epsilon})$, such that a simplex counting query can be answered in $O(\log n)$ time. Another structure has size $O(n)$ and answers simplex counting queries in $O(n^{1-1/d})$

time. These data structures also apply to searching with a hyperplane or k-flat in a set of points, to find all points that intersect it. We solve the k-flat query problem with the use of the basic lemma on parallel projections: If a k-flat and a point intersect in all projections into $(k+1)$-dimensional standard subspace, then the k-flat and the point intersect in d-space. Hence, we may treat this intersection problem in d-space as the composition of several k-flat query problems in $(k+1)$-space.

Theorem 2 *For an arbitrarily small constant $\epsilon > 0$, a set $\mathcal{P}$ of n points in d-space can be preprocessed into a data structure of size $O(n^{k+1+\epsilon})$, such that all K points of $\mathcal{P}$ that lie in a given query k-flat can be counted in $O(\log n)$ time or reported in $O(\log n + K)$ time. A structure of linear size answers counting queries in $O(n^{1-1/(k+1)+\epsilon})$ time and reporting queries in $O(n^{1-1/(k+1)+\epsilon} + K)$ time.*

Proof: By Lemma 2, the intersection query problem in the theorem is the composition of $\binom{d}{k+1}$ intersection query problems with hyperplanes in $(k+1)$-space. For each projection $\mathcal{PJ}_I$ into standard $(k+1)$-space we consider the set $\mathcal{PJ}_I(\mathcal{P})$ of points. We apply a simplex composition on this set to be able to select—for any query hyperplane in $(k+1)$-space—the points that intersect it. Hence, the application of a constant number of simplex compositions of Theorem 1 in $(k+1)$-space yields the stated bounds. □

The above result can be extended to k-simplices rather than k-flats. Corollary 2 states the desired property (that Lemma 2 is also valid for k-simplices), and we obtain:

Corollary 4 *A set $\mathcal{P}$ of n points in d-space can be preprocessed into a data structure of size $O(n^{k+1+\epsilon})$, such that all K points of $\mathcal{P}$ that lie in a given query k-simplex can be reported in $O(\log n + K)$ time or counted in $O(\log n)$ time. A structure of linear size answers counting queries in $O(n^{1-1/(k+1)+\epsilon})$ time and reporting queries in $O(n^{1-1/(k+1)+\epsilon} + K)$ time.*

3.2 Ray shooting in axis-parallel d-boxes

We study the problem of preprocessing a set $\mathcal{B}$ of n d-boxes in d-space for efficient ray shooting queries. In 3-space, this problem has been considered in [3], and they solve it with a structure of size $O(n^{2+\epsilon})$ and query time $O(\log n)$. Here we generalize this result to d-space, while maintaining the performance.

Partition the set of all facets of the d-boxes of $\mathcal{B}$ into d subsets of size $2n$, one for each orientation of the facet. Notice that we might as well perform d separate ray shooting queries in these subsets rather than one query in all boxes of $\mathcal{B}$. As a final step of the query algorithm, we choose as the answer to the query the one facet of the d subanswers that is intersected first by the query ray. Hence, we only need consider ray shooting with an arbitrary ray in the set $\hat{\mathcal{B}}$ of $(d-1)$-boxes which lie in hyperplanes perpendicular to the x_1-axis. The other sets of facets are handled analogously.

Observe that if the direction of the query ray has a positive value in the x_1-component, then the set $\hat{\mathcal{B}}$ admits a stabbing order by projection onto the x_1-axis. If two boxes of $\hat{\mathcal{B}}$ are intersected by the query ray, then the one with minimum x_1-coordinate is the answer. The converse statement is true if the direction of the query ray has a negative value in the x_1-component.

We also observe by Lemma 6 on the projections of flats, that a query ray intersects a $(d-1)$-box if and only if the ray and the box intersect in the projections into all $\binom{d}{2}$ standard 2-dimensional subspaces of d-space. Hence, we build the following structure for $\hat{\mathcal{B}}$. Let T be a binary search tree on the set $\mathcal{PJ}_1(\hat{\mathcal{B}})$ of points on the x_1-axis, obtained by projecting the facets of $\hat{\mathcal{B}}$. For all integers i, j with $1 \leq i < j \leq d$, project $\hat{\mathcal{B}}$ into the $x_i x_j$-plane, giving a set $\hat{\mathcal{B}}_{ij} = \mathcal{PJ}_{\{i,j\}}(\hat{\mathcal{B}})$ of axis-parallel line segments or rectangles. The rectangles are regarded as four edges, so we may assume that $\hat{\mathcal{B}}_{i,j}$ is a set of axis-parallel line segments in the plane. Let $\hat{\mathcal{P}}_{ij}$ be the set of points obtained by choosing one endpoint of each segment of $\hat{\mathcal{B}}_{ij}$, and let $\hat{\mathcal{P}}'_{ij}$ be the set of other endpoints. We apply a simplex composition on $\hat{\mathcal{P}}_{ij}$ and one on $\hat{\mathcal{P}}'_{ij}$ to the structure T. When performing a query, we use the line that contains the projected query ray. Hence, one can select all facets of $\hat{\mathcal{B}}_{ij}$ that intersect this line in canonical subsets using these two simplex compositions. The resulting data structure is a binary tree with $2\binom{d}{2}$ simplex compositions applied to it.

To perform a query, we project the query ray into the $x_i x_j$-plane (for $1 \leq i < j \leq d$), compute the line that contains the projected ray, and use it to select canonical nodes. After the first $2\binom{d}{2}$ layers, we have selected all facets of $\hat{\mathcal{B}}$ that are intersected by the line containing the query ray in canonical nodes, by Lemma 6. For each canonical node, we search in the associated binary search tree with the x_1-coordinate of the point where the query ray starts. The facet with larger x_1-coordinate, and minimal among these, is the answer to the query at that canonical node. The answer to the query in the full structure is the facet with the smallest x_1-coordinate among the ones found at the canonical nodes. The answer to the query in the set $\mathcal{B}$ of boxes is obtained after d such queries (one for each set of parallel facets) and choosing as the final answer the box that is hit first, among the ones found. We have obtained:

Theorem 3 *Let $\mathcal{B}$ be a set of n d-boxes in d-space. For an arbitrarily small constant $\epsilon > 0$, the set $\mathcal{B}$ can be stored in a data structure of size $O(n^{2+\epsilon})$, such that ray shooting queries with arbitrary rays can be answered in $O(\log n)$ time. A structure of linear size answers a query in $O(n^{1/2+\epsilon})$ time.*

3.3 k-Flat intersection searching in sets of j-flats

Next we consider k-flat intersection searching in sets of j-flats. We remark that even the problem of preprocessing a set L of lines in 3-space for line intersection queries is difficult. Research on the computational geometric side of lines in 3-space has appeared in [5, 6, 19, 20, 22, 23]. The solutions of Chazelle et al. [5, 6] and Pellegrini [20, 22] make use of Plücker coordinates, and can be generalized to flats in higher dimensional spaces.

