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Abstract

In this paper we give a review of the Plotkin powerdomain construction over algebraic cpo's. Algebraic cpo's are cpo's that are completely determined by their collection of finite elements. We show how one can build an algebraic cpo out of any pre-ordered set using the method of chain completion. We apply this method to define the powerdomain over an algebraic cpo. We then show how to interpret the powerdomain as a subset of the powerset of the base domain together with a suitable ordering relation. This ordering relation is the Plotkin order and it extends the Egli-Milner order. We give a necessary and sufficient condition on the base domain to ensure that the Plotkin order and the Egli-Milner order coincide. We also show how one can construct continuous functions between powerdomains out of functions between the underlying base domains. It follows that the powerdomain construction is a continuous functor on the category of algebraic cpo's.

1 Introduction

This paper arose as the author tried to digest the available literature on powerdomains. While "normal" domain theory is well-understood and has a smooth presentation, the "computational counterpart to powersets" is surprisingly difficult. Nevertheless, the powerdomain construction forms an important mathematical tool in the study of the semantics of non-deterministic and concurrent languages. It is surprising, therefore, that there does not exist a comprehensive and detailed exposition of the theory: as far as we know, apart from the unpublished [Plo81], only the defining papers by Plotkin [Plo79] and Smyth [Smy78], and the expository paper by Gunter and Scott [GS90] are widely available. This paper tries to remedy this situation. It contains an elementary exposition of the theory of (ω-)algebraic cpo's and of how one can construct powerdomains out of them.

For the powerdomain construction we use a technique that we call *chain completion.* It is closely related to ideal completion, but does not need new concepts such as 'ideals'. It is therefore accessible to anyone who is only familiar with basic domain theory. Also, it

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has a very strong underlying intuition. Specifically, in the context of the powerdomain construction, it is "convenient to work with (equivalence classes of) ω-chains rather than directed ideals", as Smyth already noted [Smy83]. While chain completion seems to be usable in most situations that arise in the practice of semantics, it seems not to be widely known. Apart from a brief description in an excercise in [Plo81] and a special case of the construction in [dV90], the author knows of no paper that deals with the general construction. We therefore discuss it at length in this paper. The construction is very powerful, yielding an algebraic cpo out of any pre-ordered set. We also give a brief comparison with the more familiar notion of ideal completion.

Given D, first of all one must define a (countable) pre-order of sets of elements from D. These elements will become finite elements in the chain-completion. It is natural to define these sets as sets of finite elements from D. Also, since this theory only applies to finite non-determinism, the sets should be finite. We then specify an order. A natural choice is the Egli-Milner order: $X \sqsubseteq_{EM} Y$ iff $\forall x \in X \exists y \in Y. x \sqsubseteq y$ and $\forall y \in Y \exists x \in X. x \sqsubseteq y$. We read this as: X approximates Y if everything X can do, Y can do better, and everything Y can do, X yields something that approximates it. We now *define* the powerdomain to be the completion of the resulting structure.

Having defined the powerdomain over some data domain D using chain completion, we then turn to the task of interpreting the abstract construction as a collection of sets of elements from D together with a suitable ordering relation. This proved technically quite involved. Let's look at one chain of finite sets of finite elements from D. It consists of sets X_i so that

$$X_1 \sqsubseteq_{EM} X_2 \sqsubseteq_{EM} \cdots \sqsubseteq_{EM} X_n \sqsubseteq_{EM} \cdots$$

A natural choice is to consider the set of all least upperbounds of chains (in D) that run through these sets. But it is easy to see that we can have equivalent chains with different such upperbound sets. So we have to define a closure operation on the sets. One problem now is that, in general, not every set of elements from D that is closed (with respect to this closure operation) actually arises as such a closure of a least upperbounds set. This implies that we have to work with chains all the time.

The next problem is that, in general, the order on the upperbound sets no longer is the Egli-Milner order, but "becomes" a new order, the so-called Plotkin order, that extends the Egli-Milner order. The Plotkin order has a "difficult" definition. A new result in this paper is the formulation of a precise order-theoretic restriction on the underlying domain D that ensures the Plotkin order to coincide with the Egli-Milner order, as opposed to the topological characterization given by Plotkin [Plo81].

