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RUU-CS-92-39
December 1992

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ISSN: 0924-3275

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Abstract

In this paper we introduce an algorithm which produces *quality* triangulations of polygonal domains with holes, for the Finite Element Method (FEM). For us *quality* triangulation means satisfaction of both of the following conditions: a) no obtuse angles. b) no 'small' angles. Triangulations satisfying only a) will be called *non-obtuse*, those satisfying only b) will be called *non-small angle*, and those which satisfy *both* conditions a) and b) will be called *nice*. Our main contributions are a) new techniques to produce *nice* triangulations of polygonal domains with holes using *non-uniform* (quadtree) underline square grid, and b) we show that the asymptotic size of a *nice* triangulation is the same as for a *non-small angle* triangulation. In other words achieving both *non-obtuse* and *non-small angle* condition at the same time is not more expensive than achieving only the *non-small angle* condition.

1 Introduction

Recently, much attention has been devoted to triangulating point configurations or polygonal regions so that all of the produced triangles are "well-shaped". Most of these efforts have been motivated by interest in the numerical solution of Partial Differential Equations (PDEs) by the Finite Element Method (FEM). The term "well-shaped" triangle may denote that no angle is arbitrarily small, or that no angle is obtuse, or both. To avoid ill-condition matrices arbitrarily small angles should be prohibited [23]. About the importance of the small angle condition the reader can consult the classical book of Ciarlet [17], or the book of Strang and Fix [47] or the paper of Bramble and Zlamal [14]. The importance of non-obtuse triangulations is cited in [47], [7], [16], [29]. The importance of the large angle condition in general is cited in Babuska and Aziz [1] for

*Part of this research was done while the author was employed by the Department of Computer Science, Rutgers University, New Brunswick, NJ 08903, U.S.A. This research was supported in part by NSF grants CCR-88-03549 and CCR-91-0473 by DIMACS under NSF grant STC-88-09648, and by the Dutch Organization for Scientific Research (N.W.O) and by the ESPRIT Basic Action No 7141 (project ALCOM II.)

two dimensions and in Krizek [27] for three dimensions. Furthermore, all triangulations should be *consistent*: a triangulation is consistent iff there is no vertex of a triangle which lies on the edge of another triangle.

no-small-angle: Every angle $\theta \geq min(c, \beta)$ where c is a positive constant and β is the sharpest angle of the polygonal domain.

non-obtuse: All the angles of any triangle are $\leq \frac{\pi}{2}$.

It should be clear that if the polygonal domain has a sharp angle then then some triangle should have this angle. However according to our constructions this sharp angle does not propagate.

The first mesh generator which guarantees quality of the element shapes is that of Baker, Grosse and Rafferty [2]. Their algorithm satisfies both quality criteria (i.e *no-small* and *non-obtuse* angle conditions at the same time), like ours, however the size of the mesh is very large since they use a uniform grid instead of a non-uniform one (i.e quadtree).

In another work, P. Chew [11] focused particularly on preventing small and large angles. His technique is based on the constrained Delaunay triangulation [8]. His algorithm produces triangles with angles between 30 and 120 degrees. However the produced mesh is quasi-uniform, i.e. $\frac{h_{max}}{h_{min}} \leq c$ where h_{max}, h_{min} are the largest and smallest edge of the triangulation respectively, and c is a constant. Quasi-uniform meshes are not desirable since they result in a huge number of elements. Very recently, Bern, Dobkin and Eppstein [5] have an algorithm which triangulates a polygon in such a way that no triangle has large angle. The size of their triangulation is $O(n \log n)$.

The first who provided a guaranteed quality mesh using quadtrees (and thus a small size mesh) are Bern, Eppstein, Gilbert [4]. Their algorithm produces $O(nA)$ triangles where n is the size of the polygon and A is the aspect ratio of the constrained Delaunay triangulation of P. They prove that this size bound is within a constant factor of optimal, but their algorithm satisfies only the *no-small* angle condition. In fact the first who provided bounds on the size of polygon quadtrees are Hunter and Steiglitz [24], however in [24] are not interested on angle conditions. They are interested only on the discretization of the polygon.

In a subsequent paper, Bern and Eppstein [6] present an algorithm for an $O(n^2)$- size *non-obtuse* triangulation of a polygonal domain. The produced triangles do not satisfy the *non-small angle* condition.

Our contribution is a mesh generator which satisfies both quality criteria, like [2], with the additional feature that the size of the mesh is "small". In fact the size of our triangulation is optimal. The "small" size is achieved using quadtrees instead of the uniform grid of [2]. This non-uniform grid, however makes the triangulation process much more complicated near the boundary. The size of our triangulation is the same order (i.e $O(nA)$) as the size of the *no-small* angle triangulation of [4]. Which means that requiring both conditions is not more expensive than requiring only the *no-small* condition. Since our triangulation is non-obtuse and any non-obtuse triangulation is a Delaunay triangulation, we can view our result as a conforming Delaunay triangulation with $O(nA)$ steiner points which satisfies the *no-small* angle condition also.

A triangulation (resp. triangle) which satisfies both the *no-small-angle* condition and the *non-obtuse* condition shall be called a *nice triangulation* (resp. *nice triangle*).

In this paper, we present an algorithm for the problem of *nice triangulation* triangulation of polygonal domains with holes.

A completely different approach has been presented by Melissaratos and Souvaine in [38]. The one we follow here does not depend heavily on resolving inconsistencies and is much more simpler than the approach in [38].

2 Main Approaches in FEM Mesh Generation

There are four main approaches in mesh generation of arbitrarily complex geometries:

- **Delaunay based mesh generators**

  Such a mesh generator takes as input a point distribution on the boundary and in the interior of the geometry and then creates the Delaunay triangulation of that point set. Then applies a postprocessing on the triangulation in order to eliminate as many as possible bad-shaped triangles. This is achieved by heuristics like Laplacian smoothing. There have been many Delaunay based mesh generators in the literature. Some representative are by Cavendish, Field and Frey [19],[21], [20], [22], Barry Joe's [28], Watson's [59], Lo's algorithm [32], Schroeder and Shephard [52], Sapidis and Perucchio [48], [49], as well as by Timothy Baker [9],[10] and Mavriplis [36], [37] for Computational Fluid Dynamics applications. Other Delaunay based mesh generators include the works of Nackman and Srinivasan [39], Jin and Wiberg [25], P. Chew [8] Steve Fortune [34], [35], Clarkson and Shor [13], Boissonnat and Teillaud [12].

  Such a method works on pointsets and not on boundary representations (except for 2-D constraint Delaunay triangulation). A consequence is that the boundary of the region is not honored by the triangulation in general (i.e there are triangles (tetrahedra) which intersect the boundary of the geometry). Each "gray" triangle can be included in the domain. That results in a bad approximation of the geometry. Another option is to triangulate the part of the "gray" triangle which lies in the interior of the geometry. However that leads to poorly shaped triangles.

- **Medial axis transform based mesh generators**

  Take as input a boundary description of the geometry; compute its medial axis; apply postprocessing like Laplacian smoothing. This kind of method was introduced by Patrikalakis and Gursoy [41], Gursoy [18] and by Srinivasan, Nackman, Tang, Meshkat [55].

  This method honors the boundary of the geometry, however it seems that there are difficulties for extending the method in 3-D.

- **Advancing front method** [31], [42],[43], [33]

  Start with a discretization of the geometry boundary and proceeds to the interior adding triangles until the hole geometry is covered. This method because of its algorithmic and programming simplicity has been dominated in CFD applications. A characteristic of this method is that node distribution and linking of the nodes is done concurrently.

- **Quadtree-Octree based** [57], [58], [50], [44], [15], [53].

  Quadtree type mesh generators have the advantage that they distribute nodes "fairly" and automatically. That means they distribute "many" nodes both on the boundary and the interior of complicated parts of the geometry and "fewer" nodes elsewhere. This "fair" distribution results in optimal size meshes. Also, after the creation of the quadtree, the linking of nodes in order to create the triangulation requires only linear time in the number of nodes something which is not true for the Delaunay triangulation. Unfortunately, creating a quadtree in a satisfactory resolution is time consuming. In addition the quadtree imposes artificial orientations in the geometry, [55].

Among all these different approaches to mesh generation, it is not clear which, if any, is going to dominate in the future. None of the major mesh generators cited above, however, guarantees quality of the element shapes.

3 Definitions and Preliminaries

We begin this section with a collection of essential definitions:

Definition: A $\triangle ABC$ is called *non-small angle* iff there exists a constant $\theta_0 \in (0, \frac{\pi}{2})$ *independent* of the coordinates of A, B, C such that $\angle A, \angle B, \angle C \in (\theta_0, \pi)$.

Definition: A triangulation is called *non-small-angle* iff all its triangles are non-small angle.

Definition: A $\triangle ABC$ is *non-obtuse* iff it is a) non-small angle and b) $\angle A, \angle B, \angle C \leq \frac{\pi}{2}$.

Definition: A triangulation is called *non-obtuse* iff all its triangles are non-obtuse.

Definition: A $\triangle ABC$ is *nice* iff it is non-obtuse and non-small-angle.

Definition: An edge e of the polygon of type h (v resp.) if the line through e makes an angle $< \frac{\pi}{4}$ with the horizontal (vertical resp.).

Definition: Two edges e_1, e_2 of a polygon P adjacent to vertex v are called *vertically separable* (resp. *horizontally separable* iff a vertical (resp. horizontal) line through v cuts the interior angle at v into two angles each of them is greater or equal to $\frac{\pi}{4}$.

Definition: A polygonal region Q is called *semirectilinear* if every h (v resp.) type edge e is adjacent either to horizontal (vertical resp.) edge or to another h (v resp.) type edge e' such that e, e' are horizontally (vertically resp.) separable.

Definition: A polygon vertex v at which the conditions for semirectilinearity are violated is called an *improper* vertex.

4 Algorithm Overview

The algorithm has two phases: The *preprocessing phase* and the *triangulation phase.* During the preprocessing phase we discretize the domain using a quadtree. This quadtree subdivision is a standard process and has been used many times in the literature, not only in mesh generation of the FEM method, but in Computer graphics, image processing and elsewhere. It is important to say that the first who gave a bound on the size of the quadtree which represents a polygonal region is Hunter and Steiglitz in 1976 [24]. The root of the quadtree is a square containing the polygonal domain. Subdivide the square into four squares. Check whether a square lies entirely inside or outside P or whether it intersects the boundary of P. In the first case stop the subdivision. In the second case further testing is needed: If a q-tree node intersects only one edge or two adjacent edges of P stop. Otherwise we subdivide. Any two non-adjacent edges of P should not intersect adjacent nodes of the q-tree. The above condition guarantees satisfactory clearance between boundary parts something necessary in order to apply our construction rules.

Any polygon vertex should lie in a local uniform grid of constant size. This condition makes easier the vertex triangulation process.

We use the following terminology and facts:

Definition: A planar subdivision created by a quadtree is called *balanced* or (*restricted*) if any two adjacent squares have sides which are equal or one is twice as the other

Restricted quadtrees have been used for surface rendering purposes also [46].

From now on we are going to work only with *balanced* quadtrees.

Fact: [45], [46]. An unbalanced quadtree subdivision with n regions can be converted to a balanced one by dividing larger squares until the balancing condition is satisfied. The resulting balance subdivision has $O(n)$ size. Hanan Samet, in his books [45] and [46], credits this problem to David Mount.

In the triangulation phase we triangulate the interior of P using the non-uniform grid. The difficulty in the second phase is to triangulate grid cells which are intersected by the polygon boundary since interior cells are triangulated trivially. The triangulation phase itself has two parts:

- Triangulate *near polygon edges*
- Triangulate *near polygon vertices.*

In section 5 we describe how to triangulate near edges and in 6 we describe how to triangulate near vertices.

a) Discretize the domain using a standard quadtree algorithm.

b) Remove every acute vertex v of P by inserting an artificial edge AB such that ΔvAB is isosceles with $vA = vB$ where A, B lie on the edges of P incident to vertex v. See section 6. What remains is a polygon P^* with no acute angles. (see fig. 1)

c) Remove every vertex v of P^* by cutting off a rectilinear region around v. Triangulate that rectilinear region as shown in section 6.

   What remains is a semirectilinear polygon P^{**}. (see fig. 2)

d) for each edge e of P^{**} triangulate e. This is shown in section 5.

e) What remains after step d) is a rectilinear polygon P^{***} which can be triangulated trivially.

f) Triangulate the cut isosceles triangles ΔvAB of step b) using the Steiner points on AB created by the triangulation of P^*. This is shown in section 6.

g) Resolve the inconsistencies between the triangulation of P^* and the isosceles triangles ΔvAB.

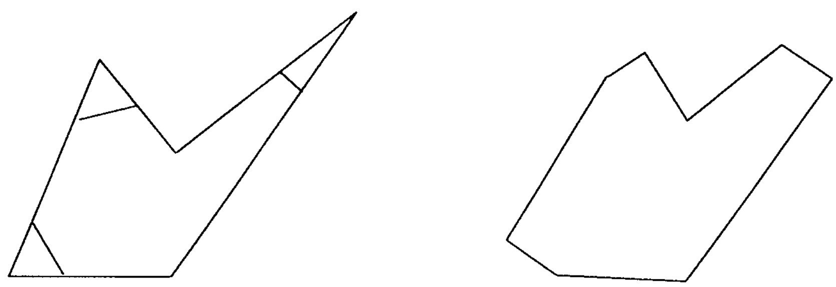

Figure 1: Polygons P and P*

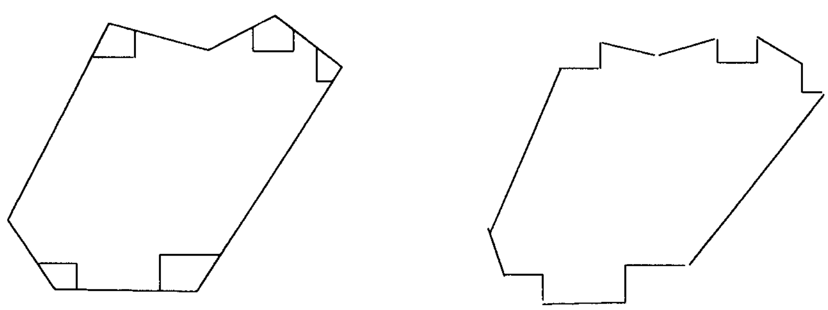

Figure 2: Polygons P* and P**

5 Triangulating near Polygon edges.

5.1 General remarks on edge triangulation

With a uniform grid, *nice* triangulation is straightforward [2]. The triangulation of Bern et al [4] on a non-uniform grid to satisfy the *non-small angle* condition requires greater care but is not complicated. Our goal here is to satisfy non-obtusenes as well.

In order to offer the the reader some intuition to our technique let's consider the following examples in fig. 3.

In fig.3.a we see that the *nice* triangulation using a uniform grid is easy.

In fig.3.b we see that a *non-small angle* triangulation using a non-uniform grid is achieved again easily. However if you want to *nice* triangulate the same configuration this is impossible unless you use extra Steiner points in the interior or the boundary of the configuration. But addition of Steiner points does not come for free.

It may require addition of Steiner points in neighbor cells which eventually may lead in an orderwise increase of the size of our triangulation something undesirable.

If we are more careful and insert Steiner points only at midpoints of quadtree edges then the increase will be constantwise. In fig. 3.f we see an inappropriate way of creating a *nice* triangulation. The insertion of Steiner point P makes necessary the insertion of points P_1, P_2, P_3, P_4. Thus we need four triangles per cell.