The incidence of a point and a plane in 3-space is described by one linear equation, and this is generally true for points and hyperplanes in d-space. However, the incidence of a point and a line in 3-space is not described by one linear equation. This is not surprising, because a line in 3-space is determined by four parameters, whereas a point or a plane is determined by only three. If we assume that the line ℓ is not horizontal, then ℓ is, for instance, uniquely determined by its intersection points with the planes $x_3 = 0$ and $x_3 = 1$. In each of these planes, two parameters are needed to specify the intersection points, which totals up to four parameters. From this discussion one could hope that for two lines in 3-space, the intersection can be described as a linear equation in 4-space. Unfortunately, this is not the case. The equation is not linear but quadratic. In this subsection we will look at a transform that turns the test for intersection of two lines in 3-space into a linear equation in 6-space. More generally, we will consider intersections between j-flats and k-flats in higher dimensional space. The transform that makes these intersection tests into linear ones is called the Plücker transform or Grassmann transform. Previous descriptions and applications to computational geometry have been given by Stolfi [24], Chazelle et al. [5, 6] and Pellegrini [20]. The latter papers only use the Plücker transform for lines in 3-space.

Let f be a k-flat in d-space. Then f is determined by $k+1$ points (it is the smallest linear variety that contains these points), or f is determined by the intersection of $d-k$ hyperplanes with linearly independent normal vectors. In order to describe the Plücker transform one should consider the problem in *oriented projective space* (see e.g. [24]). Then f is a $(k+1)$-flat in $(d+1)$-space, and f is determined by either $k+1$ points in $(d+1)$-space, or by $d-k$ hyperplanes in $(d+1)$-space. Assume that $p_i = (p_{i0}, p_{i1}, \ldots, p_{id})$, for $0 \leq i \leq k$, are the $k+1$ points that determine f. Then f is described by the matrix M_f:

$$M_f = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0d} \\ p_{10} & p_{11} & \cdots & p_{1d} \\ \vdots & \vdots & & \vdots \\ p_{k0} & p_{k1} & \cdots & p_{kd} \end{pmatrix}$$

This matrix is called the *simplex representation* of f. Clearly, there are many simplex representations of the same flat f (unless $k = 0$). The *Plücker coordinates* of f are obtained by taking all $(k+1) \times (k+1)$ submatrices of M_f, and computing the determinants. The value of each determinant gives one coordinate in the Plücker representation of f. Since M_f is a $(k+1) \times (d+1)$ matrix, f has $\binom{d+1}{k+1}$ Plücker coordinates. A line in 3-space, for instance, has 6 Plücker coordinates.

Let g be a j-flat in d-space with $j = d - k - 1$. In oriented projective space, g is a $(j+1)$-flat in $(d+1)$-space, determined by either $j+1$ points, or $d-j$ hyperplanes. The flat g also has a simplex representation, being a matrix M_g. Suppose that we compute the Plücker coordinates of g. (Notice that—because of the assumption $d = j + k + 1$—there are $\binom{d+1}{j+1} = \binom{d+1}{k+1}$ Plücker coordinates of g, which is the same number as for f.) A proper rearrangement of the Plücker coordinates, and the negation of some of them, yields the *Plücker coefficients* of g (see [5, 6, 14, 20, 24] for more details). One can prove:

Lemma 8 *Flats f and g intersect (or are parallel) if and only if the dot product of the Plücker coordinates of f and the Plücker coefficients of g is 0.*

Consequently, if we consider a k-flat to be a point in $\binom{d+1}{k+1}$-space with as coordinates its Plücker coordinates, then we can consider a j-flat to be a hyperplane in $\binom{d+1}{k+1}$-space, and f and g intersect or are parallel if and only if the point is incident to the hyperplane in the space called *Plücker space.*

The dimension of the Plücker space can be reduced by one, by observing that the hyperplane always passes through the origin. In fact, the Plücker coordinates and coefficients can be regarded as the homogeneous coordinates and coefficients in projective oriented $(\binom{d+1}{k+1} - 1)$-space. (Intuitively, we project all hyperplanes and points in $\binom{d+1}{k+1}$-space onto a sphere centered at the origin. The surface of the sphere can be regarded as projective oriented $(\binom{d+1}{k+1} - 1)$-space, and the incidences are preserved in this projection.) We define π to be the transform that maps a k-flat to the point in projective oriented $(\binom{d+1}{k+1} - 1)$-space defined by its Plücker coordinates. Similarly, if $j = d - k - 1$, we define ϖ to be the transform that maps a j-flat to the hyperplane in projective oriented $(\binom{d+1}{k+1} - 1)$-space defined by its Plücker coefficients. We then have:

Lemma 9 *A k-flat f and a j-flat g in $(j+k+1)$-space intersect or are parallel if and only if the point $\pi(f)$ is incident to the hyperplane $\varpi(g)$ in projective oriented D-space, where $D = \binom{j+k+2}{k+1} - 1$.*

Remark: It should be noted that not every point in D-space is the image of some k-flat. This is intuitively clear because a non-degenerate k-flat in $(k+j+1)$-space is determined by only $(k+1)\cdot(j+1)$ parameters. In fact, the image of all k-flats is an algebraic surface of dimension $(k+1)\cdot(j+1)$ in D-space, called the *Grassmann manifold* or *Grassmannian.* For more details we refer to [14, 24].

For any k-flat and j-flat, we can consider their intersection in the so-called *join* of the two flats, which is the space spanned by the base vectors of both flats, together with a vector from the one flat to the other (see e.g. Stolfi [24]). The join is an at most $(k+j+1)$-dimensional linear variety. If a j-flat and a k-flat intersect, then this happens in the join, because both flats are completely contained in the join. However, if we consider k-flat intersection search in a set of j-flats, we cannot use the join, because at the time of preprocessing, the join is not known yet, and furthermore, for each j-flat we have a different join with the k-flat. In order to deal with the problem in $(k+j+1)$-space, Lemma 3 shows that we may consider the intersection in a constant number of standard $(k+j+1)$-dimensional subspaces, which are the $\binom{d}{k+j+1}$ linear subspaces spanned by $k+j+1$ of the base vectors of d-space. Thus these subspaces are independent of the j-flats and k-flats.

From Lemma 9 and Theorem 1 we immediately obtain the following:

Theorem 4 *Let S be a set of j-flats in d-space. For an arbitrarily small $\epsilon > 0$, one can preprocess S for k-flat intersection queries into a data structure of size $O(n^{D+\epsilon})$, such that all j-flats that intersect a query k-flat can be counted in $O(\log n)$ time, where $D = \binom{k+j+2}{j+1} - 1$ (see also Table 1).*

$k \backslash j$	1	2	3	4
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2	9	19	34	55
3	14	34	69	125
4	20	55	125	251

$k \backslash j$	1	2	3	4
1	4	7	11	15
2	7	14	23	35
3	11	23	42	72
4	15	35	72	138

Table 1: The exponents D (left) and D' (right) as in the two theorems for small values of j and k and any value of d.

The results of the above theorem can be improved. The remark after Lemma 9 states that the image of all k-flats is a $((k+1)\cdot(j+1))$-dimensional hypersurface in D-space. As in the case of lines in 3-space [3, 5, 6, 20, 22], this can be exploited. This hypersurface, the Grassmann manifold, is algebraic and of constant degree, so we can use the results on the complexity of the zone of such a surface, which is $O(n^{\lfloor (D+(k+1)\cdot(j+1))/2 \rfloor} \log n)$, see [2, 21]. Unfortunately, the structure that exploits the fact that only the zone of the Grassmann manifold is interesting is not based on cuttings. Therefore, we cannot use the simplex compositions of Theorem 1. Instead, we use a related structure (as was done for lines in space in [3, 5, 6, 20, 22]) based on Clarkson's random sampling method [10]. We refer to the cited papers for the details, and obtain:

Theorem 5 *Let S be a set of j-flats in d-space. For an arbitrarily small constant $\epsilon > 0$, we can preprocess S for k-flat intersection queries into a data structure of size $O(n^{D'+\epsilon})$, such that all j-flats that intersect a query k-flat can be counted in $O(\log^{O(1)} n)$ time, where $D' = \lfloor (\binom{k+j+2}{j+1} - 1 + (k+1)\cdot(j+1))/2 \rfloor$ (see also Table 1).*

4 Conclusions

We presented several geometric lemmas on the parallel projection of k-flats. They showed that the containment of a point in a k-flat in d-space can be reduced to d instances of the containment of a point in a k-flat in $(k+1)$-space. Or, more generally, the intersection of a k-flat and a j-flat can be reduced to a constant number of instances of the same problem in $(k+j+1)$-space. Furthermore, it is also possible to obtain such a characterization for the above-below relation of a k-flat and a j-flat. This relation can be considered as several instances of the intersection, the non-intersection and the above-below relation in $(k+j+1)$-space. All these results hold for general flats. When we are considering orthogonal flats, then we can do considerably better. The intersection of an orthogonal k-flat and a general j-flat reduces to $(j+1)$-space, for instance.

The lemmas on projections were applied in combination with known results on cuttings, partition trees and other structures, in order to generalize them to higher dimensional spaces (but while maintaining the dimensions of the flats). One of the applications is an efficient solution to ray shooting in axis-parallel d-boxes in d-space (an $O(n^{2+\epsilon})$ size structure with query time $O(\log n)$). Another application concerned k-flat search in a set of j-flats, which generalizes the work on lines in 3-space [5, 19, 23]. Perhaps the most interesting observation is that the *dimension of the stored objects and query objects* is more relevant to the complexity of the solution than the *dimension of the space* in which the objects lie. In fact, the dimension of the space only appears in the bounds of the applications as a constant, and becomes invisible in the $O(..)$ notation. Like the widely applied duality transformations [7, 12], projection is a useful tool for several problems in computational geometry.

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The results of the above theorem can be improved. The remark after Lemma 9 states that the image of all k-flats is a $((k+1)\cdot(j+1))$-dimensional hypersurface in D-space. As in the case of lines in 3-space [3, 5, 6, 20, 22], this can be exploited. This hypersurface, the Grassmann manifold, is algebraic and of constant degree, so we can use the results on the complexity of the zone of such a surface, which is $O(n^{\lfloor (D+(k+1)\cdot(j+1))/2 \rfloor} \log n)$, see [2, 21]. Unfortunately, the structure that exploits the fact that only the zone of the Grassmann manifold is interesting is not based on cuttings. Therefore, we cannot use the simplex compositions of Theorem 1. Instead, we use a related structure (as was done for lines in space in [3, 5, 6, 20, 22]) based on Clarkson's random sampling method [10]. We refer to the cited papers for the details, and obtain:

Theorem 5 *Let S be a set of j-flats in d-space. For an arbitrarily small constant $\epsilon > 0$, we can preprocess S for k-flat intersection queries into a data structure of size $O(n^{D'+\epsilon})$, such that all j-flats that intersect a query k-flat can be counted in $O(\log^{O(1)} n)$ time, where $D' = \lfloor (\binom{k+j+2}{j+1} - 1 + (k+1)\cdot(j+1))/2 \rfloor$ (see also Table 1).*

Let g be a j-flat in d-space with $j = d-k-1$. In oriented projective space, g is a $(j+1)$-flat in $(d+1)$-space, determined by either $j+1$ points, or $d-j$ hyperplanes. The flat g also has a simplex representation, being a matrix M_g. Suppose that we compute the Plücker coordinates of g. (Notice that—because of the assumption $d = j + k + 1$—there are $\binom{d+1}{j+1} = \binom{d+1}{k+1}$ Plücker coordinates of g, which is the same number as for f.) A proper rearrangement of the Plücker coordinates, and the negation of some of them, yields the *Plücker coefficients* of g (see [5, 6, 14, 20, 24] for more details). One can prove:

Lemma 8 *Flats f and g intersect (or are parallel) if and only if the dot product of the Plücker coordinates of f and the Plücker coefficients of g is 0.*

Consequently, if we consider a k-flat to be a point in $\binom{d+1}{k+1}$-space with as coordinates its Plücker coordinates, then we can consider a j-flat to be a hyperplane in $\binom{d+1}{k+1}$-space, and f and g intersect or are parallel if and only if the point is incident to the hyperplane in the space called *Plücker space.*

The dimension of the Plücker space can be reduced by one, by observing that the hyperplane always passes through the origin. In fact, the Plücker coordinates and coefficients can be regarded as the homogeneous coordinates and coefficients in projective oriented $(\binom{d+1}{k+1} - 1)$-space. (Intuitively, we project all hyperplanes and points in $\binom{d+1}{k+1}$-space onto a sphere centered at the origin. The surface of the sphere can be regarded as projective oriented $(\binom{d+1}{k+1} - 1)$-space, and the incidences are preserved in this projection.) We define π to be the transform that maps a k-flat to the point in projective oriented $(\binom{d+1}{k+1} - 1)$-space defined by its Plücker coordinates. Similarly, if $j = d - k - 1$, we define ϖ to be the transform that maps a j-flat to the hyperplane in projective oriented $(\binom{d+1}{k+1} - 1)$-space defined by its Plücker coefficients. We then have:

Lemma 9 *A k-flat f and a j-flat g in $(j+k+1)$-space intersect or are parallel if and only if the point $\pi(f)$ is incident to the hyperplane $\varpi(g)$ in projective oriented D-space, where $D = \binom{j+k+2}{k+1} - 1$.*

Remark: It should be noted that not every point in D-space is the image of some k-flat. This is intuitively clear because a non-degenerate k-flat in $(k+j+1)$-space is determined by only $(k+1)\cdot(j+1)$ parameters. In fact, the image of all k-flats is an algebraic surface of dimension $(k+1)\cdot(j+1)$ in D-space, called the *Grassmann manifold* or *Grassmannian.* For more details we refer to [14, 24].