In fig. 3.g we see a better way to triangulate. We need only two triangles per cell which intuitively means that we gain a factor of two in our mesh, something very important.

However if we prefer not to insert the additional Steiner point P, we need to *expand* our triangulation to the previous cell. (In fig. 3.d the perpendicular MT on vw intersects the previous cell.) Since the previous cell has already been triangulated, the *expansion* will lead either to a *non-nice* triangulation or, even worse, to a non-valid one (i.e intersecting triangle edges). That will require to *repair* the already completed triangulations of the previous cells. The main question is whether such *repairs* propagate. This approach will be called *approach A*.

Surprisingly, we are able to prove that modifications are necessary only to the previous cell and do not propagate to other cells. These modifications and the related proofs are provided in section 5.3.

In contrast if we use the *insertion* approach, which will be called *approach B*, then not only we may double the size of the mesh but the propagated point P may create inconsistent triangulation in a vertex region which should be resolved.

We will present both approaches in parallel and since we have mentioned the advantages and disadvantages of each we will leave the implementor to make a choice depending on the application. The reader who is going to follow *Approach B* can skip the section 5.3. In section 5.2 we refer to *Approach A* or *Approach B* if there is need to distinguish the two approaches.

We consider intersections of an h (v resp.) type edge with vertical (horizontal resp.) grid edges only. We are going to work with type h (since the type v) is symmetric. Let v, w be two consecutive intersection points of an edge e with two vertical q-tree edges.

5.2 Edge-triangulation rules

Lemma 5.1 *Line segment vw lies entirely either: a) in one square or b) in the union of two squares.*

Proof: Easy consequence of the q-tree conditions and the slope of vw. □

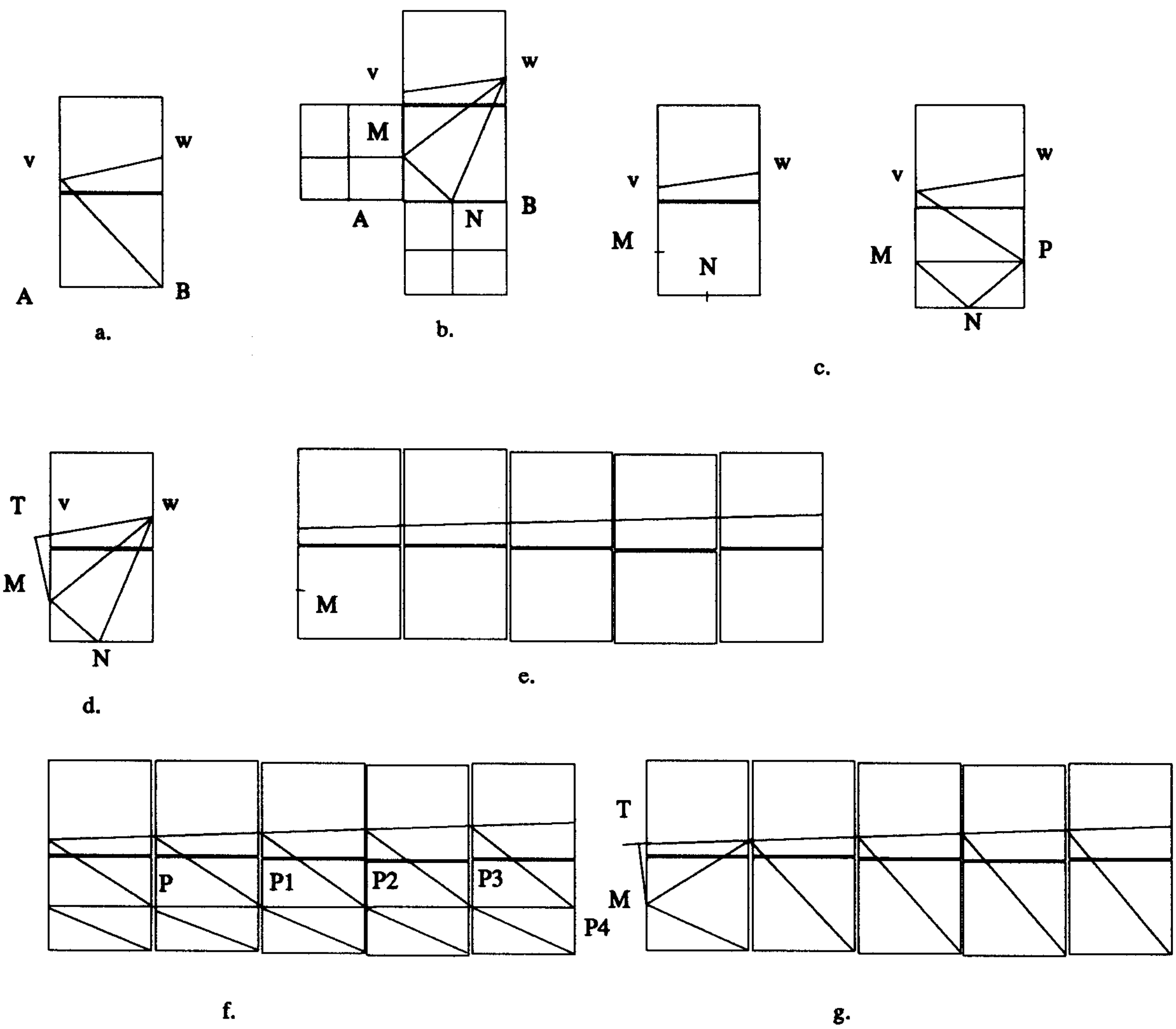

Figure 3: Options in edge triangulation

Let v belong to square s_1 and w belong to a square s_2. Lemma 5.1 there are four cases, to which we will refer by their labels.

a) s_1 and s_2 are the same square.

b) s_1 and s_2 have the same size and s_2 is above of s_1.

c) s_2 is above of s_1 and s_2 is half of s_1.

d) s_2 is above of s_1 and s_2 is double size of s_1.

(see fig.4).

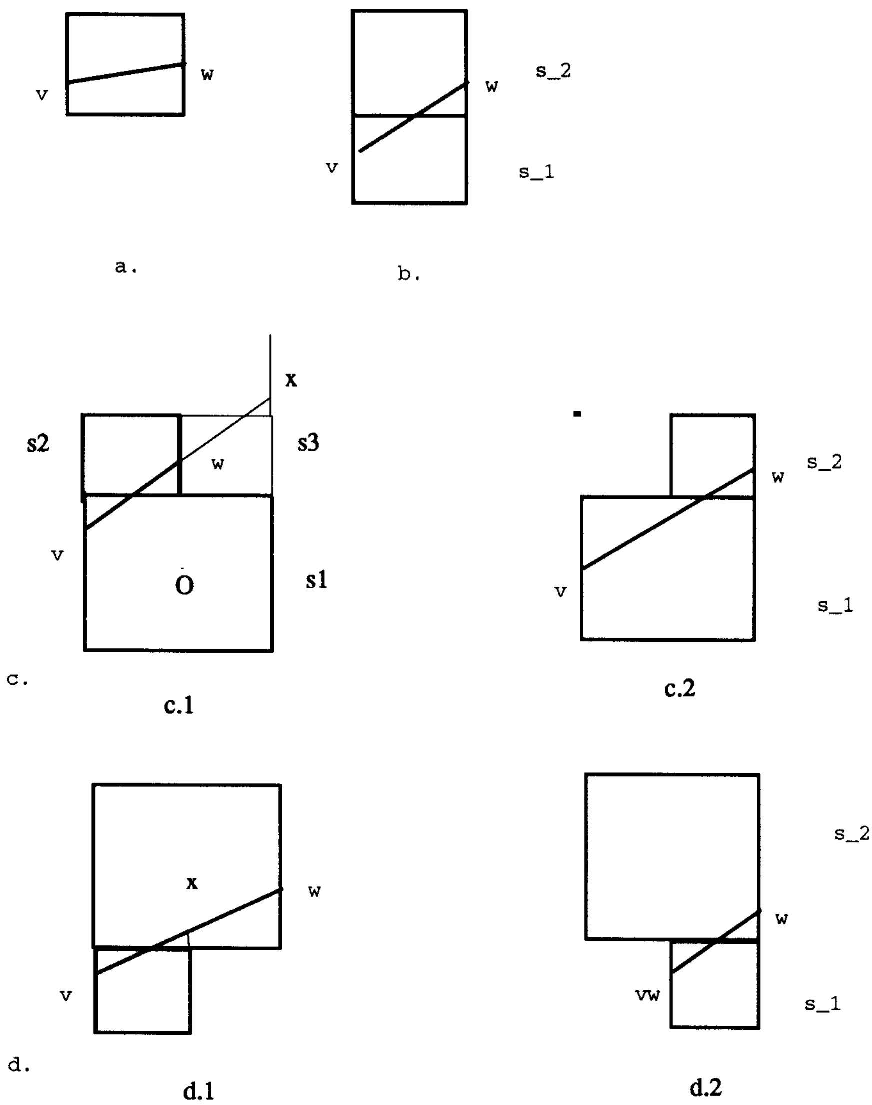

Figure 4: General cases

From now on we refer to the above cases as $a), b), c), d)$ respectively. We need some definitions and notation.

For a square s A, B, C, D are the vertices in ccw order. M, N, P, Q are the midpoints of DA, AB, BC, CD respectively. O the center of the square and g the side length. Let v, u points on sides AD, DC respectively of the square. The pentagon $vuCBA$ is called *cut-square*. O' is defined as the intersection point of the horizontal from P and the vertical from u. R is the midpoint of MD. K is defined the point inside the square $ABCD$ such that $MK = \frac{g}{4}$ and MK is parallel to AB. If u is on QC and $uc = \Theta(g)$ then such a cut-square is called *regular*. A *regular cut-square* may have at most two Steiner points M and P. A Steiner point like M (P resp.) exists iff there are two left (right resp.) neighbors of s and $vM \geq \frac{g}{4}$ ($wP \geq \frac{g}{4}$ resp.). Steiner point N exists iff s has two below neighbors. We define the "below" function as follows: Let v be a point on a vertical edge AB of a quadtree square s with length side g. Let M be the midpoint of AB. if M is a q-tree node and V is above M and $vM \geq \frac{g}{4}$ then $b(v) = M$, else if $vA \geq \frac{g}{2}$ then $b(v) = A$ else $b(v) = \omega$ which means a quadtree node which lies on the box below s. In the following we present a lemma which shows how to nice triangulate a *regular cut-square* using the steiner points if they exist.

Lemma 5.2 *A regular cut-square can always be nice triangulated.*

Proof: Consult fig. 5.

For each of the cases we are going to consider two subcases: In the first subcase we assume that Steiner point N does not exist and in the second subcase we assume that N exists.

1) v is on MD. 1.a) No steiner points on the sides of $ABCD$. Then connect B to v and u. We have to prove that ΔvuB is non-obtuse. $\angle vuB \leq \angle RuB$ since $Mv < \frac{g}{4}$. But $\angle RuB < \frac{\pi}{2}$. (otherwise M would exist as a steiner point). If N is a Steiner point then connect N to v and u, and B to u.

1.b) Steiner point M exists. Then steiner point P (i.e the midpoint of BC) should exist also. This is true since the existence of M implies $Mv \geq \frac{g}{4}$ which implies $wC \leq \frac{g}{4}$. Then connect O to v, M, u, P, A, B. If N is a Steiner point then in addition to the above connections do connect N to O.

1.c) Only P exists. Then connect O to v, u, A, B, P. If N is a Steiner point then add the additional connection between O and N. Resolve the obtuse $\angle vOu$ by taking the perpendicular from O. 2) v is on MA.

2.a) No steiner points. Connect B to v, u. Can be proved easily that ΔvuB is non-obtuse. If N is a Steiner point then connect N to v, u and B to u. $\angle uvN \leq \angle QvN \leq \frac{\pi}{2}$. Resolve the probable obtuse $\angle uNv$ by taking the perpendicular from N.

2.b) Steiner point P. Connect P to v, u. Connect v to B. If N is a Steiner point then connect O to v, u, P, B, N and connect P to u, N and v to N. Since u is not very close to C, according to *cut-square* property, then $\angle vOu$ is not close to π thus ΔvOu does not have small angles.

□

Consider two squares $ABCF$ and $FCDE$ (with FC common side) with side length g and let v, w points on AF and DC respectively such that slope of $vw \leq 1$. Let M, N, P be the midpoints of AF, AB, BC respectively and let G be a point on wC such that $wG = \Theta(g)$. Question: given the quadrilateral $vABw$ (or equivalently called two square

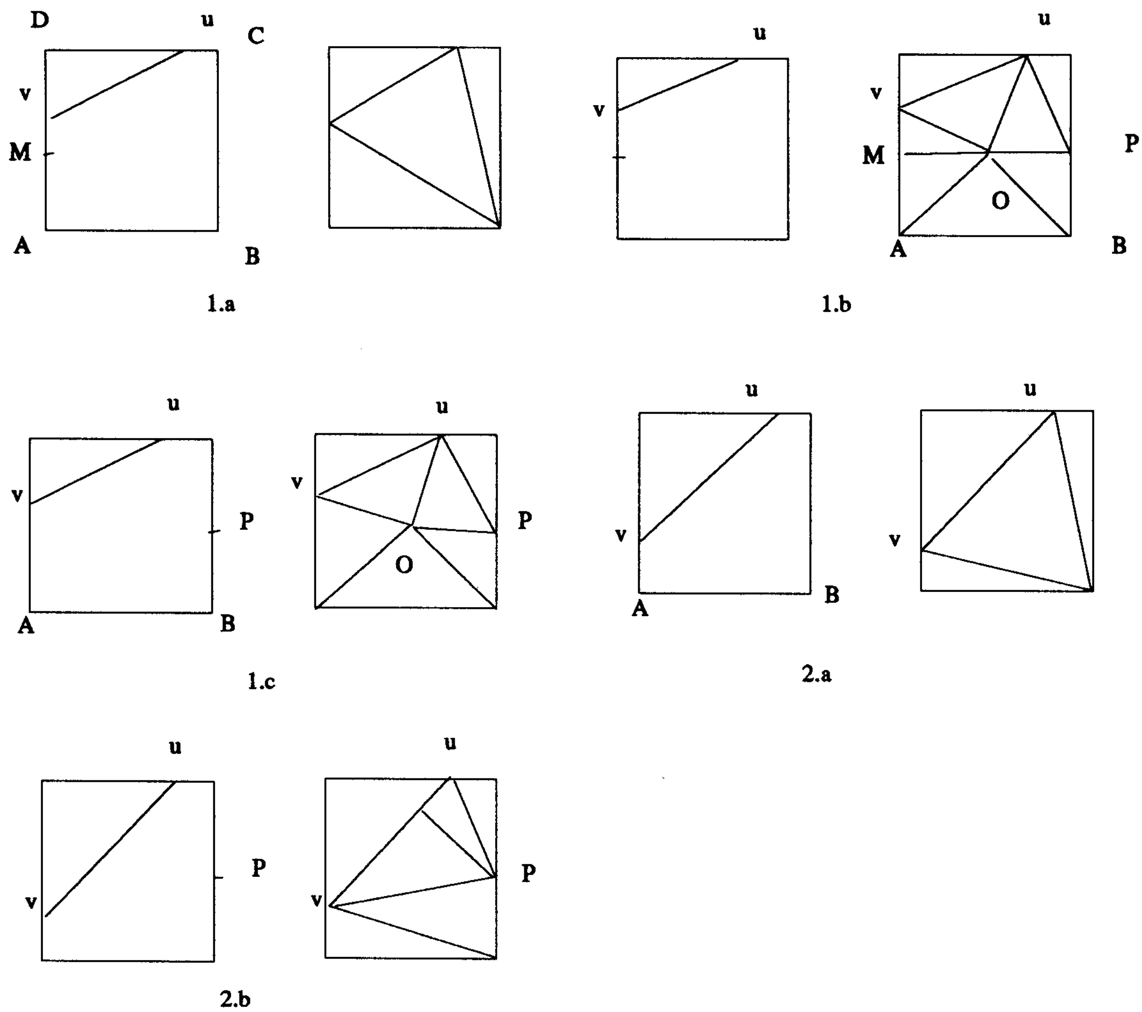

Figure 5: Lemma 5.2

configuration) and a subset (possibly empty) of M, N, P, C, G can you nice triangulate $vABw$ using no additional steiner points on its rectilinear sides? The following lemma gives a positive answer. Let e be an h type edge. Let v be the intersection point of e and a quadtree edge ab.