For any k-flat and j-flat, we can consider their intersection in the so-called *join* of the two flats, which is the space spanned by the base vectors of both flats, together with a vector from the one flat to the other (see e.g. Stolfi [24]). The join is an at most $(k+j+1)$-dimensional linear variety. If a j-flat and a k-flat intersect, queries is difficult. Research on the computational geometric side of lines in 3-space has appeared in [5, 6, 19, 20, 22, 23]. The solutions of Chazelle et al. [5, 6] and Pellegrini [20, 22] make use of Plücker coordinates, and can be generalized to flats in higher dimensional spaces.

The incidence of a point and a plane in 3-space is described by one linear equation, and this is generally true for points and hyperplanes in d-space. However, the incidence of a point and a line in 3-space is not described by one linear equation. This is not surprising, because a line in 3-space is determined by four parameters, whereas a point or a plane is determined by only three. If we assume that the line ℓ is not horizontal, then ℓ is, for instance, uniquely determined by its intersection points with the planes $x_3 = 0$ and $x_3 = 1$. In each of these planes, two parameters are needed to specify the intersection points, which totals up to four parameters. From this discussion one could hope that for two lines in 3-space, the intersection can be described as a linear equation in 4-space. Unfortunately, this is not the case. The equation is not linear but quadratic. In this subsection we will look at a transform that turns the test for intersection of two lines in 3-space into a linear equation in 6-space. More generally, we will consider intersections between j-flats and k-flats in higher dimensional space. The transform that makes these intersection tests into linear ones is called the Plücker transform or Grassmann transform. Previous descriptions and applications to computational geometry have been given by Stolfi [24], Chazelle et al. [5, 6] and Pellegrini [20]. The latter papers only use the Plücker transform for lines in 3-space.

Let f be a k-flat in d-space. Then f is determined by $k+1$ points (it is the smallest linear variety that contains these points), or f is determined by the intersection of $d - k$ hyperplanes with linearly independent normal vectors. In order to describe the Plücker transform one should consider the problem in *oriented projective space* (see e.g. [24]). Then f is a $(k+1)$-flat in $(d+1)$-space, and f is determined by either $k+1$ points in $(d+1)$-space, or by $d-k$ hyperplanes in $(d+1)$-space. Assume that $p_i = (p_{i0}, p_{i1}, \ldots, p_{id})$, for $0 \le i \le k$, are the $k+1$ points that determine f. Then f is described by the matrix M_f:

$$M_f = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0d} \\ p_{10} & p_{11} & \cdots & p_{1d} \\ \vdots & \vdots & & \vdots \\ p_{k0} & p_{k1} & \cdots & p_{kd} \end{pmatrix}$$

This matrix is called the *simplex representation* of f. Clearly, there are many simplex representations of the same flat f (unless $k = 0$). The *Plücker coordinates* of f are obtained by taking all $(k+1) \times (k+1)$ submatrices of M_f, and computing the determinants. The value of each determinant gives one coordinate in the Plücker representation of f. Since M_f is a $(k+1) \times (d+1)$ matrix, f has $\binom{d+1}{k+1}$ Plücker coordinates. A line in 3-space, for instance, has 6 Plücker coordinates.

lie in hyperplanes perpendicular to the x_1-axis. The other sets of facets are handled analogously.

Observe that if the direction of the query ray has a positive value in the x_1-component, then the set $\hat{\mathcal{B}}$ admits a stabbing order by projection onto the x_1-axis. If two boxes of $\hat{\mathcal{B}}$ are intersected by the query ray, then the one with minimum x_1-coordinate is the answer. The converse statement is true if the direction of the query ray has a negative value in the x_1-component.

We also observe by Lemma 6 on the projections of flats, that a query ray intersects a $(d-1)$-box if and only if the ray and the box intersect in the projections into all $\binom{d}{2}$ standard 2-dimensional subspaces of d-space. Hence, we build the following structure for $\hat{\mathcal{B}}$. Let T be a binary search tree on the set $\mathcal{PJ}_1(\hat{\mathcal{B}})$ of points on the x_1-axis, obtained by projecting the facets of $\hat{\mathcal{B}}$. For all integers i, j with $1 \le i < j \le d$, project $\hat{\mathcal{B}}$ into the $x_i x_j$-plane, giving a set $\hat{\mathcal{B}}_{ij} = \mathcal{PJ}_{\{i,j\}}(\hat{\mathcal{B}})$ of axis-parallel line segments or rectangles. The rectangles are regarded as four edges, so we may assume that $\hat{\mathcal{B}}_{i,j}$ is a set of axis-parallel line segments in the plane. Let $\hat{\mathcal{P}}_{ij}$ be the set of points obtained by choosing one endpoint of each segment of $\hat{\mathcal{B}}_{ij}$, and let $\hat{\mathcal{P}}'_{ij}$ be the set of other endpoints. We apply a simplex composition on $\hat{\mathcal{P}}_{ij}$ and one on $\hat{\mathcal{P}}'_{ij}$ to the structure T. When performing a query, we use the line that contains the projected query ray. Hence, one can select all facets of $\hat{\mathcal{B}}_{ij}$ that intersect this line in canonical subsets using these two simplex compositions. The resulting data structure is a binary tree with $2\binom{d}{2}$ simplex compositions applied to it.

To perform a query, we project the query ray into the $x_i x_j$-plane (for $1 \le i < j \le d$), compute the line that contains the projected ray, and use it to select canonical nodes. After the first $2\binom{d}{2}$ layers, we have selected all facets of $\hat{\mathcal{B}}$ that are intersected by the line containing the query ray in canonical nodes, by Lemma 6. For each canonical node, we search in the associated binary search tree with the x_1-coordinate of the point where the query ray starts. The facet with larger x_1-coordinate, and minimal among these, is the answer to the query at that canonical node. The answer to the query in the full structure is the facet with the smallest x_1-coordinate among the ones found at the canonical nodes. The answer to the query in the set $\mathcal{B}$ of boxes is obtained after d such queries (one for each set of parallel facets) and choosing as the final answer the box that is hit first, among the ones found. We have obtained:

Theorem 3 *Let $\mathcal{B}$ be a set of n d-boxes in d-space. For an arbitrarily small constant $\epsilon > 0$, the set $\mathcal{B}$ can be stored in a data structure of size $O(n^{2+\epsilon})$, such that ray shooting queries with arbitrary rays can be answered in $O(\log n)$ time. A structure of linear size answers a query in $O(n^{1/2+\epsilon})$ time.*

3.3 k-Flat intersection searching in sets of j-flats

Next we consider k-flat intersection searching in sets of j-flats. We remark that even the problem of preprocessing a set L of lines in 3-space for line intersection time. These data structures also apply to searching with a hyperplane or k-flat in a set of points, to find all points that intersect it. We solve the k-flat query problem with the use of the basic lemma on parallel projections: If a k-flat and a point intersect in all projections into $(k+1)$-dimensional standard subspace, then the k-flat and the point intersect in d-space. Hence, we may treat this intersection problem in d-space as the composition of several k-flat query problems in $(k+1)$-space.

Theorem 2 *For an arbitrarily small constant $\epsilon > 0$, a set $\mathcal{P}$ of n points in d-space can be preprocessed into a data structure of size $O(n^{k+1+\epsilon})$, such that all K points of $\mathcal{P}$ that lie in a given query k-flat can be counted in $O(\log n)$ time or reported in $O(\log n + K)$ time. A structure of linear size answers counting queries in $O(n^{1-1/(k+1)+\epsilon})$ time and reporting queries in $O(n^{1-1/(k+1)+\epsilon} + K)$ time.*

Proof: By Lemma 2, the intersection query problem in the theorem is the composition of $\binom{d}{k+1}$ intersection query problems with hyperplanes in $(k+1)$-space. For each projection $\mathcal{PJ}_I$ into standard $(k+1)$-space we consider the set $\mathcal{PJ}_I(\mathcal{P})$ of points. We apply a simplex composition on this set to be able to select—for any query hyperplane in $(k+1)$-space—the points that intersect it. Hence, the application of a constant number of simplex compositions of Theorem 1 in $(k+1)$-space yields the stated bounds. □

The above result can be extended to k-simplices rather than k-flats. Corollary 2 states the desired property (that Lemma 2 is also valid for k-simplices), and we obtain:

Corollary 4 *A set $\mathcal{P}$ of n points in d-space can be preprocessed into a data structure of size $O(n^{k+1+\epsilon})$, such that all K points of $\mathcal{P}$ that lie in a given query k-simplex can be reported in $O(\log n + K)$ time or counted in $O(\log n)$ time. A structure of linear size answers counting queries in $O(n^{1-1/(k+1)+\epsilon})$ time and reporting queries in $O(n^{1-1/(k+1)+\epsilon} + K)$ time.*

3.2 Ray shooting in axis-parallel d-boxes

We study the problem of preprocessing a set $\mathcal{B}$ of n d-boxes in d-space for efficient ray shooting queries. In 3-space, this problem has been considered in [3], and they solve it with a structure of size $O(n^{2+\epsilon})$ and query time $O(\log n)$. Here we generalize this result to d-space, while maintaining the performance.

Partition the set of all facets of the d-boxes of $\mathcal{B}$ into d subsets of size $2n$, one for each orientation of the facet. Notice that we might as well perform d separate ray shooting queries in these subsets rather than one query in all boxes of $\mathcal{B}$. As a final step of the query algorithm, we choose as the answer to the query the one facet of the d subanswers that is intersected first by the query ray. Hence, we only need consider ray shooting with an arbitrary ray in the set $\hat{\mathcal{B}}$ of $(d-1)$-boxes which *in d-space intersect if and only if for all $I \in \mathcal{I}$, we have that $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect (in $(j+1)$-space).*

Lemma 7 *For $0 \leq k \leq d-1$ and $0 \leq j \leq d-1$, a non-vertical orthogonal k-flat f lies above a non-vertical general j-flat g if and only if (i) f and g do not intersect, (ii) $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$ do intersect, and (iii) for at least one subset I of $\{1, \ldots, d-1\}$ of size $j+1$, $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$.*

We remark that all given lemmas and corollaries also hold for geometric objects that lie inside the flats, rather than the whole flats themselves. For instance, a triangle t in d-space can be reconstructed from all $\binom{d}{3}$ projections into standard 3-space.

3 Applications

Before we give applications of the projection lemmas, we state a general result on the performance of multi-layer structures. Similar results can be found in [1, 11, 13, 15, 25]. If a data structure consists of a main tree that is used for simplex queries, and each node stores an associated structure for some query problem, then we say that a simplex composition has been applied to (the data structure for) that query problem. The main tree can be used to select all points that lie in a given query simplex in a small number of subsets that are associated to certain nodes. At these nodes, the associated structure is queried. The answers to the query are those objects that are an answer both in the main tree and in the associated structure. The performance of any data structure to which a simplex composition is applied, is as follows:

Theorem 1 *Let $\mathcal{P}$ be a set of n points in d-space ($d \geq 2$), and let $\mathcal{S}$ be a set of n objects. Let T' be a data structure on $\mathcal{S}$ of size $O(f(n))$ and with query time $O(g(n))$. For an arbitrarily small constant $\epsilon > 0$, the application of simplex composition on $\mathcal{P}$ to T' results in a data structure*

(i) *of size $O(n^\epsilon(n^d + f(n)))$ and query time $O(\log n + g(n))$,*

(ii) *of size $O(n + f(n))$ and query time $O(n^\epsilon(n^{1-1/d} + g(n)))$,*

(iii) *of size $O(m^\epsilon(m + f(n)))$ and query time $O(n^\epsilon(g(n) + n/m^{1/d}))$, for any $n \leq m \leq n^d$,*

assuming that $f(n)/n$ is non-decreasing and $g(n)/n$ is non-increasing.

3.1 k-Simplex intersection searching in sets of points

We stated in the introduction that efficient solutions to simplex query problem on a set $\mathcal{P}$ of n points in d-space exist. It can be solved with a data structure of size $O(n^{d+\epsilon})$, such that a simplex counting query can be answered in $O(\log n)$ time. Another structure has size $O(n)$ and answers simplex counting queries in $O(n^{1-1/d})$

Let I' be the subset for which $f' = \mathcal{PJ}_{I'}(f)$ lies above $g' = \mathcal{PJ}_{I'}(g)$, as in (iii). By definition, f' and g' are non-vertical, and there is a vertical line ℓ' such that $p' = \ell' \cap f'$ lies above $q' = \ell' \cap g'$. We have that ℓ' is a vertical line in the image space of $\mathcal{PJ}_{I'}$. We now consider $\ell'' = \mathcal{PJ}_{I'}(\ell)$ in the same image space. Let $p'' = \mathcal{PJ}_{I'}(p)$ and $q'' = \mathcal{PJ}_{I'}(q)$. Let L be the 2-flat which contains ℓ'' and ℓ' in the image space of $\mathcal{PJ}_{I'}$ (see Figure 2). If p does not lie above q, then p'' does not lie above q'', and the line segments $\overline{p''p'}$ and $\overline{q''q'}$ intersect in L. Hence, f' and g' intersect (they contain these line segments), which contradicts the statement that f' lies above g'. Consequently, p lies above q. □

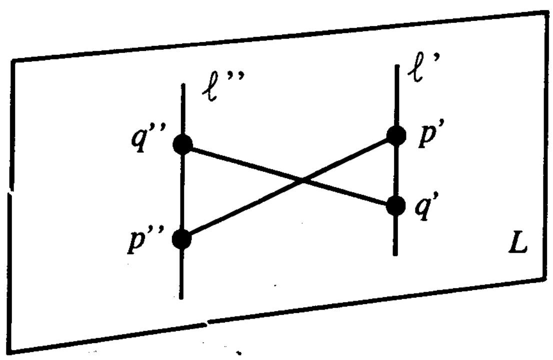

Figure 2: Situation inside L.