Lemma 5.3 *A two square configuration can always be nice triangulated.*

Proof:

Consult figures 6. We have the following cases: 1) If b(w)=G then we consider the following two cases: 1.a) If $Fu < \frac{g}{2}$ (implies also that $Fv < \frac{g}{2}$, otherwise the slope of vw would be larger than one) then: Connect O to v, R, C, A, B. Connect R to G, C. If $\angle RGw$ is obtuse then resolve it by taking the perpendicular from G to vw. If $\angle RGw$ is non-obtuse then $RT \geq GC$ thus connect G to T. In case N exists then just connect N to O.

1.b) If $Fu \geq \frac{g}{2}$ then connect G to u. If $\angle uGw$ is obtuse then resolve it by taking the perpendicular from G to vw. Then triangulate the pentagon $vABCu$ as in *lemma* 5.2.

2) If $b(w) = C$ then: 2.a) If $Fu < \frac{g}{2}$ (implies also that $Fv < \frac{g}{2}$, otherwise the slope of vw would be larger than one) then: Connect O to v, R, C. The existence of N again does not create any difficulties. We just add the connection between N and O.

2.b) If $Fu \geq \frac{g}{2}$ then connect C to u and C to w and then triangulate pentagon $vABCu$ as in lemma 5.2.

3) Assume that $b(v) = M, A, \omega$ and $b(w) = P, B, \omega$, then the following constructions guarantee nice triangulations.

3.a) $b(v) = M$ and $b(w) = P$. Then connect P to M and w. The existence of N does not create difficulties. Just connect N to M, P.

3.b) $b(v) = M$ and $b(w) = B$. If we use *approach B* then insert point P and we have case 3.a.

Approach A: Let T be the perpendicular projection of M onto vw. Connect M to T, B. The resulting triangles are $\triangle TMw, \triangle MwB, \triangle MAB$ which are clearly *nice*.

The newly created perpendicular edge MT lies in the adjacent grid cell. Reasonable questions are: a) what modifications should be done to the previous cell to in order maintain its triangulation *nice*, b) Is there any guarantee that the modifications do not propagate indefinitely?

Answers to these questions are given in lemma 5.10.

However if N exists then the resulting triangles are $\triangle TMw, \triangle MwN, \triangle MAN, \triangle NBw$.

It is clear that $\triangle MAN, \triangle NBw$ are *nice*. We have to prove first that $\angle wMN \leq \frac{\pi}{2}$ and second that $\rho = \angle TwM$ is *non-small*.

Since $b(w) = B$ implies that $wC < \frac{g}{2}$ and thus $wP < g$ which implies directly that $\angle wMP < \frac{\pi}{4}$ and thus $\angle wMN \leq \frac{\pi}{2}$.

To prove that ρ is *non-small*, Let z be the perpendicular projection of w onto the line through AD. Let $x = zD$, $y = Mv$, $\theta = \angle zwM$, $\phi = \angle zwv$ and thus $\rho = \theta - \phi$. Let also $g = AD$.

We have

$$\tan(\theta) = \frac{x + \frac{g}{2}}{g}$$

$$\tan(\phi) = \frac{x - y + \frac{g}{2}}{g}$$

Thus

$$\tan(\rho) = \frac{\frac{y}{g}}{1 + \frac{(x+\frac{g}{2})(x-y+\frac{g}{2})}{g^2}}$$

Thus

$$f(x,y) = \tan(\rho) = \frac{4gy}{5g^2 + 4x^2 - 4gx - 2gy - 4xy}$$

where $0 \leq x \leq \frac{g}{2}$ and $\frac{g}{4} \leq y \leq \frac{g}{2}$.

Clearly

$$f(x,y) \geq f(x, \frac{g}{4}) = \frac{2g^2}{9g^2 + 8x^2 - 10gx}.$$

Let $h(x) = 9g^2 + 8x^2 - 10gx$.

Then $h'(x) = 16x - 10g$, thus $h(x)$ is decreasing in $[0, \frac{5g}{8}]$ and increasing in $[\frac{5g}{8}, \frac{g}{2}]$ thus $h(x)$ attains its global maximum for either $x = 0$ or $x = \frac{g}{2}$. In fact the maximum is achieved for $x = \frac{g}{2}$ and $h(\frac{g}{2}) = \frac{1}{3}$. Thus $f(x,y) \geq \frac{1}{3}$.

Again we need to consider the modification in the previous cell because of the perpendicular MT. This is proved in lemma 5.10.

3.c) $b(v) = M$ and $b(w) = \omega$. (infeasible since $wB \geq vA$).

3.d) $b(v) = A$ and $b(w) = B$.

If N does not exist just connect B to v. the resulting triangles are $\triangle vAB, \triangle vwB$ which clearly are *nice*.

If N exists then:

If $vA < \frac{g}{2}$ then the resulting triangles are $\triangle vAN, \triangle vNw, \triangle wNB$. It can be proved that $\angle wvN \leq \frac{\pi}{2}$.

If $vA \geq \frac{g}{2}$ then

Approach B: Insert Steiner point P and we have the case $b(v) = A$ and $b(w) = P$, i.e case 3.h) of the lemma.

Approach A: Let H be a point on AF such that $AH = \frac{g}{4}$. Let T be the perpendicular projection of H onto the line through vw. The triangles of our construction are $\triangle THw, \triangle HNw, \triangle HAN, \triangle wNB$. The last two are clearly *nice*. It remains to prove that for the first two. We need to prove that $\angle vwH$ is *non-small*. This can be done either directly as in the previous case or indirectly by using the *non-small angle triangle* lemma. We choose the second approach.

Since $b(v) = A$ implies that $vA \geq \frac{g}{2}$ and thus H is between A and v and $vH \geq \frac{g}{4}$.

Let $\rho = \angle vHw$. Since w lies between D and C then $\tan(\rho)$ lies in $[\frac{4}{7}, \frac{4}{3}]$. Since Hw lies in $[\frac{5g}{4}, \frac{\sqrt{65}}{4}g]$ then implies that $\triangle vHw$ has *non-small* angles.

We need also to prove that $\angle HNw \leq \frac{\pi}{2}$. It is clear that $\angle HNw \leq \angle HNC = \frac{\pi}{2}$. The last equality comes from the similarity of $\triangle HAN, \triangle CNB$.

Here also we have to prove that HT does not create problems in the previous cell.

This is again proved in lemma 5.11.

3.e) $b(v) = A$ and $b(w) = \omega$. (infeasible since $vA \leq wB$)

3.f) $b(v) = \omega$ and $b(w) = P$.

If N does not exist then:

Connect P to v, resolve the obtuse $\angle vPw$. Connect v to B. Apply lemma 5.6 for ΔvAB.

If N exists then:

Let R be the intersection of vw with the vertical from N. Connect R to v, N, P and N to v, P. Apply lemma 5.6 for ΔvAN.

$$RN = \frac{Av + Bw}{2}.$$

Since $Av < \frac{g}{2}$ and $Bw < \frac{3g}{2}$ implies that $RN < g$. We also have $wP < g$. Then ΔvRN, ΔRPN ΔwRP are non-obtuse.

3.g) $b(v) = \omega$ and $b(w) = B$

If N does not exist then, connect v to B and apply lemma 5.6 for ΔvAB.

If N does exists then, connect N to v, w and apply lemma 5.6 for ΔvAN.

We have to prove that $\angle wvN \leq \frac{\pi}{2}$.

Let $\angle wvN = \omega$. Let vz be a horizontal ray directed to the right of v. Let also $\theta = \angle wvz$ and $\phi = \angle Nvz$. Then $sign(\tan(\omega)) = sign(\tan(\theta + \phi) = sign(1 - \tan(\theta)\tan(\phi))$.

Let $x = vA$ and $y = wC$ where $0 \leq x, y < \frac{g}{2}$.

$$\tan(\theta) = \frac{g - x + y}{g}$$

and $\tan(\phi) = \frac{2x}{g}.Thus$

$$sign(1 - \tan(\theta)\tan(\phi)) = sign(1 - \frac{2x}{g}\frac{g - x + y}{g}) = sign(g^2 - 2gx + 2x^2 - 2xy).$$

Let $f(x, y) = g^2 - 2gx + 2x^2 - 2xy$. Then $f(x, y) \geq f(x, \frac{g}{2}) = g^2 - 3gx + 2x^2 = h(x)$. The zeros of the quadratic equation $h(x) = 0$ are $\frac{g}{2}$ and g. Thus $h(x) > 0$ for $x < \frac{g}{2}$, which implies that $\omega < \frac{\pi}{2}$. In case $\angle vNw > \frac{\pi}{2}$, resolve it with the perpendicular from N onto vw.

3.h) $b(v) = A$ and $b(w) = P$

If N does not exist then connect P to v, A.

If N exists then connect P to N, v and N to v.

□

Lemma 5.4 *A one square configuration can be nice triangulated.*

Proof: The constructions are similar to those of lemma 5.3 and thus for the sake of brevity we ask the reader to consult fig. 7. Again in some cases the suggested construction penetrates the previous cell. In lemma 5.13 we prove that this modification does not propagate. □