2.2 Orthogonal k-flats

Let an orthogonal k-flat be a k-dimensional affine subspace of d-space, spanned by the translation of k base vectors of d-space. In similar ways as above it is possible to obtain projection lemmas for orthogonal k-flats. For example, one can show that in order to test the intersection of an orthogonal k-flat and a general j-flat, one can consider a constant number of instances of the problem in $(j+1)$-space. We state the analogs of the Lemmas 2, 3 and 4 below. The proofs of these lemmas follow in the same way as in the general case.

Lemma 5 *For $0 \le k \le d-1$, let $\mathcal{I}$ be the collection of d subsets of $\{1, \ldots, d\}$ of size 1. For an orthogonal k-flat f in d-space, we have*

$$\bigcap_{I \in \mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f.$$

Lemma 6 *For $0 \le k \le d-1$ and $0 \le j \le d-1$, let $\mathcal{I}$ be the collection of $\binom{d}{j+1}$ subsets of $\{1, \ldots, d\}$ of size $j+1$. An orthogonal k-flat f and a general j-flat g*

The second projection lemma considers the intersection of a k-flat f and a j-flat g in d-space. It shows that f and g intersect if and only if $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect for all $\mathcal{PJ}_I$ ranging over all possibilities of projecting into standard $(k+j+1)$-space. As a concrete example, two lines in 4-space intersect if and only if they intersect in all four projections into standard 3-dimensional subspaces.

Lemma 3 *For $k \geq 0$, $j \geq 0$ and $j+k \leq d-1$, let $\mathcal{I}$ be the collection of $\binom{d}{k+j+1}$ subsets of $\{1,\ldots,d\}$ of size $k+j+1$. A k-flat f and a j-flat g in d-space intersect if and only if for all $I \in \mathcal{I}$, we have that $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect (in $(k+j+1)$-space).*

Proof: Let $\vec{u}_1,\ldots,\vec{u}_k$ be a set of base vectors for f, and let p be a point in f. Then $f = p + a_1\vec{u}_1 + \cdots + a_k\vec{u}_k$ for reals $a_1,\ldots,a_k$. Similarly, let $g = q + b_1\vec{v}_1 + \cdots + b_j\vec{v}_j$ for a point q in g, a base $\vec{v}_1,\ldots,\vec{v}_j$ of g, and reals $b_1,\ldots,b_j$. Define F as follows. Take a maximal set of independent vectors from $\vec{u}_1,\ldots,\vec{u}_k,\vec{v}_1,\ldots,\vec{v}_j$—let it be $\vec{w}_1,\ldots,\vec{w}_m$. Then $F = p + c_1\vec{w}_1 + \cdots + c_m\vec{w}_m$ for reals $c_1,\ldots,c_m$. We have the following relations for f, g and F: (i) $f \subseteq F$, (ii) g and F are parallel, that is, all vectors of g also appear in F after the appropriate translation, and (iii) g and f intersect if and only if $g \subseteq F$.

The important observation is: for all $I \in \mathcal{I}$, we have either $\mathcal{PJ}_I(g) \cap \mathcal{PJ}_I(F) = \emptyset$, or $\mathcal{PJ}_I(g) \subseteq \mathcal{PJ}_I(F)$. (The observation is true by (ii).) The proof can be completed as in Corollary 3. □

A flat is vertical if and only if it contains a line parallel to the x_d-axis. We say that a non-vertical k-flat f lies above a non-vertical j-flat g if and only if f and g do not intersect, and there is a vertical line intersecting f and g for which the intersection point with f lies above the intersection point with g.

Lemma 4 *For $k \geq 0$, $j \geq 0$ and $j+k \leq d-1$, a non-vertical k-flat f lies above a non-vertical j-flat g if and only if (i) f and g do not intersect, (ii) $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$ do intersect, and (iii) for at least one subset I of $\{1,\ldots,d-1\}$ of size $k+j+1$, $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$.*

Proof: 'only if': Assume that f lies above g. Then (i) and (ii) are true by definition. Let ℓ be a vertical line that intersects f in a point p and g in a point q. Then $\mathcal{PJ}_I(p)$ and $\mathcal{PJ}_I(q) \in \mathcal{PJ}_I(\ell)$, and since $\mathcal{PJ}_I$ is a parallel projection perpendicular to the x_d-axis, $\mathcal{PJ}_I(p)$ lies above $\mathcal{PJ}_I(q)$. Consequently, for any subset I as in the lemma, $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$ or they intersect. Assume that for all subsets I they intersect. Because $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$ also intersect, we conclude by Lemma 3 that f and g intersect, a contradiction. Hence, there is at least one subset I for which $\mathcal{PJ}_I(f)$ lies above $\mathcal{PJ}_I(g)$.

'if': Assume that (i), (ii) and (iii) of the lemma hold. Let ℓ be any vertical line that contains an intersection point of $\mathcal{PJ}_d(f)$ and $\mathcal{PJ}_d(g)$, thus $\ell \subseteq \mathcal{PJ}_d^{-1}(\mathcal{PJ}_d(f) \cap \mathcal{PJ}_d(g))$. Let $p = \ell \cap f$ and let $q = \ell \cap g$. We have to show that p lies above q.

by induction, $\dim(\bigcap_{I\in\tilde{\mathcal{I}}_1}\tilde{F}_I) \le k$, where $\tilde{\mathcal{I}}_1 = \{\ \{a_1,\ldots,a_k\} \mid \{a_1,\ldots,a_k,d\} \in \mathcal{I}_1\}$ and $\tilde{F}_I = \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(\tilde{f}))$. Consequently, $\dim(\bigcap_{I\in\mathcal{I}_1} F_I) \le k+1$, because the d-th base vector is added.
Claim: There is a subset $J \in \mathcal{I}_2$ of indices for which F_J does not contain a line parallel to the x_d-axis.
Proof: Take the projection $\tilde{f}$ of f onto the hyperplane $x_d = 0$. Choose any set $\mathcal{K}$ of k linearly independent vectors in $\tilde{f}$. Choose a set $\mathcal{J}$ of $(d-1)-k$ base vectors such that $\mathcal{J}\cup\mathcal{K}$ spans the hyperplane $x_d = 0$ (which is $(d-1)$-space). The set J of indices that corresponds to the base vectors that are not in the set $\mathcal{J}$ is the required set of the claim.