Lemma 5.5 *A non-cross two-square configuration can be nice triangulated.*

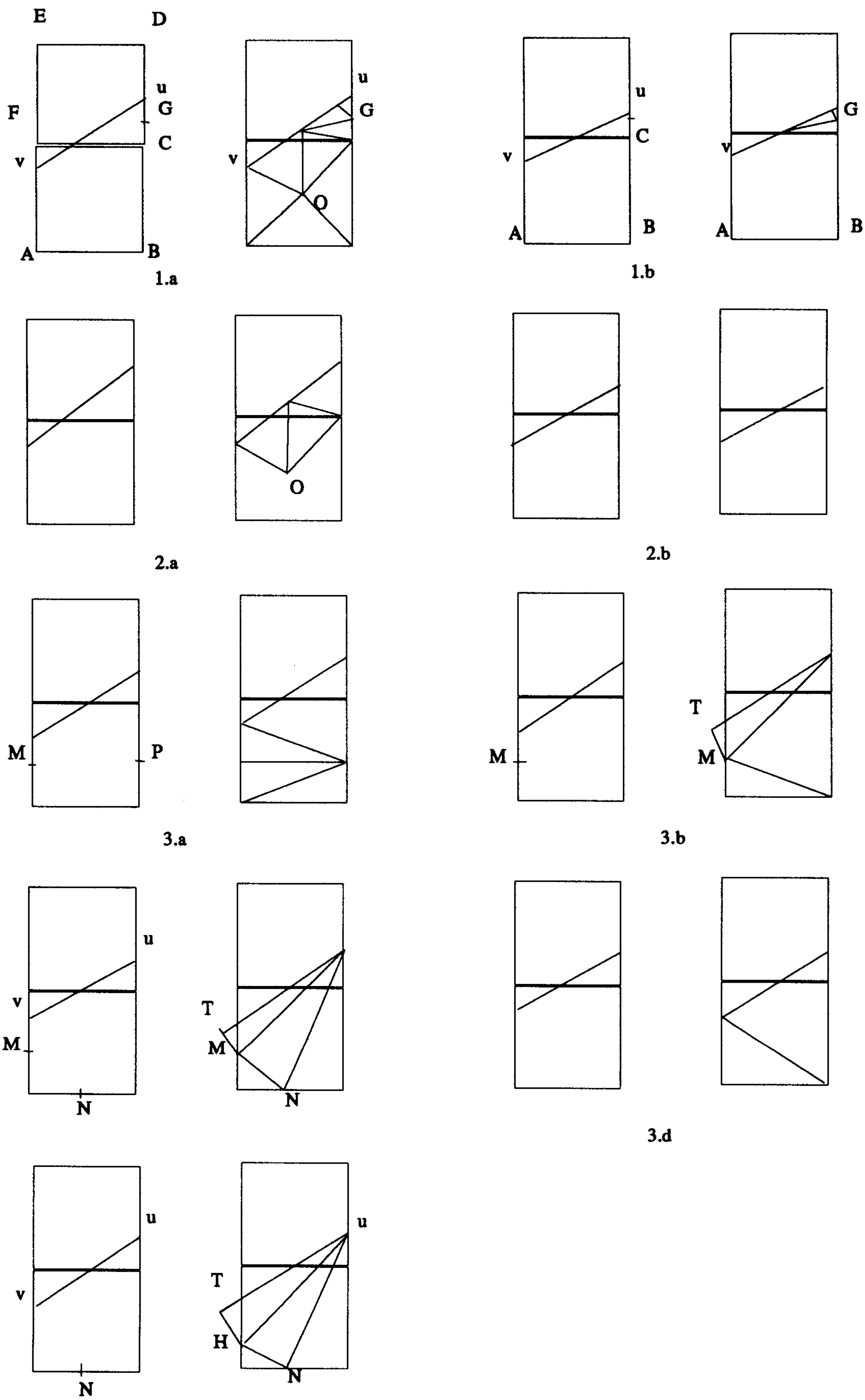

Figure 6: Lemma 5.3

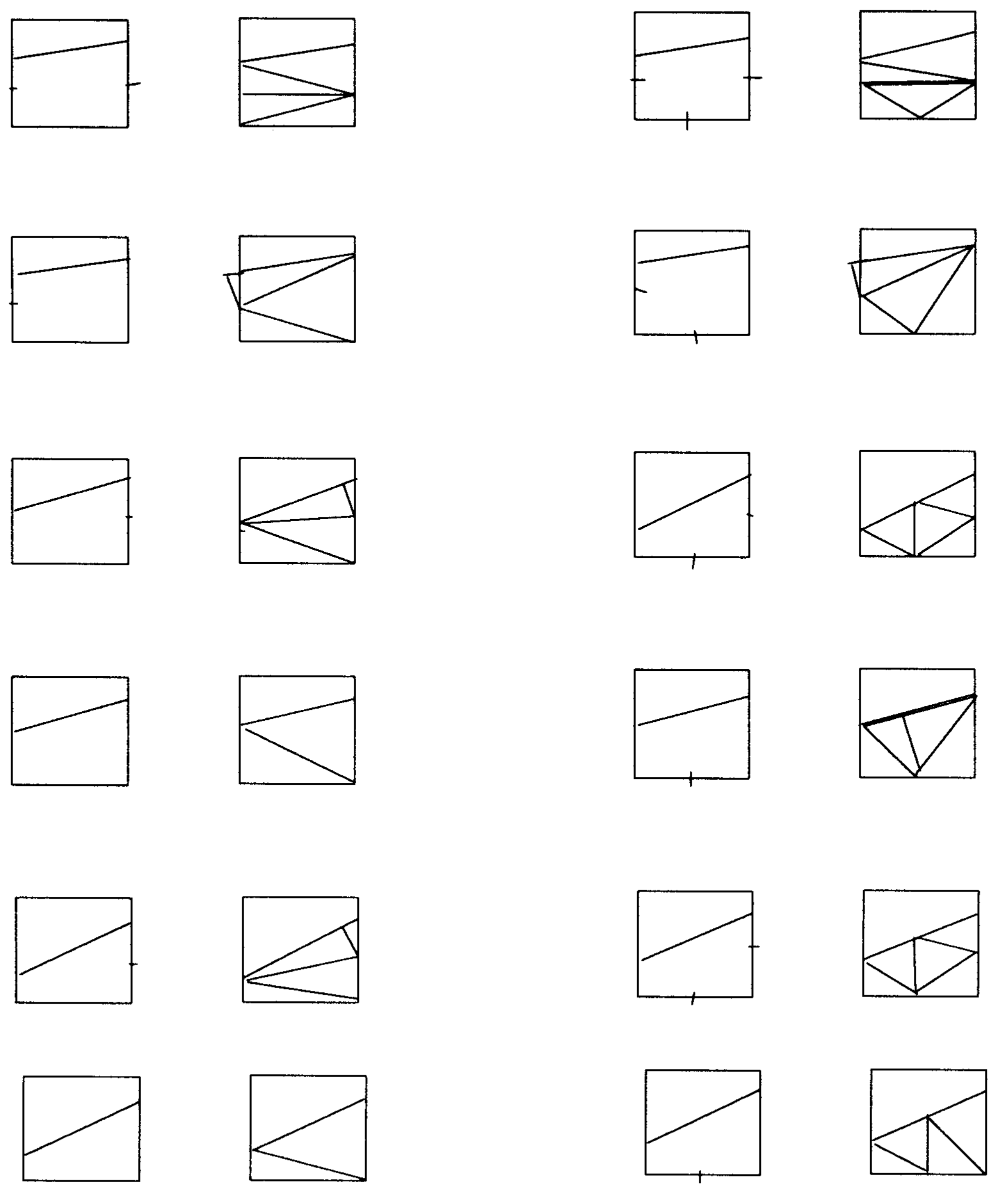

Figure 7: Lemma 5.4

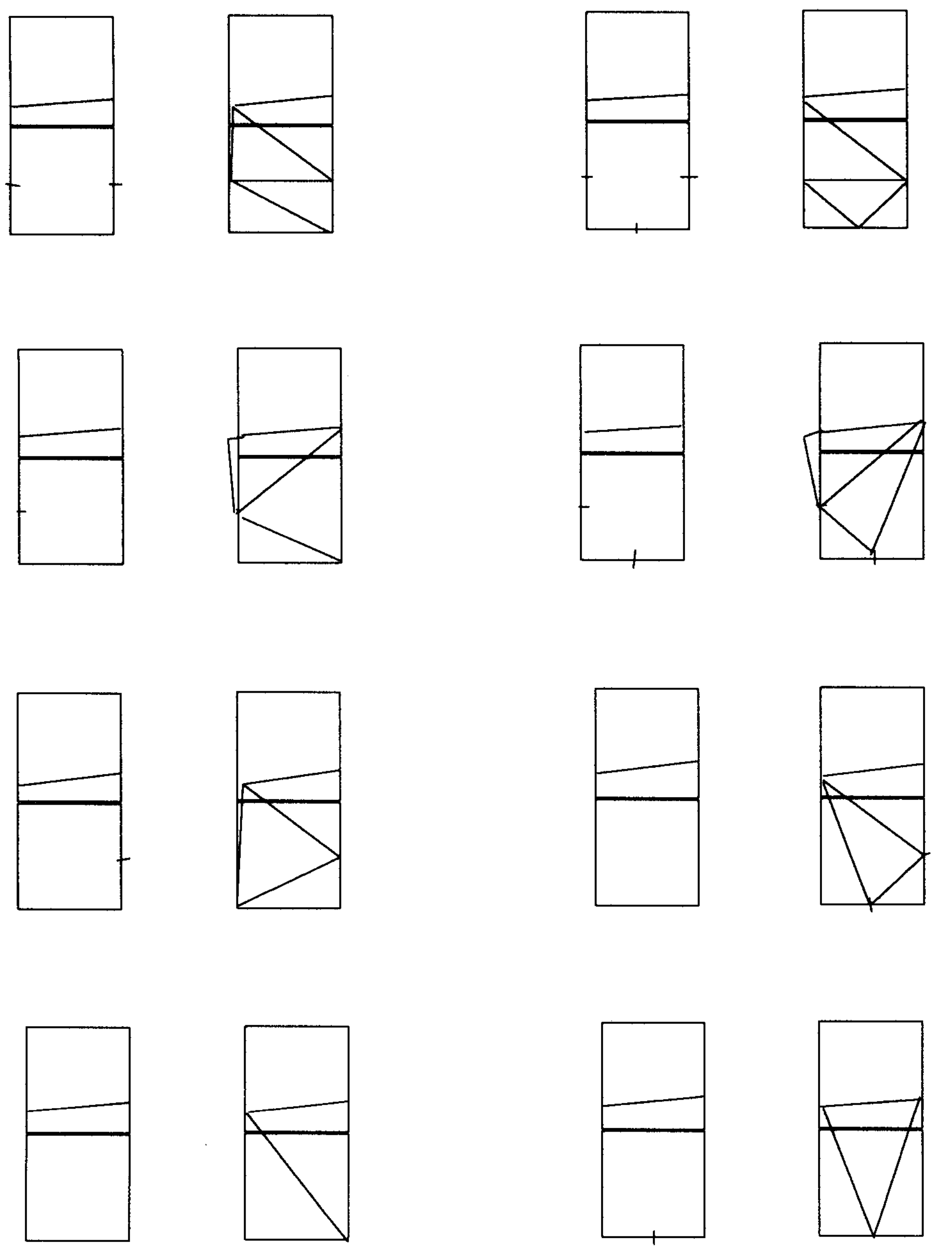

Figure 8: Lemma 5.5

Proof:

The idea of the constructions is similar to those of lemma 5.3 thus consult fig. 8.

□

Lemma 5.6 *Given a square $FCDE$ of size g and a point v on EF such that $Fv < \frac{g}{2}$ then the square below $FCDE$ can be used to create nice triangulations.*

Proof:

In the quad-tree subdivision the square below $FCDE$ may have size g, $\frac{g}{2}$ or $2g$. We invoke the current lemma only from the cases $3.f$ and $3.g$ of lemma 5.3 and thus the $\frac{g}{2}$ case is excluded.

Since vC consists part of internal boundary, we are not allowed to insert Steiner points on it. Many of the cases can be proved by simple constructions although some of them require nontrivial ones.

Case A: The square below $FCDE$ has size $2g$. Let $ABGF$ the below square. See fig. 9.a.

Case B: The square below $FCDE$ has size g. Let $ABCF$ be the square below $FCDE$. Let also $\{M, N, P\}$ be the midpoints of AF, AB, BC respectively. We have to nice triangulate the quadrilateral $vABC$ subject to the presence of a subset of the Steiner points $\{M, N, P\}$.

a) $S = \{\Omega\}$. See fig. 9.b Connect C to A.

b) $S = \{M\}$. Connect M to C, B. See fig. 9.c

c) $S = \{N\}$. Connect N to v, C. Since $Fv < \frac{g}{2}$ implies $\angle vCF < \angle NCB$ implies $\angle vCN < \frac{\pi}{2}$. See fig. 9.d

d) $S = \{P\}$. Let K a point such that $PK = \frac{g}{4}$. Connect K to v, C, B, A. Since $Fv < \frac{g}{2}$ implies $\angle vCF < \angle KCP$ implies $\angle vCK < \frac{\pi}{2}$. See fig. 9.e

Also $\angle vKC \leq \angle FKC \leq \frac{\pi}{2}$. Since $Av < \frac{3g}{2}$ and the distance of K from Av equals $\frac{3g}{2}$ then $\angle vKA < \frac{\pi}{2}$.

e) $S = \{P, N\}$. Let H be a point on AF such that $FH = \frac{g}{2}$. Connect H to C, P, N, v. Clearly all created triangles are nice. See fig. 9.f

However we need to prove that the newly created Steiner point H does not create inconsistencies indefinitely. We prove that in lemma 5.17.

f) $S = \{M, N\}$. Connect M to N, C. See fig. 9.g

g) $S = \{M, N, P\}$. Connect M to C, P, N. See fig. 9.h

□

Lemma 5.7 *Steiner point H does not create inconsistencies indefinitely.*

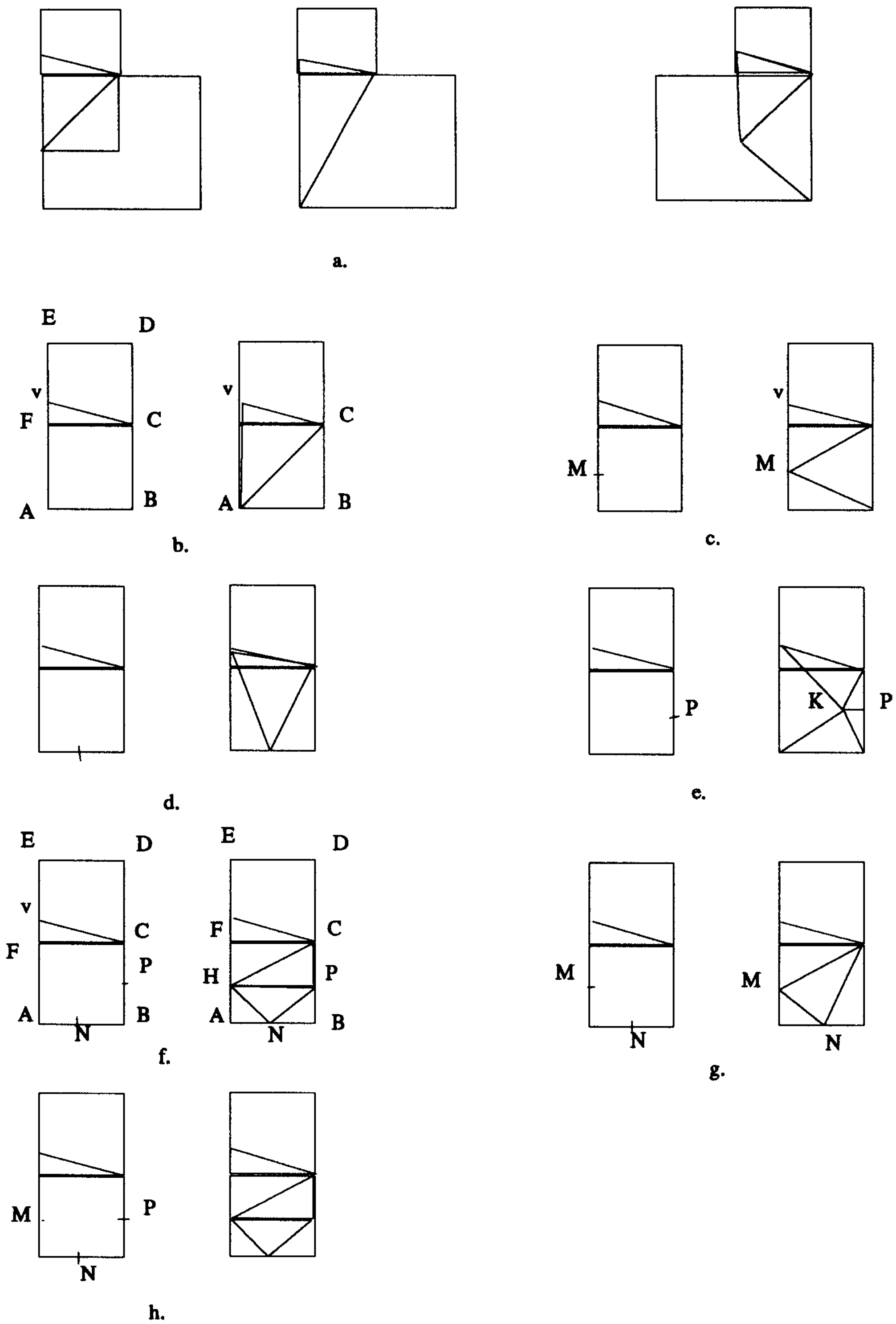

Figure 9: Lemma 5.6

Proof: The proof is based on the fact that that we have used *approach B* for edge triangulation.

We have the following possibilities:

a) The left adjacent square is of of half size. Then H already existed and thus no need for change.

b) The left adjacent square is of the same size. Then the previous configuration is any of the configurations of lemma 5.3 such that $b(w) = B$. That implies it corresponds to cases 3.b, 3.d, or 3.g of that lemma.

b.1) If we have case 3.b, i.e $b(v) = M$, then H already exists as P using the terminology of lemma 5.3. Thus no modifications required. The possible existence of Steiner point N does not create any problems.

b.2) If we have case 3.d, i.e. $b(v) = A$, then we distinguish two subcases: First if Steiner point N exists then H already exists as P according to case 3.d of lemma 5.3. Thus no modifications required. Second, if Steiner point N does not exist, then the insertion of point H will require modifications in that cell. We need to connect H to A and v and delete edge vB.

b.3) If we have case 3.g, i.e.$b(v) = \omega$, then again we have to distinguish whether Steiner point N exists or not. If N does not exist then connect H to v and w and resolve the obtuse $\angle vPw$ by taking the perpendicular from H onto vw.

If N exists then: Let R be the intersection of vw and the vertical from N. Delete the edge Nw and insert edges RN, RH, HN. As it is proved in lemma 5.3, since $RN < g$ guarantees the non-obtuseness of $\angle RHN$.

c) The left adjacent square is of double size. Similar techniques to those of step b) can be applied which therefore are omitted.

□

Lemma 5.8 *Let v, w be two consecutive intersection points of an edge e with two vertical q-tree edges. We can nice triangulate segment vw using $O(1)$ additional Steiner points.*

Proof:

Refer to the general cases of fig.4. Case a) can be triangulated according to lemmas 5.4 and 5.5 and case b) can be triangulated according to the constructions of lemma 5.3.

For case c): case c.1: Let x be the intersection of the line through v and w and the line through the right vertical side of s_1. Let s_3 be the square which is the right neighbor of s_2. If x lies on the boundary of s_3 then triangulate the union of s_1, s_2, s_3 according to *lemma* 5.3. If x is above s_3 then connect O to w and v, where O is the center of the square s_1. Note in this case $vA \geq \frac{g}{2}$ since otherwise the slope of vw would be greater than one. That implies $\triangle vAO$ is non-obtuse. Then treat the triangulation below wx recursively.

case c.2: According to lemma 5.3 Case d): Case d.1: Triangulate separately below vx and xw according to *lemma* 5.3. Case d.2: Triangulate below vw according to lemma 5.3. □

By applying lemma 5.8 along the whole boundary of P we have:

Theorem 5.9 *We can nice triangulate the semirectilinear polygon P^{**} using $O(nA)$ additional Steiner points.*

5.3 Repairing Triangulations of the Previous Cell

Lemma 5.10 *The perpendicular from M onto vw in lemma 5.3 does not propagate modifications indefinitely.*

Proof: Consult fig. 10.

We need to change the *previous* configuration if the *current* configuration is case 3.b of lemma 5.3.

The *current* configuration is 3.b. The existence or not of the Steiner point N does not make any difference here.

Then the *previous* configuration can be a) cases of lemma 5.4 such that $b(w) = B$ and without the N Steiner point (since existence of N implies violation of the balancing condition of the q-tree) or b) cases of lemma 5.3 such that $b(w) = C$.

a) The relevant cases of lemma 5.4 are 2.a, 6.a, 9.a. and the resulting transformations are shown in 14.

b) The relevant cases of lemma 5.3 are 2.a and 2.b. It is clear that the perpendicular CT on vw maintains the triangulations *nice*.

□

Lemma 5.11 *The perpendicular from H onto vw in lemma 5.3 does not propagate modifications indefinitely.*

Proof:

1) The left cell is of the same size.

1.a) One-square configuration with $b(w) = B$.

1.a.1) $b(v) = M$ fig. 11.a fig. 11.b

1.a.2) $b(v) = A$ fig. 11.c fig. 11.d

1.a.3) $b(v) = \omega$ fig. 11.e fig. 11.f

1.b) Two-square configuration with $b(w) = C$.

As discussed in lemma 5.10 these inconsistencies can be remedied. The triangulation of the previous cell is done according to case 2 of the lemma. Particularly we cannot have case 2.b. Since we require $vA \geq \frac{g}{2}$ which in case 2 notation means $wC \geq \frac{g}{2}$. However case 2.b requires $uC \leq \frac{g}{2}$ which implies the slope of vw is greater than one, a contradiction.

Now we have to change the previous cell triangulation since we have the new point H on wc.

Thus consider the configuration of case 2.a with the additional points H and T where T as defined before and the fact that $wC \geq \frac{g}{2}$ which is not necessary for case 2.a in general.

If $\angle RHC \leq \frac{\pi}{2}$ then just connect R to H.

If $\angle RHC > \frac{\pi}{2}$ then implies $RQ > HC = \frac{g}{4}$. That means QR is not very small thus connect Q to R, v, S, H where S is the perpendicular projection of Q onto the line

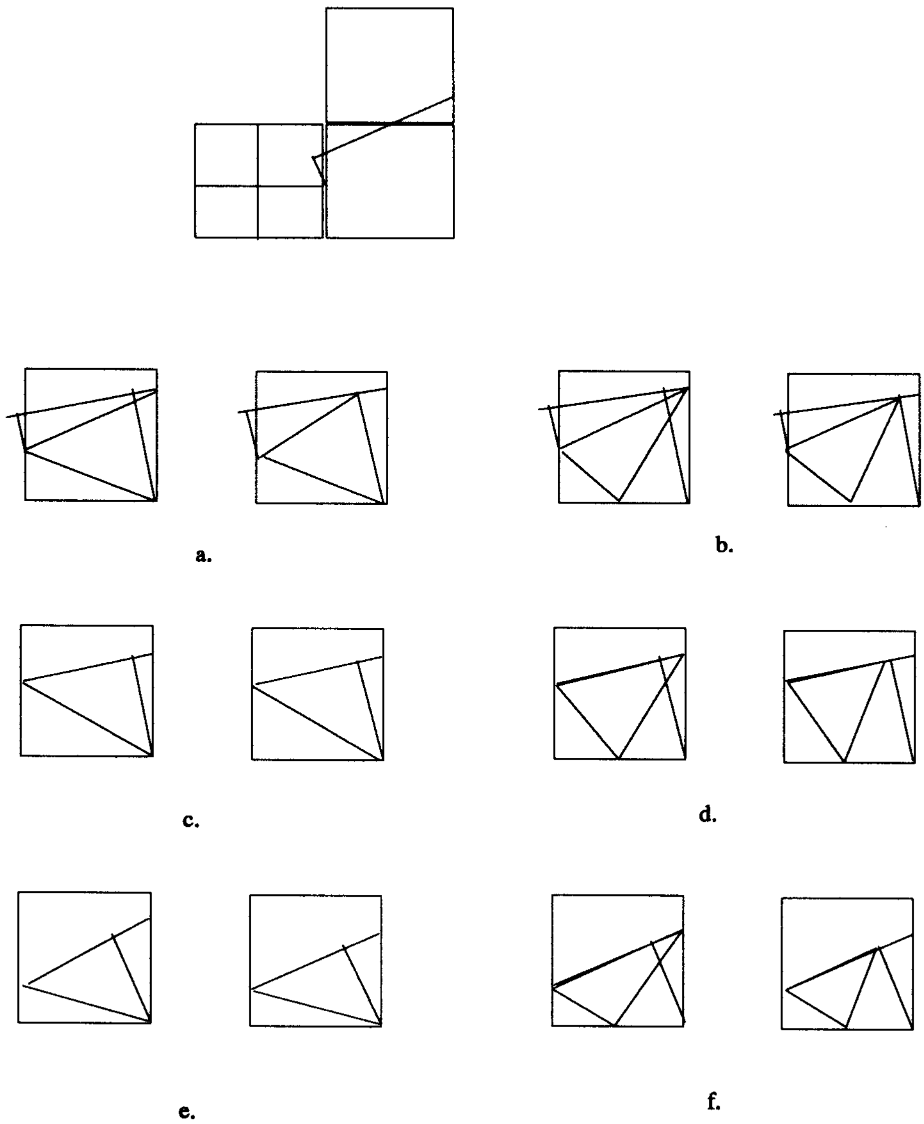

Figure 10: Lemma 5.10

through vw. It is easy to show that the created triangles are nice. Also by the construction is clear that the modifications do not propagate further.

The right ΔwTH is nice since $wH \geq \frac{g}{4}$ and $\angle TwH$ does not approach $\frac{\pi}{2}$. This is important because if $\angle TwH$ approaches $\frac{\pi}{2}$ then ΔwTH is not nice.

2) The left cell has half size:

2.a) The previous configuration is case $b(w) = B$ of lemma 5.3.

2.a.1) $b(v) = M$ fig. 12.a fig. 12.b

2.a.2) $b(v) = A$ fig. 12.c fig. 12.d

2.b) The previous configuration is case $b(w) = B$ of lemma 5.5.

2.b.1) $b(v) = M$ fig. 12.e fig. 11.f

2.b.2) $b(v) = A$ fig. 12.g fig. 12.h

□

Lemma 5.12 *The perpendicular from M onto vw in lemma 5.5 does not propagate modifications indefinitely.*

Proof:
The configuration of the previous cell can be either:

a) *two-square* with $b(w) = B$ and without Steiner point N which are only cases 3.b and 3.d of lemma 5.3.

a.1) Case of lemma 5.3 such that $b(v) = M$ and $b(w) = B$ without the N Steiner point. See fig. 6 for the appropriate transformation.

a.2) Case of lemma 5.3 such that $b(v) = A$ and $b(w) = B$ without the N Steiner point. See fig. 6

b) A configuration of lemma 5.5 with $b(w) = B$.

b.1) Case $b(v) = A$ $b(w) = B$ of lemma 5.5. See fig. 8 for the transformation.

b.2) Case $b(v) = M$ $b(w) = B$ of lemma 5.5. See fig. 8

□

Lemma 5.13 *The perpendicular from M onto vw in lemma 5.4 does not propagate modifications indefinitely.*

Proof: Consult fig. 14.
The previous configuration may be either a) cases from lemma 5.4 such that $b(w) = B$ and without the N Steiner point or b) cases from lemma 5.3 such that $b(w) = C$.

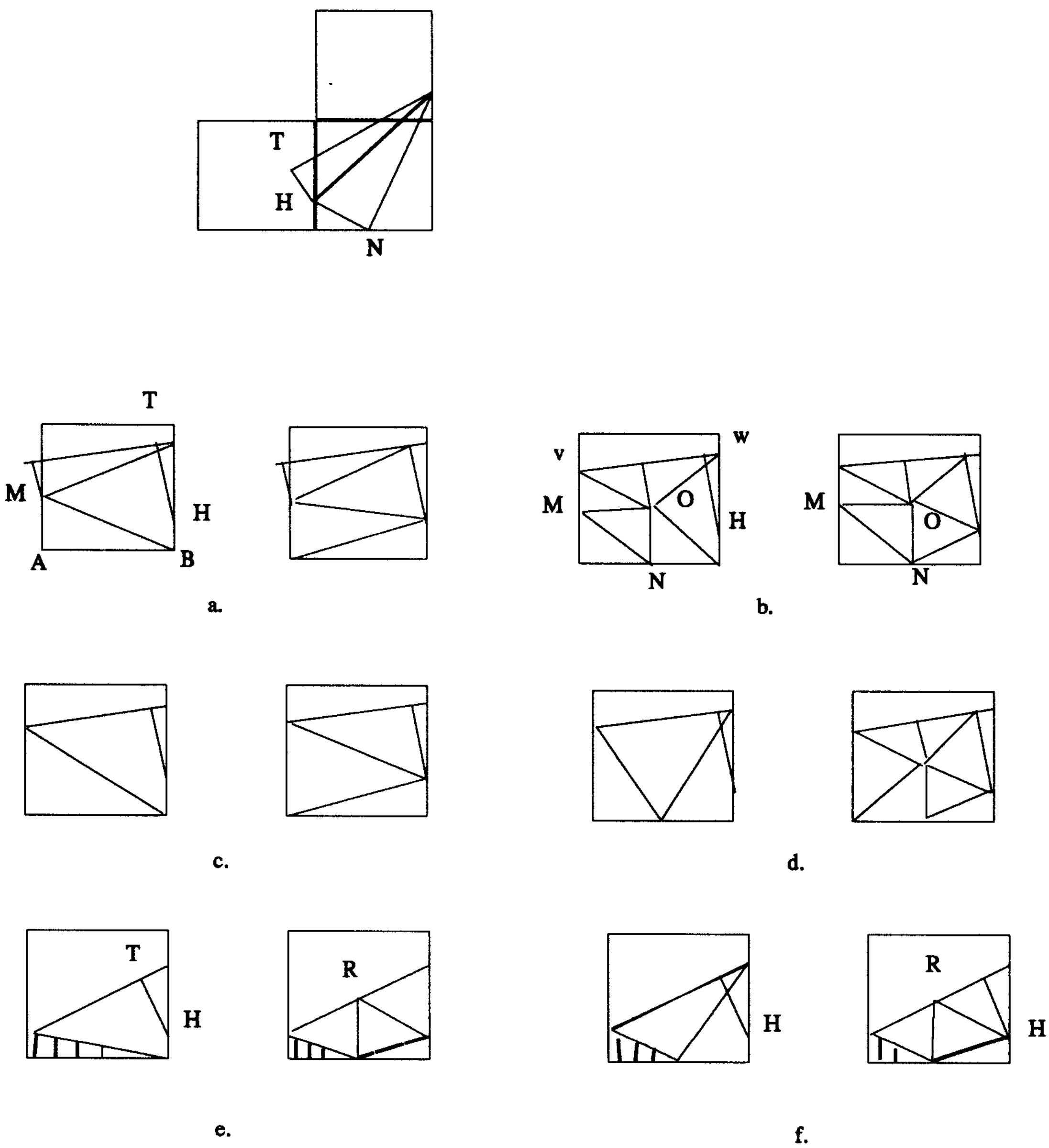

Figure 11: Lemma 5.11 The left square has the same size

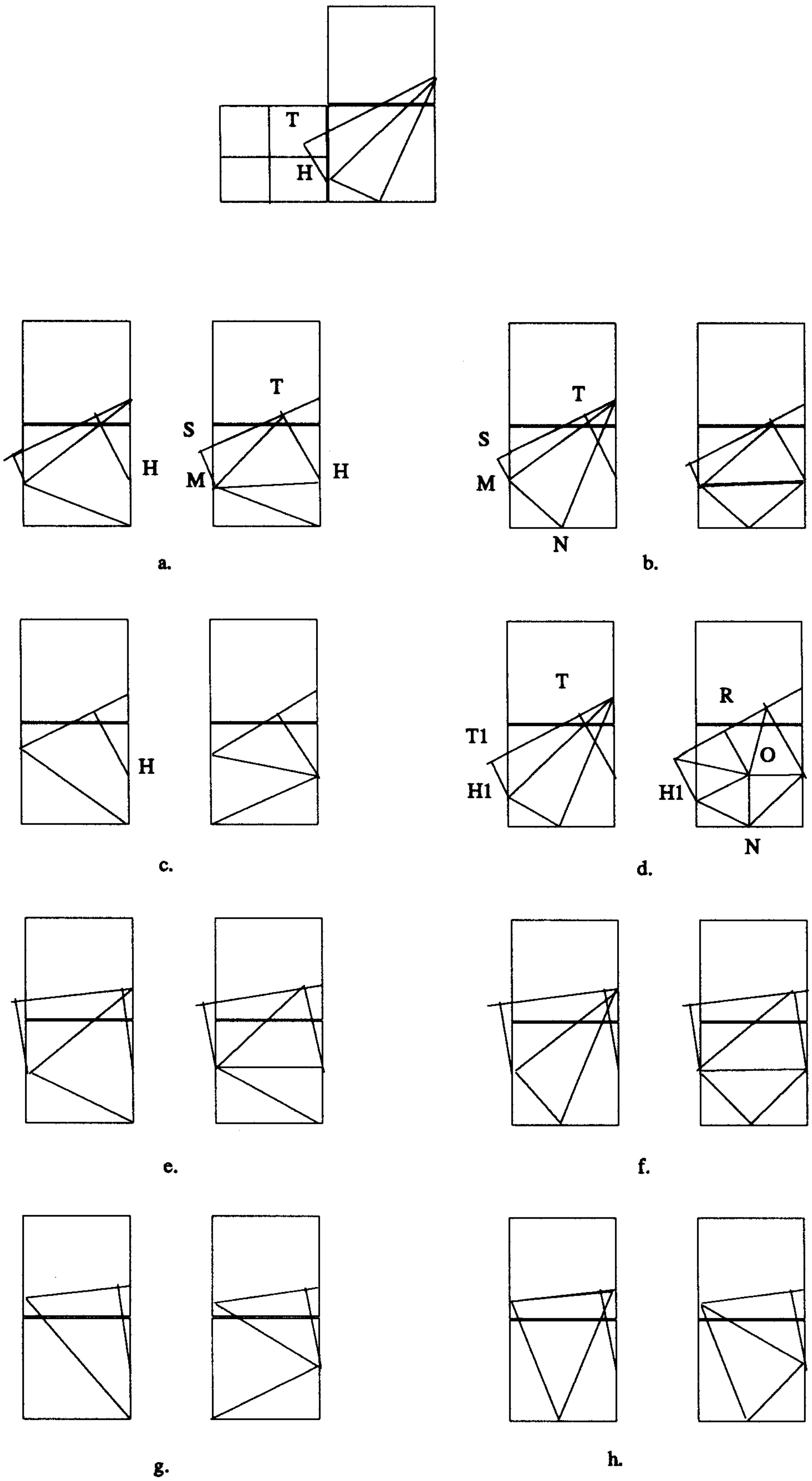

Figure 12: Lemma 5.11 cont.