Because $\bigcap_{I\in\mathcal{I}_1} F_I$ is a $(k+1)$-flat which contains a line parallel to the x_d-axis, and F_J is a j-flat with $j \le d-1$ which does not contain a line parallel to the x_d-axis, we have $\dim(\bigcap_{I\in\mathcal{I}_1} F_I \cap F_J) \le k$. Thus also $\dim(\bigcap_{I\in\mathcal{I}} F_I) \le k$, which proves the induction step.

Second, assume that f contains a line parallel to the x_d-axis. Let $\tilde{f}$ be the projection of f onto the hyperplane $x_d = 0$. Then $\tilde{f}$ is a $(k-1)$-flat in $(d-1)$-space, and by induction we have $\dim(\bigcap_{I\in\tilde{\mathcal{I}}_1}\tilde{F}_I) \le k-1$. Consequently, $\dim(\bigcap_{I\in\mathcal{I}_1} F_I) \le k$, because the d-th base vector is added. Thus also $\dim(\bigcap_{I\in\mathcal{I}} F_I) \le k$, which proves the induction step. □

Corollary 1 *With the notation of the above lemma, there exists a subcollection $\mathcal{I}'$ of $\mathcal{I}$ of size d such that*

$$\bigcap_{I\in\mathcal{I}'} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f.$$

Proof: Follows immediately from the proof of the above lemma. □

Corollary 2 *With the notation of the above lemma,*

$$\bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(g)) = g$$

holds when g is a j-flat and $j \le k$.

Corollary 3 *With the notation of the above lemma, we have for a point p in d-space: $p \in f$ if and only if for all $I \in \mathcal{I}$: $\mathcal{PJ}_I(p) \in \mathcal{PJ}_I(f)$.*

Proof: The 'only if' follows from Lemma 1. Next, suppose that for all $I \in \mathcal{I}$: $\mathcal{PJ}_I(p) \in \mathcal{PJ}_I(f)$. Then $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(p)) \subseteq \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$ for all $I \in \mathcal{I}$, by definition of $\mathcal{PJ}_I^{-1}$. By the previous corollary, $\bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(p)) = p$, thus $p \in \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$ for all $I \in \mathcal{I}$, and $p \in \bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f$ by Lemma 2. □

is precisely f again. In fact, rather than taking all $\binom{d}{k+1}$ possibilities, d projections and inverse mappings suffice.

As a concrete example of the first projection lemma, let ℓ be any line in 3-space. Project ℓ into the x_1x_2-plane, the x_1x_3-plane and the x_2x_3-plane, and we add to the projected line the base vectors $\vec{x}_3$, $\vec{x}_2$ and $\vec{x}_1$, respectively, then the intersection of these three flats (usually planes) is the original line ℓ again. See Figure 1 for an example.

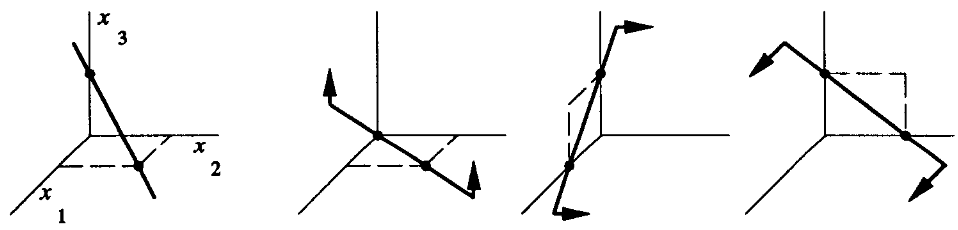

Figure 1: Left: a line in 3-space which intersects the x_3-axis in the one fat point and the x_1x_2-plane in the other fat point. The other three figures show the line projected in the x_1x_2-plane, the x_1x_3-plane and the x_2x_3-plane. The arrows show how the projected line is mapped back into 3-space by the inverse mappings.

Lemma 2 *For $0 \leq k \leq d-1$, let $\mathcal{I}$ be the collection of $\binom{d}{k+1}$ subsets of $\{1,\ldots,d\}$ of size $k+1$. For a k-flat f in d-space, we have*

$$\bigcap_{I\in\mathcal{I}} \mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f)) = f,$$

where $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$ is the maximal region F_I for which $\mathcal{PJ}_I(F_I) = \mathcal{PJ}_I(f)$.

Proof: Since $f \subseteq F_I$ for every I, we have that $f \subseteq \bigcap_{I\in\mathcal{I}} F_I$. Because f is a k-flat, it is sufficient to prove that the dimensionality $\dim(\bigcap_{I\in\mathcal{I}} F_I) \leq k$. We use induction on d, and there are two base cases.

Base case (i): If $k = d-1$, then $\mathcal{I} = \{1,\ldots,d\}$. Hence, $\mathcal{PJ}_I$ and $\mathcal{PJ}_I^{-1}$ are the identity transforms, and the statement is trivial.

Base case (ii): If $k = 0$, then f is a point $(p_1,\ldots,p_d)$, and the flats F_I are d hyperplanes with mutually perpendicular normal vectors, which intersect in one point, namely, f.

Assume that $0 < k < d-1$. Split $\mathcal{I}$ into $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_1$ contains all subsets of $\mathcal{I}$ that exclude d, and $\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1$. Notice that $\mathcal{I}_1$ and $\mathcal{I}_2$ are non-empty.

First, assume that f does not contain a line parallel to the x_d-axis. Let $\tilde{f}$ be the projection of f onto the hyperplane $x_d = 0$. Then $\tilde{f}$ is a k-flat in $(d-1)$-space, and structure that stores a set of j-flats in d-space for k-flat intersection queries. The structure is a generalization of the results on lines in 3-space [2, 5, 24], and uses the Plücker transform and random sampling.

2 Projections

In this section we discuss aspects of the mapping called projection. A projection reduces the dimension of the space. We describe the *standard parallel projection* $\mathcal{PJ}$, which is most useful for our purposes. The projection $\mathcal{PJ}$ removes a subset of $d-k$ coordinates of any point in d-space, thereby giving a point in k-space. The linear subspace that is the image space of the projection $\mathcal{PJ}$ is spanned by k of the base vectors of d-space, and is called a *standard k-dimensional subspace* or *standard k-space.* Let $p = (p_1, \ldots, p_d)$ be any point in d-space, $d \geq 2$. We define the projection $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(p)$, where $1 \leq i_1 < \cdots < i_k \leq d$ is a subset of the coordinates, to be the point in k-space of which the i_j-coordinates are maintained and the other coordinates are omitted. For any geometric object o, consisting of a collection of points in d-space, we define $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(o)$ to be the set of points $\{(z_1, \ldots, z_k) \mid (z_1, \ldots, z_k) = \mathcal{PJ}_{\{i_1,\ldots,i_k\}}(p) \text{ where } p \in o\}$ in k-space. Projections have the following property:

Lemma 1 *Let o_1 and o_2 be two objects in d-space. If o_1 and o_2 intersect, then $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(o_1)$ and $\mathcal{PJ}_{\{i_1,\ldots,i_k\}}(o_2)$ intersect in standard k-space, for any $1 \leq i_1 < \cdots < i_k \leq d$.*

The inverse of Lemma 1 is not necessarily true. In any (non-trivial) projection, certain information is lost. However, there are situations in which it is possible to obtain information about the intersection of two objects from their intersections in a number of projections. In particular, this is the case for objects in higher dimensional space that do not use all the dimensions. Such objects are k-flats. In the following subsections we consider general k-flats first, and then we study orthogonal k-flats.