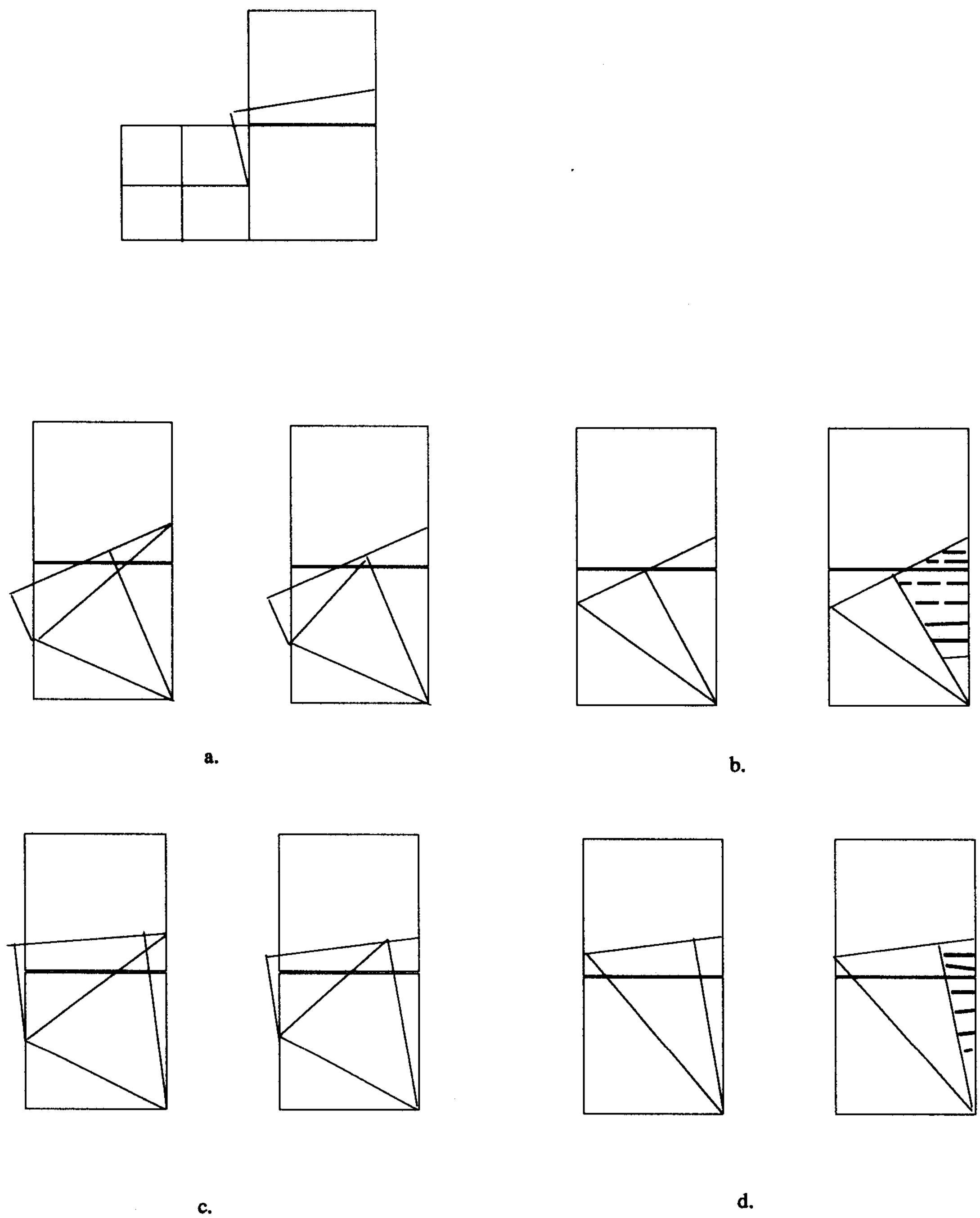

Figure 13: Lemma 5.12

a) a.1) The previous configuration is $b(v) = M$ and $b(w) = B$ see fig. 14 for the transformation.

a.2) The previous configuration is $b(v) = A$ and $b(w) = B$ see fig. 14

a.3) The previous configuration is $b(v) = \omega$ and $b(w) = B$ see fig. 14

b) Can be treated easily.

□

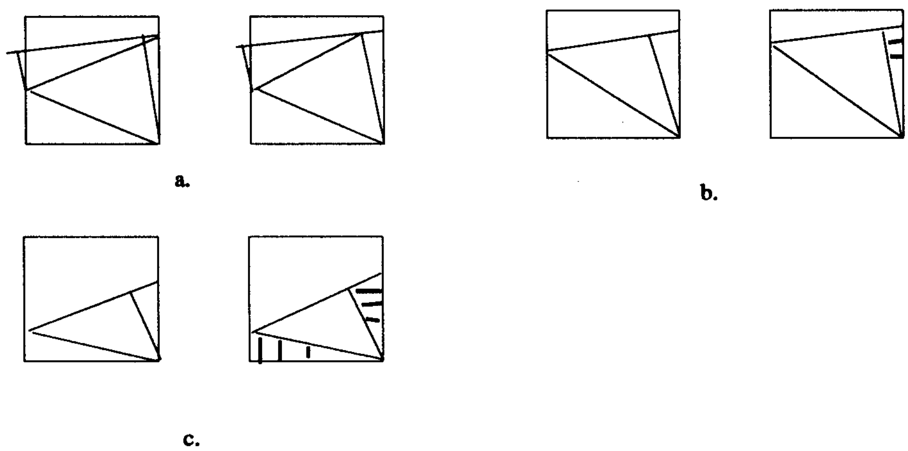

Figure 14: Lemma 5.13

Lemma 5.14 *Let $ABCDEF$ be a two-square configuration with square size g and let v (resp. w) points on EF (resp. DC) such that $Fv < \frac{g}{2}$ and $wc < \frac{g}{2}$. Let also BT be the perpendicular projection of B onto vw and let M be the midpoint of AF. Then $\angle TMB \leq \frac{\pi}{2}$.*

Proof: See fig. 15.a. Let G be a point on DC such that MB is perpendicular to MG. Let R be the intersection of vw with MG. Let $x = Rw$ and $y = vR$. Let Q be the perpendicular projection of v onto DC. Let $w = wQ$ and $z = QC$.

It is sufficient to prove that $x(x+y) \geq w(w+z+g)$. where z, w in $[0, \frac{g}{2})$.

From the similarity of $\triangle MvR, \triangle RwG$ we get that

$$\frac{x}{y} = \frac{\frac{g}{2} + g - w - z}{\frac{g}{2} + z} (1)$$

It is also true that $(x+y)^2 = g^2 + w^2$ (2)

From (1) we get

$$\frac{x}{\frac{3}{2}g - z - w} = \frac{y}{\frac{1}{2}g + z} = \frac{x+y}{(2g - w)} (3)$$

From (2) and (3) we get

$$x(x+y) = \frac{\frac{3}{2}g - z - w}{(2g - w)}(w^2 + g^2)$$

Then it is sufficient to prove that
$(\frac{3}{2}g - z - w)(w^2 + g^2) \geq w(2g - w)(w + z + g)$ (4)
Let $u = z + w$ and $v = w$ then u, v in $[0, \frac{g}{2})$.
Then (4) becomes
$(\frac{3}{2}g - u)(v^2 + g^2) \geq v(2g - v)(u + g)$ (5)
(5) becomes $3gv^2 + 3g^3 - 2g^2u - 4guv - 4g^2v + 2gv^2 \geq 0$ (6)
Let $f(u, v) = 3gv^2 + 3g^3 - 2g^2u - 4guv - 4g^2v + 2gv^2$.
We need to prove that $f(u, v) \geq 0$.
We have that $\frac{\partial f}{\partial u} = -4gv - 2g^2 < 0$ (7)
From (7) we have that $f(u, v) \geq f(\frac{g}{2}, v) = g(5v^2 - 6gv + 2g^2) = gh(v)$ where $h(v) = 5v^2 - 6gv + 2g^2$.
But $h'(v) = 10v - 6g$. Then for v in $[0, \frac{g}{2})$ $h'(v) < 0$ which implies that $h(v) > h(\frac{g}{2}) = \frac{g^2}{2} > 0$ Q.E.D.

□

Lemma 5.15 *Let $ABCDEF$ be a two-square configuration with square size g and let v (resp. w) points on AF (resp. DC) such that $Fv < \frac{g}{2}$ and $wc < \frac{g}{2}$. Let also BT be the perpendicular projection of B onto vw and let M be the midpoint of AF. Then $\angle TMB \leq \frac{\pi}{2}$.*

Proof:
Similar techniques like lemma 5.14 apply and therefore the proof is omitted. See fig. 15.b.

□

Lemma 5.16 *Let $ABCDEF$ be a two-square configuration with square size g and let v (resp. w) points on AF (resp. DC) such that $Fv < \frac{g}{2}$ and $wc < \frac{g}{2}$. Let H be a point on AF such that $AH = \frac{g}{4}$, M, P the midpoints of AF and BC respectively, and S be the midpoint of vw Let PQ, OR, HT be the perpendicular projections of P, O, H onto vw. Let K be a point on MP such that $MK = \frac{g}{4}$. Then $\angle QOP, \angle TOH, \angle RHO \leq \frac{\pi}{2}$.*

Proof: See fig. 15.c.

a) $\angle QOP \leq \frac{\pi}{2}$:

It is sufficient to show that $Qw \leq Qv$ which is equivalent to $\tan(\theta) \geq \tan(\phi_1 + \phi_2)$ (1)

But

$$\tan(\theta) = \frac{1}{\tan(\phi_1)}, \tan(\phi_1) = \frac{x + y}{g}, \tan(\phi_2) = \frac{\frac{g}{2} - y}{g}$$

where x, y in $[0, \frac{g}{2}]$.
Thus (1) becomes

$$\frac{g}{x + y} \geq \frac{\frac{x+y}{g} + \frac{g}{2} - y}{1 - \frac{(x+y)(\frac{g}{2}-y)}{g^2}}$$

which is

$$\frac{1}{x+y} \geq \frac{x+\frac{g}{2}}{g^2-(x+y)(\frac{g}{2}-y)}(2)$$

Since $g^2-(x+y)(\frac{g}{2}-y)>0$ (because $\phi_1+\phi_2<\frac{\pi}{2}$) (2) becomes $g^2-(x+y)(\frac{g}{2}-y)-(x+y)(x+\frac{g}{2}) = g^2-(x+y)(g+x-y) = g^2-gx-gy-x^2+y^2 = 2((\frac{g}{2})^2-x^2)+(x-\frac{g}{2})^2+(y-\frac{g}{2})^2 \geq 0$. Q.E.D.

b) $\angle RHO \leq \frac{\pi}{2}$: True since R is above MO and between v and w.

c) $\angle TOH \leq \frac{\pi}{2}$:

If $\angle THO \geq \frac{\pi}{2}$ then we are done. If $\angle THO < \frac{\pi}{2}$ then it suffices to show that OT lies between the rays OH and OG which is true by construction of vw.

□

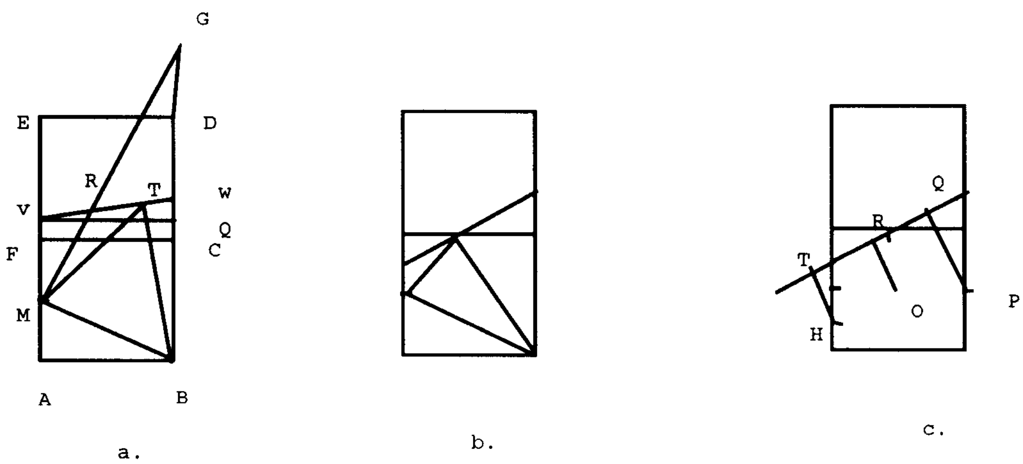

Figure 15: figures a, b, c

Lemma 5.17 *Steiner point H does not create inconsistencies indefinitely.*

Proof: The proof is based on the fact that that we have used *approach B* for edge triangulation. We have the following possibilities:

a) The left adjacent square is of of half size. Then H already existed.

b) The left adjacent square is of double size.

c) The left adjacent square is of the same size. Then the previous configuration is any of the configurations of lemma 5.3 such that $b(w) = B$.

c.1) $b(v) = M$ with and without the N Steiner point appear in fig. 16.a and fig. 16.b respectively.

c.2) $b(v) = A$ with and without the N Steiner point appear in fig. 16.c and fig. 16.d respectively.

c.3) $b(v) = \omega$ with and without the N Steiner point appear in fig. 16.e and fig. 16.f respectively.

All possibilities appear in fig. 16.

□

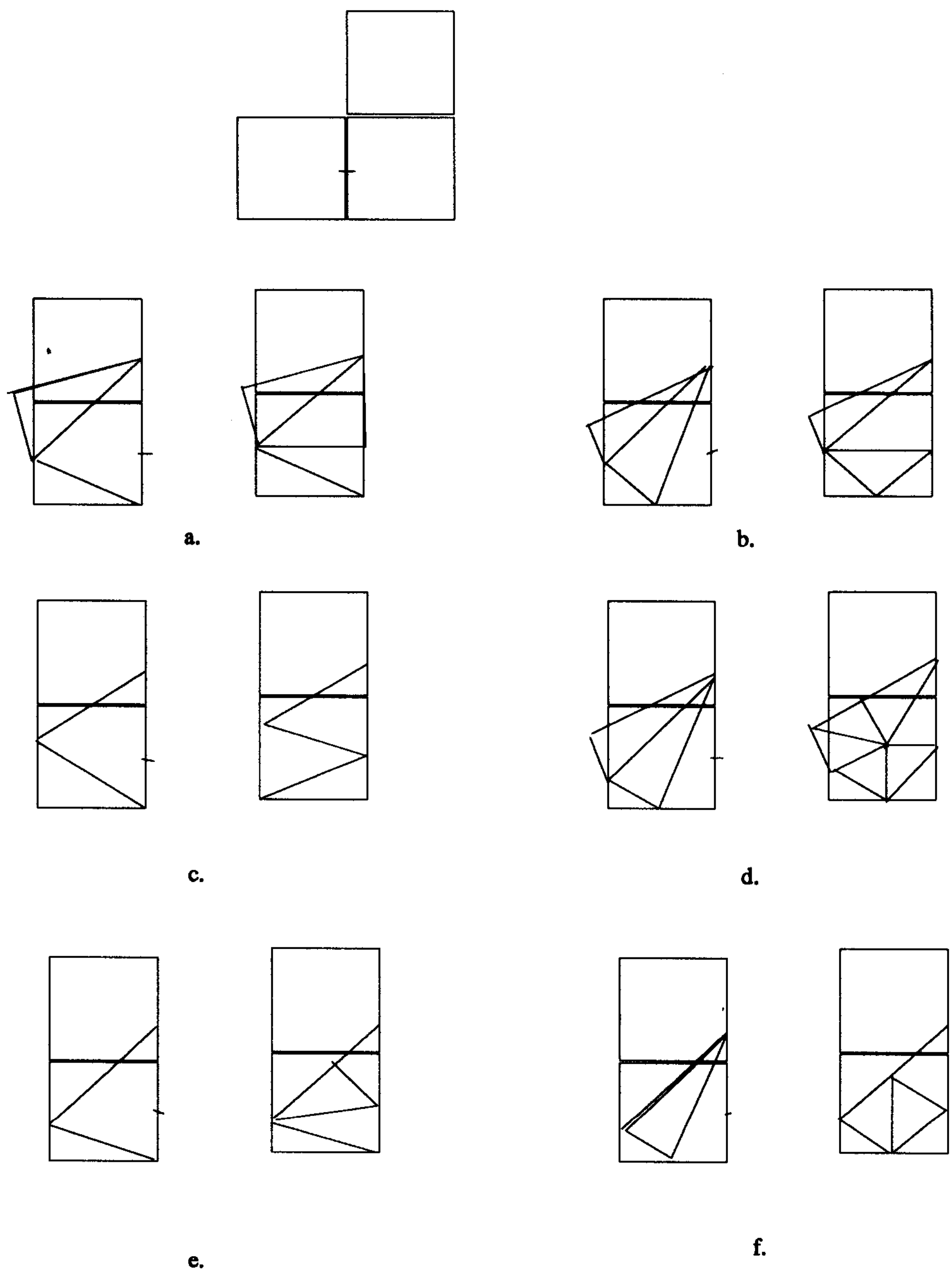

Figure 16: Lemma 5.17

6 Triangulating near Polygon vertices

In triangulating near polygon edges we always assumed that we have enough *clearance* below the edge. However this is not true when we need to triangulate close to a vertex. Thus we need to treat vertices in a different way. Vertices are classified into two basic classes: *acute* vertices and *obtuse* vertices.

In subsection 6.1 we show how to do nice triangulation near acute vertices and in subsection 6.2 we show how to do nice triangulation near obtuse vertices.

6.1 Triangulating near acute vertices

Acute vertex triangulation is done by cutting of an isosceles triangle. After cutting this isosceles triangle by the cutting edge AB we may need to separate points A and B.

To be more precise we will require that a) AB intersects *constant* number of q-tree cells and b) all these cells and their neighbors are of the same size. In fact the requirement is a uniform grid of constant size along AB. This can be achieved by further q-tree subdivisions. The above mentioned subdivisions will create Steiner points on AB. Thus we need to further triangulate $\triangle OAB$ to resolve the inconsistencies.

We are going to make the following two assumptions on the grid structure in the vicinity of AB. First the grid is uniform and second AB intersects $O(1)$ number of grid cells. The triangulation of $\triangle OAB$ can be achieved using a uniform grid whose axis are parallel and perpendicular (resp.) to the base of the isosceles triangle.

The following lemma describes how to triangulate the isosceles triangle using a uniform grid.

Lemma 6.1 *Given an isosceles triangle $\triangle OAB$ with Steiner points on AB $v_1, v_2, .., v_l$ such that $v_1 = A$ and $v_l = B$. and $v_i v_{i+1} = g$ for $i = 2, ...l-2$ and $v_1 v_2$ as well as $v_{l-1} v_l$ are between g and $2g$. Then there is a nice triangulation of $\triangle OAB$ using all the Steiner points on AB.*

Proof: Consult fig.17.

Consider a uniform grid with one axis parallel to AB and the other perpendicular to AB and grid spacing equal to g. Call the former axis horizontal and the latter vertical.

Starting from vertex O and moving down to AB let gl be the first horizontal gridline such that contains a one or two *safe* gridpoint(s). A gridpoint M will be called *safe* iff the horizontal distances of M from the triangle sides OA and OB is $[g, 2g)$. (Such a gridline always exists). Call C (D resp.) the intersection of OA (OB resp.) with gl. Then AC and BD can be treated as well separated edges and the triangulation is done according to the lemmas in section 5. What remains is to show how to triangulate $\triangle ACD$ using the *safe* gridpoints.

a) Only one safe point M on gl. Assume W.L.O.G that $MC \leq MD$. Let E be the intersection of AC and the parallel from M to AD and let F be the intersection of AD and the parallel from E to CD. We can easily prove that $\triangle AEF, \triangle ECM, \triangle MEF, \triangle MFD$ are nice.

b) Two safe gridpoints M, N on gl. Let gm be the gridline which is immediately above gl. Let G, H the intersection points of gm with OA and OB respectively. Then gm does not have a *safe* gridpoint. (otherwise we would have considered gm instead gl). Let $MNxy$ be the grid cell above gl and let Q be the midpoint of xy. Then connect Q to M, N, H, G and G to M, C and H to N, D. Then apply the constructions of a) in $\triangle AGH$ using point Q.

□

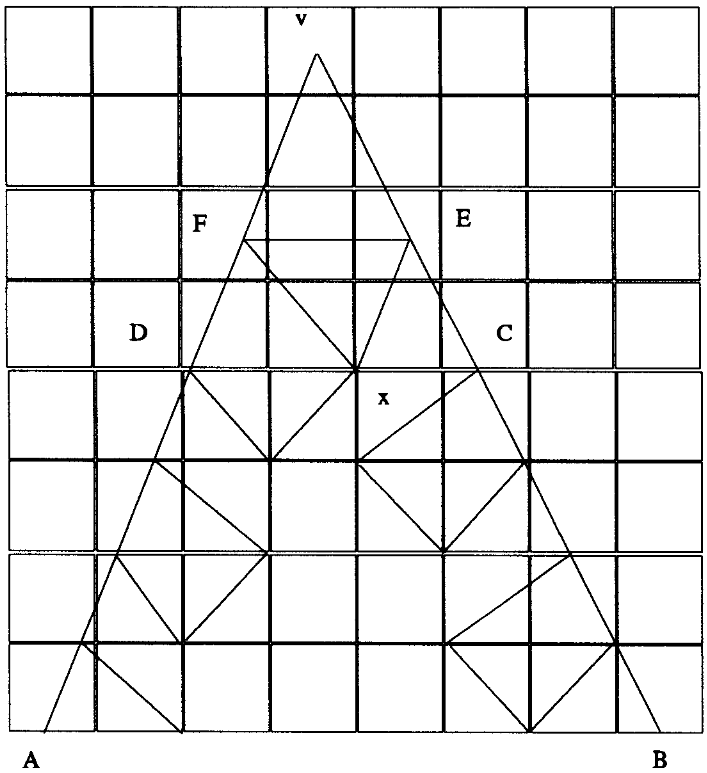

Figure 17: Triangulation of an isosceles triangle using uniform rotated grida

Lemma 6.2 *Given the conditions of lemma 6.1 the isosceles $\triangle OAB$ can be nice triangulated using a tree like grid.*

Proof: The proof is along the lines of the proof of lemma 6.1 and thus omitted. It should be clear that the number of produced triangles is $O(l)$ i.e does not depend on the $\angle AOB$. See fig. 18.

□

However the question is "where" we cut the isosceles triangle? If we cut very close to the vertex v, then the cutting edge will be very small (smaller than the minimum feature size of the polygon) which may result in an uncontrolled number of q-tree subdivisions. If the cutting edge is not "very small" we say that satisfies the *non-small cut* condition. In order for the cutting edge to be not "very small" , the cutting edge should be of the order of the the q-tree square which includes vertex v. Let AB be the cutting edge and

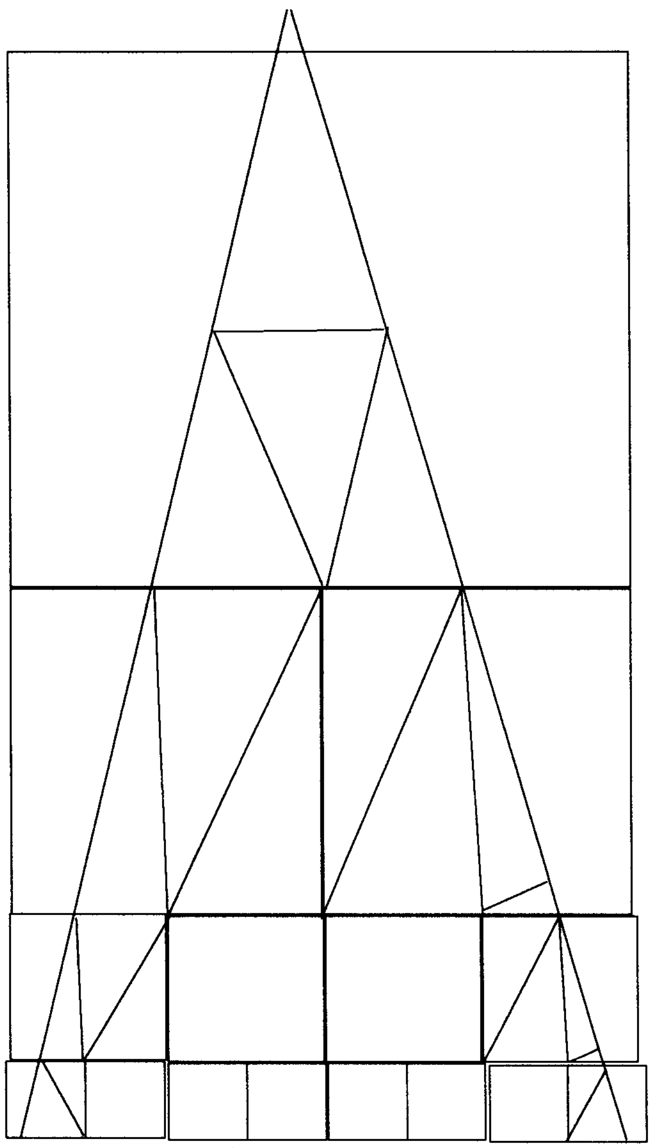

Figure 18: Triangulation using a tree-like grid

let θ be the acute angle at vertex v. After specifying points A and B we may have to separate A and B in the q-tree subdivision. If AB intersects $O(1)$ squares of the q-tree, then A and B satisfy the *constant clearance* condition.

Lemma 6.3 *Let O be an acute vertex of a polygon P. Let θ be the corresponding angle and A_v be the aspect ratio of any triangle of a triangulation of P which has O as a vertex. Then $\frac{1}{\tan\theta} \leq A_v$.*

Proof: Let $\triangle OAB$ be such a triangle and let ϕ, ρ be its smallest angle and its angle at vertex O respectively. It is well known that $\frac{1}{\tan\phi} \leq A_v$. But $\frac{1}{\tan\theta} \leq \frac{1}{\tan\rho} \leq \frac{1}{\tan\phi}$ Q.E.D. □

Lemma 6.4 *Under the non-small cut condition the number of additional q-tree subdivisions to achieve constant clearance of A and B for all acute vertices is $O(nA)$.*

Proof: Let $OA = b$ and $AB = x$. Then $x = 2b\sin\frac{\theta}{2}$. It is well known [45] that given two points A and B in a square of side s such that $AB = x$ then the number N of q-tree squares to separate AB is $O(log\frac{s}{x})$. Under the *non-small cut* condition $b = \Theta(s)$ which implies $N = O(log(\frac{1}{\sin\frac{\theta}{2}}))$.

We can prove that $\ln(\frac{1}{\sin\frac{\theta}{2}})) \leq \frac{1}{\tan\theta})$. Then as a simple consequence of lemma 6.3 we have that $\frac{1}{\tan\theta} \leq A$ and thus summing over all polygon vertices we get the result. □

Lemma 6.5 *The number of triangles required to nice triangulate a cuted isosceles triangle with acute angle θ using a uniform grid is $O(\frac{1}{\tan\theta})$.*

Proof: Let $AB = a$ and $vA = h$ and let g be the size of the grid cell. Then the number of triangles is $O(\frac{ah}{g^2})$. Since $h = \frac{a}{tan\theta}$ we get that number of triangles is $O((\frac{a}{g})^2\frac{1}{\tan\theta})$. However since we assume *constant clearance* between A and B implies that $\frac{a}{g}$ is constant Q.E.D. □

Lemma 6.6 *The total number of nice triangles to triangulate the acute vertices of the polygon is $O(nA)$.*

Proof: According to lemma 6.5 the number of created triangles for acute vertex i is $O(\frac{1}{\tan\theta_i})$. From lemma 6.3 we have that $\frac{1}{\tan\theta_i} \leq A$. Q.E.D □

6.2 Triangulation near non-acute vertices

The classification of *obtuse* vertices is done in the following way; Consult fig.19.

A vertex v is of *type A* iff a horizontal or vertical line through v cuts the interior angle at v into two angles each of them is greater or equal to $\frac{\pi}{4}$. Note that reflex vertices are of type A.

Assume W.L.O.G. that vA is in the eighth octant like fig.19. If vB is in the second octant then vertex v is classified as vertex of *type B*. If vB is in the third octant then v is of *type C*.

For the triangulation of the vertex regions we will assume that a uniform grid of constant size is available around each vertex. Each of the above types of obtuse vertices is classified as case (i-j) where $i, j = 1, 2, 3$ according to the following criterion. Let v be a vertex and l be the ray with origin v which contains one of the incident edges e of v. Assume W.L.O.G. that e_1 is an h (v resp.) edge. Let A be the intersection of l and the second vertical (horizontal resp.) grid line. A can lie on the boundary of cell 1,2, or 3. Then edge e can be classified as case-1, case-2, case-3 respectively. A vertex v is case(i-j) iff one of the incident edges is case-i and the other of case-j.

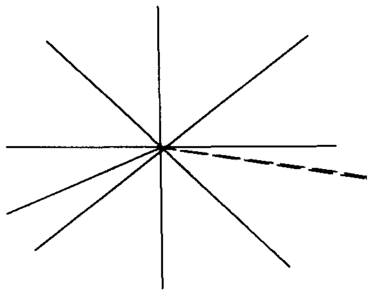

Figure 19: Types of obtuse vertices

Lemma 6.7 *Type A vertices can be nice triangulated.*

Proof: The triangulation of this type is the easiest since we can always bring the perpendicular from O to the second horizontal (vertical resp.) gridline without creating small angles. Consult figure 20. □

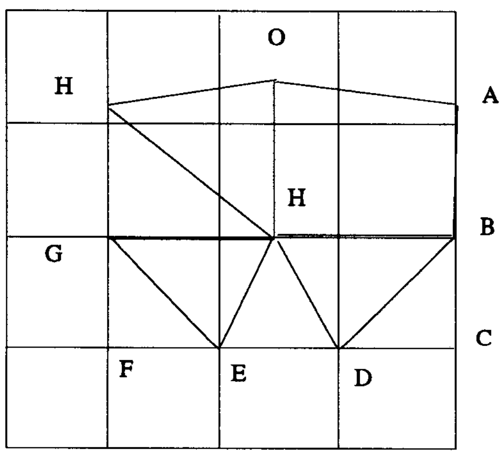

Figure 20: Triangulation of type A vertices

Lemma 6.8 *Type B vertices can be nice triangulated.*

Proof: Consult figure 21. There are the following types of type B vertices:

Case(1-1): Let D a point on CB such that $DO = DA$. If $\angle ODA > \frac{\pi}{2}$ then resolve it by the bisector DM of OA. Then $\triangle ODC, \triangle ODA, \triangle ADB$ are nice. However point D

is not a grid vertex. This inconsistency can be resolved by techniques similar to those in the proof of lemma 7.1.

Case(1-2): If $AB \geq \frac{g}{2}$ then construction is like case(1-1). Otherwise connect D to E, A, O, C. Resolve the probably obtuse angle ODA.

Case (1-3): Not feasible, since that would imply that vertex O is acute.

Case(2-1): Connect D to C, O, A, B.

Case(2-2): Connect D to C, O, A, B.

Case(2-3): Let F and be the perpendicular projections of O on the second and third horizontal gridline below O. Let G be the perpendicular projection of C on ED. Then connect D to F, A, B. Connect G to C, F, E.

Case(3-2): Connect D to C, O. Resolve the obtuse $\angle ODC$. Connect E to O, A, B.