2.1 General k-flats

For a set $I \subseteq \{1, \ldots, d\}$, we define $\mathcal{PJ}_I^{-1}$ to be the inverse mapping of the projection $\mathcal{PJ}_I$ in the following way: $\mathcal{PJ}_I^{-1}(p)$ is the collection of points q in d-space for which $\mathcal{PJ}_I(q) = p$. The mapping $\mathcal{PJ}_I^{-1}$ *adds the base vectors $\vec{x}_j$*, where $j \notin I$, to a flat f' that lies in the image space of $\mathcal{PJ}_I$. The result is the subspace spanned by the flat f' and the vectors $\vec{x}_j$ which are translated to originate in f'. For a k-flat f and a set I, we define F_I to be $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$, or in words, the flat that is obtained if f is projected with $\mathcal{PJ}_I$ into a standard subspace, and then mapped back into d-space with $\mathcal{PJ}_I^{-1}$. Obviously, f is contained in F_I.

The first projection lemma that we prove states that if $\mathcal{PJ}_I$ ranges over all possibilities of projecting into standard $(k+1)$-space, then the intersection of the F_I

well, we show that k-flat searching can be reduced to a 'combination' of d searches in $(k+1)$-dimensional space. This search is performed with a k-flats, thus hyperplanes. Furthermore, we present two more basic lemmas on projections, which we explain below.

Let d-dimensional space (or d-space for short) be equipped with a set of d base vectors $\vec{x}_1, \ldots, \vec{x}_d$. Let a k-dimensional (linear) subspace be standard if it is spanned by k of the base vectors, originating at the origin. Thus d-space has $\binom{d}{k}$ standard k-subspaces. We use parallel projections, denoted $\mathcal{PJ}_I$, which simply 'remove' one or more of the base vectors. The subscript I denotes the set of base vectors that span the image space, and are maintained. The inverse mapping $\mathcal{PJ}_I^{-1}$, applied to an object in the image space of $\mathcal{PJ}_I$, yields the region of d-space that maps onto the object. For a k-flat f in d-space, define F_I to be $\mathcal{PJ}_I^{-1}(\mathcal{PJ}_I(f))$. The first projection lemma that we prove states that if $\mathcal{PJ}_I$ ranges over all possibilities of projecting into standard $(k+1)$-space, then the intersection of the F_I is precisely f again. In fact, rather than taking all $\binom{d}{k+1}$ possibilities, only d of the F_I suffice to obtain precisely f. The second projection lemma considers the intersection of a k-flat f and a j-flat g in d-space. It shows that f and g intersect if and only if $\mathcal{PJ}_I(f)$ and $\mathcal{PJ}_I(g)$ intersect for $\mathcal{PJ}_I$ ranging over all possibilities of projecting into standard $(k+j+1)$-space. The third projection lemma states that a k-flat f lies above a j-flat g if and only if f and g do not intersect, they do intersect in the projection on the hyperplane $x_d = 0$, and for at least one projection into standard $(k+1)$-space (including the x_d-coordinate), the projection of f lies above the projection of g.

The dimensions of the spaces into which we project are sharp in the sense that the lemmas generally do not hold when we map to a lower dimensional space.

We also show that if one of the flats is axis-parallel, then we can do better. In that case, the dimension of the axis-parallel flat plays no role in the dimension of the standard subspaces in which the intersection can be regarded.

The applications of the results on projections are obtained by using multi-layer data structures (see e.g. [1, 11, 13, 15, 25]). Multi-layer structures allow to express more complicated query problems as a combination of several more simple query problems. In our case, it is possible to express certain d-dimensional query problems as the combination of several lower dimensional query problems. This results in a gain in efficiency. In the first application, we study searching with a k-flat or a k-simplex in a set of points in d-space. By the first projection lemma, this is equivalent to a combination of $\binom{d}{k+1}$ searches with a k-flat or k-simplex in a set of points in $(k+1)$-space. Therefore, multi-layer structures solve the problem using $O(n^{k+1+\epsilon})$ storage rather than $O(n^{d+\epsilon})$ storage. The second application is a data structure for ray shooting in axis-parallel boxes in d-space. In this case, we can reduce the problem to a combination of 2-dimensional line intersection queries and ray shooting in axis-parallel line segments, to obtain an $O(n^{2+\epsilon})$ size data structure which supports ray shooting queries in $O(\log n)$ time. This result is a generalization of a result of de Berg et al. [3], who considered the 3-dimensional problem. We close with a data

The Power of Parallel Projection*

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Abstract

It is shown that from d projections of a k-flat into k-dimensional linear subspaces one can still reconstruct the k-flat. Furthermore, we show that a k-flat and a j-flat intersect in d-space if and only if they intersect in $\binom{d}{k+j+1}$ linear subspaces of dimension $(k+j+1)$ (and which are independent of the k-flat and the j-flat). Applications of these projection results are k-dimensional simplex searching for a set $\mathcal{S}$ of n points in d-dimensional space with a structure of size $O(n^{k+1+\epsilon})$ and $O(\log n)$ query time, for arbitrarily small positive ϵ. A second application is ray shooting in axis-parallel boxes in d-dimensional space, with a structure of size $O(n^{2+\epsilon})$ and $O(\log n)$ query time. Thirdly, we obtain a structure for k-flat intersection searching in a set of j-flats.

1 Introduction

Well-studied is the simplex query problem: preprocess a set $\mathcal{P}$ of n points in d-dimensional space into a data structure, such that for any query simplex s, all points of $\mathcal{P}$ that lie inside s can be counted or reported efficiently. The research of a long list of authors has lead to data structures of size $O(n^{d+\epsilon})$ and query times $O(\log n)$ or $O(\log n + K)$, where K is the number of answers, see e.g. [8, 16, 18]. (In this paper, ϵ denotes an arbitrarily small positive constant that should be chosen beforehand.) Other data structures have size $O(n)$ and query time $O(n^{1-1/d})$, plus $O(K)$ in the reporting case [9, 17, 18]. In a certain restricted model of computation, this is close to the best one could hope for [4]. These data structures are also the best known for searching with a hyperplane in a set of points, to find all points that intersect it, and it is easy to see that searching with a k-flat yields the same results (a k-flat is a k-dimensional affine subspace of d-space, see [12]). But can we do better with a k-flat? In this paper we answer this question to the affirmative. With the use of a basic lemma on parallel projections, which has other applications as

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