Case(3-3): See Figure. □

Lemma 6.9 *A type C vertex can be nice triangulated.*

Proof: The classification of type C vertex is the same as type B. Similar techniques as in type B can be applied. Thus most of the description is omitted. However we will selectively describe some complicated constructions. Consult figure 22. Consider case (1-2). Connecting simply C to O and B does not always work since $\angle BOC$ may be obtuse. Resolving it using the perpendicular OT from O onto BC also does not work since we need then to resolve Steiner point T. Point T can be resolved temporarily by the perpendicular TF onto BD. Then we have to resolve F and this is the problem. The suggested construction is the following: Let K be the midpoint of CD and let T be the perpendicular projection of O onto BD. If $\angle BOC$ is obtuse then shift point C towards K until either $\angle BOC$ becomes $\frac{\pi}{2}$ or C coincides with K whichever happens first. If point K is reached first, which is the interesting case, let θ be the angle of OC with the horizontal ray through O. Then θ is *non-small*, since the the vertical distance of K from the horizontal ray through O and the distance of O from AK are of the order of the size of the grid cell. However $\angle BOT \geq \theta$ which implies that $\triangle BOT$ is nice. Thus the suggested triangles are $\triangle BOT, \triangle TOK, \triangle TKD, \triangle AOK$. (see fig. 22 case(1-2)).

Case (2-1) is symmetric.

In case (3-1) a similar situation arises. Initially we try to connect O to E. If $\angle EOA$ is obtuse we apply the above mentioned method in the quadrilateral $OECA$ by shifting E towards the midpoint K of EC.

Case (3-3) is infeasible since that would imply vertex angle $\geq \frac{\pi}{2}$.

Case (2-3) If $\angle OED > \frac{\pi}{2}$ then connect O to the midpoint K of BC.

Case(3-2) is symmetric to case (2-3). □.

Thus we get the following:

Theorem 6.10 *Every non-acute vertex region of a polygonal domain P can be nice triangulated using $O(1)$ additional Steiner points per vertex.*

Finally we have :

Theorem 6.11 *Every vertex region, acute or not can be nice triangulated by adding at most $O(1)$ Steiner points.*

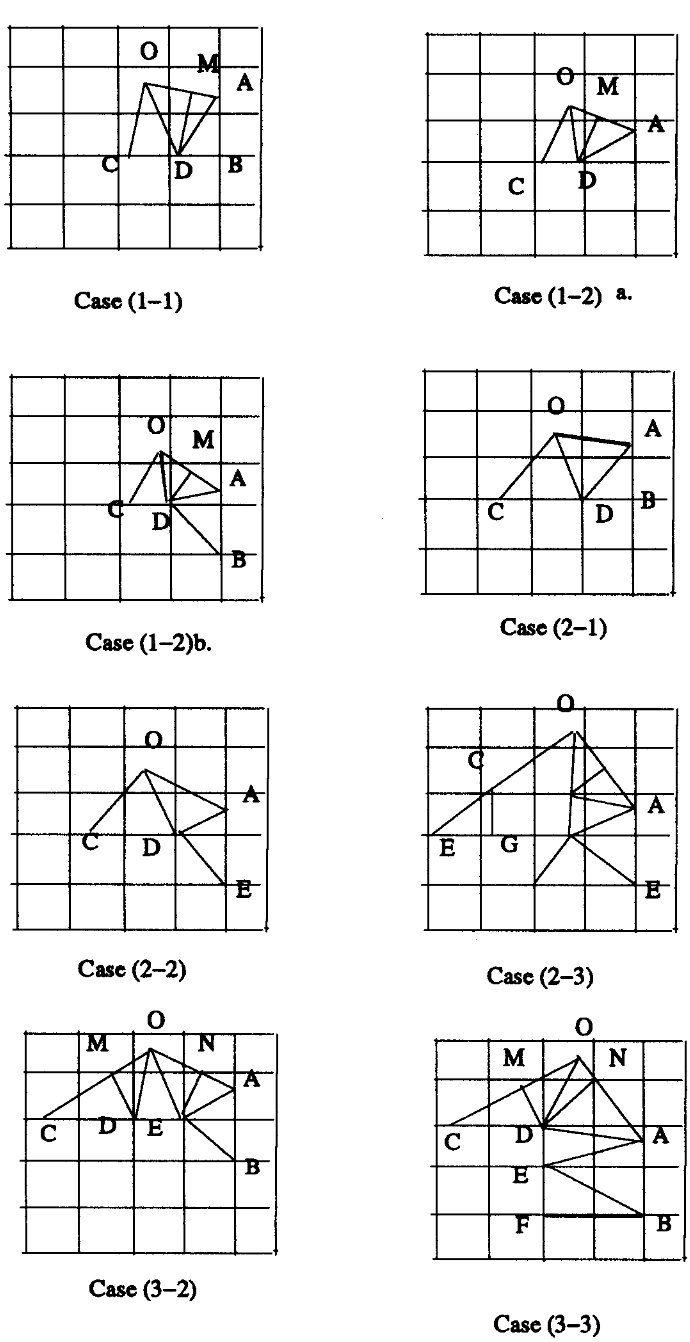

Figure 21: Triangulation of type B vertices

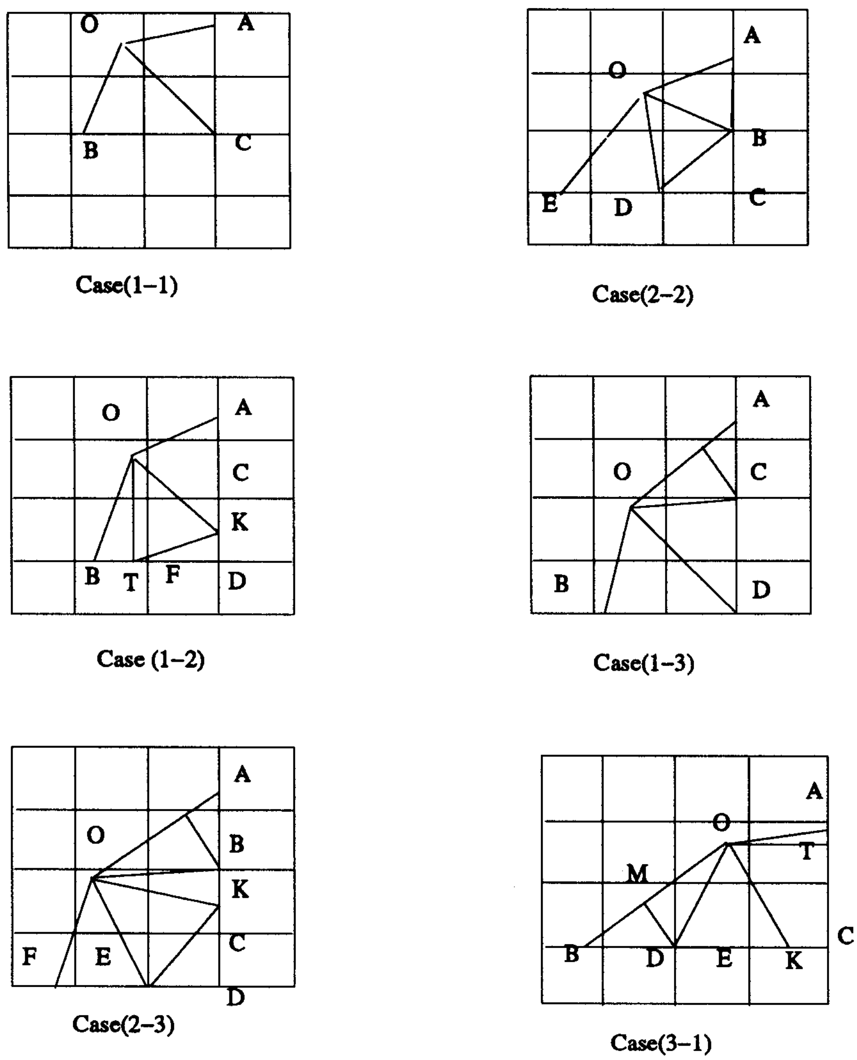

Figure 22: Triangulation of type C vertices

7 Matching the triangulations of different regions (fixing inconsistencies)

So far we have shown how to triangulate independently *near edges* (section 5), near *obtuse* and *acute* vertices (section 6).

However these independent triangulation may lead to inconsistencies.

We have two types of inconsistencies:

a) Inconsistencies between the triangulation of an acute vertex region of P and the triangulation of an non-acute vertex region of P^*.

   Let AB be a cutting edge of an acute vertex O. Assume that the *reduced* polygon P^* as well as the isosceles triangle $\triangle OAB$ have been triangulated. Recall that obtuse vertices A, B of P^* and edge AB have been triangulated using a uniform grid.

   However triangulation of the obtuse vertices A, B of P^* introduces additional Steiner points on the cutting edge AB which do not match with vertices of the triangulation of $\triangle OAB$.

b) Inconsistencies between the triangulation of obtuse vertex regions and triangulation of edges.

   These come from the different way we create a Steiner point when we have a uniform or a non-uniform grid. In the vertex triangulation process we use a uniform grid in contrast to the edge triangulation process.

In the following lemma we shoe how we resolve inconsistencies of type a). Similar techniques are applicable for type b) also.

Lemma 7.1 *The inconsistencies between triangulations of an acute vertex region O of polygon P and the triangulation of the corresponding non-acute vertices A, B of the reduced polygon P^* can be resolved.*

Proof: According to lemma 6.8 and lemma 6.9 there exists at most one Steiner point on Av_2 and at most one on $v_{l-1}B$. We will treat the case of a Steiner point S on $v_{l-1}B$. Similarly the other case can be treated.

a) type 1:

   1.a $\angle QSB \leq \frac{\pi}{2}$ then connect S to R, Q, v_{l-1}, B. See fig. 23.a.

   1.b $\angle QSB > \frac{\pi}{2}$ then the same as 1.a but resolve the obtuse angle. See fig. 23.b.

b) type 2:

   2.a $\angle QSB > \frac{\pi}{2}$ then connect v_{l-1} to T, Q, S. See fig. 23.c.

   2.b $\angle QSB \leq \frac{\pi}{2}$ then let M be a point on QR such that $Mv_{l-1} = MS$. Then connect v_{l-1} to T, M, S and S to M, Q, B. See fig. 23.d. Since point M is not a quadtree vertex creates inconsistency which should be resolved. The appropriate transformations are shown in fig. 23.e, 23.f, 23.g.

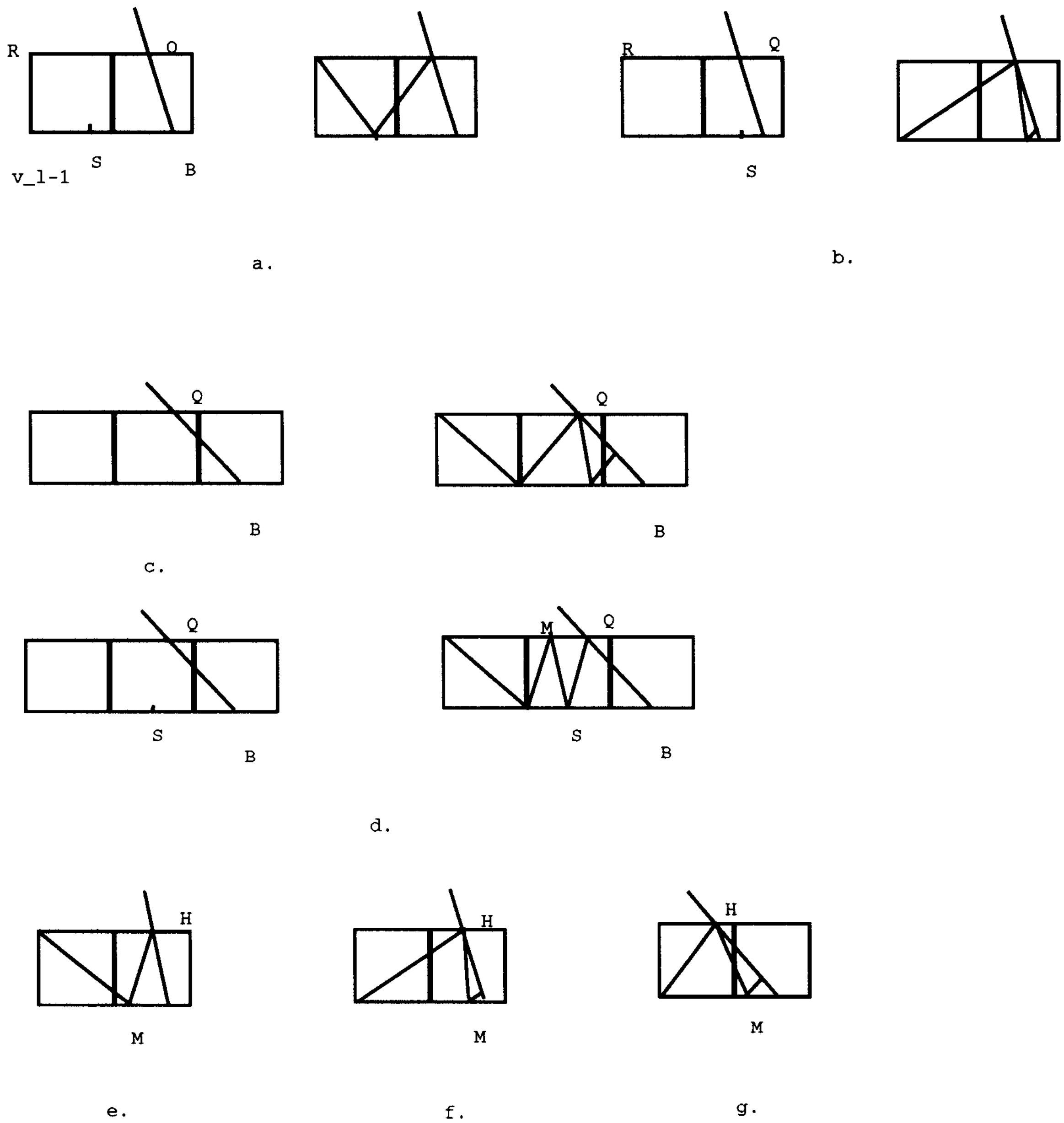

Figure 23: Matching different triangulations

8 Putting Things Together : Size and Quality of the produced Triangular Mesh

In Step a. of the algorithm the size of the quadtree is $O(nA)$ according to [4].

In steps b. and f. the number of additional triangles is $O(n)$ since we can nice triangulate each acute vertex using constant number of triangles.

Triangulating the rectilinear region of every non-acute vertex in step c. requires $O(1)$ additional Steiner points per vertex thus a total of $O(n)$ additional points.

In step d. we add at most $O(1)$ number of Steiner points per quadtree cell which intersects the boundary of the geometry, according to the constructions of section 5.2, thus a total $O(nA)$ additional points.

Resolving inconsistencies in step g. requires $O(1)$ additional Steiner points per vertex region, according to constructions of lemma 7.1.

Thus we get the following:

Theorem 8.1 *The number of triangles required to produce a triangulation of a polygonal domain with holes P, such that every triangle satisfies both the non-small angle and non-obtuse angle condition is $O(nA)$ where n is the size of P and A is the aspect ratio of the constraint Delaunay triangulation of P.*

9 Conclusions

We presented a new algorithm for producing non-obtuse no-small angle triangulations for a polygon with holes. An important open problem is whether a quality triangulation is possible for both the polygon and the holes.

10 Acknowledgements

I would like to thank Mark Overmars of Utrecht University, and Nicolas Patrikalakis of Massachusetts Institute of Technology for reading the manuscript and providing useful comments. I would like also to thank Chee Yap of Courant Institute, Dias Badekas of Rutgers, Vijay Srinivasan and Lee Nackman of IBM Thomas Watson Research Center, Brenda Baker, Eric Grosse, Steve Fortune and Ken Clarkson of AT & T Bell Laboratories for useful discussions on mesh generation.

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