The aim of the present paper is modest: we only want to give a concise introduction to the 'classical' theory of the powerdomain construction. We feel that the paper contains most of the theory needed in the practice of building denotational semantics for non-determinism. More advanced theory on powerdomains, like the algebraic characterisation of powerdomains by Hennessy and Plotkin [HP79] or Heckmann [Hec91], the topological description of powerdomains by Smyth [Smy83], and the logical approach by Winskel [Win85] or Abramsky [Abr91], is not covered. We feel that an elementary exposition of the theory reported in those papers requires a full length paper itself. Furthermore, Vickers [Vic89] has already covered most of this theory from a slightly different perspective in great detail. Likewise, other approaches to the concept of powerdomain, like the categorical notion of Abramsky [Abr90], the 'mixed' powerdomain of Gunter [Gun91], and the game theoretic approach by Moschovakis [Mos91] are not treated. Nonetheless we feel that a thorough understanding of the theory presented in this paper is a prerequisite for an understanding of that work.

Furthermore, there seems to be a general misunderstanding in the literature of how to interpret, or represent, the powerdomain by a collection of subsets of the underlying domain together with a suitable ordering relation. We try to remedy this situation by developing this interpretation in quite some detail.

Apart from its purpose as an introductionary text, the paper contains a few new results. We have given a new order-theoretic description of the Lawson compact cpo's. We also give a negative answer to a question posed by Plotkin [Plo81] whether or not all 'closed' (in a suitable sense) subsets of a ω-algebraic cpo are present in the powerdomain on that cpo.

The paper is organized as follows. In section 2 we give the necessary definitions of complete partial orders and such. In particular, we define the notion of algebraicity. In section 3 we define the method of chain completion and show that it constitutes a functor from the category of pre-orders with monotonic functions, to the category of cpo's with continuous functions. In section 4 we use this completion procedure to define the powerdomain $\mathcal{P}^*(D)$ of an algebraic cpo D. In section 5 we show an isomorphism between a subset of $\mathcal{P}D$ with a suitable ordering relation, and the powerdomain $\mathcal{P}^*(D)$. This section can be seen as giving an interpretation of the (rather abstractly formulated) powerdomain. In section 6 we identify a class of domains for which the Plotkin order and the Egli-Milner order coincide. In section 7 we study some functions related to powerdomains.

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2 Domains

In this section we give a short review of notions relating to cpo's.

Definition 2.1 *1. A set D with a relation* $\sqsubseteq \subseteq D \times D$ *is a* pre-ordered set *iff*

(a) $\forall d \in D . d \sqsubseteq d$ *(reflexivity);*

(b) $\forall d_1, d_2, d_3 \in D . d_1 \sqsubseteq d_2 \sqsubseteq d_3 \Longrightarrow d_1 \sqsubseteq d_2$ *(transitivity)*

2. It is called a partially ordered set *iff moreover* $\forall d_1, d_2 \in D . d_1 \sqsubseteq d_2 \sqsubseteq d_1 \Longrightarrow d_1 = d_2$ *(anti-symmetry).*

We will call the order relation $\sqsubseteq$ a pre-order, or partial order, respectively. Sometimes we call a pre-ordered set (partially ordered set) simply a pre-order (partial order). In case $x \sqsubseteq y$ in a (pre-/partial) order, we say that x *approximates* y. Intuitively, y is "more defined" than x, or has more "information content" (for a discussion regarding these notions, see [Sco76, Sco81]). In case $x \not\sqsubseteq y$ and $y \not\sqsubseteq x$ we say that x and y are *incomparable*. In case $x \sqsubseteq y$ and $y \sqsubseteq x$, then x and y are *isomorphic*, denoted by $x \equiv y$. In case the structure in question is a partial order, we have $x \equiv y$ iff $x = y$.

In the sequel we assume that all pre-orders/partial orders have a least element denoted by $\perp$, that is, $\perp \sqsubseteq x$ for all $x \in D$. Using the above terminology, $\perp$ is the least defined element, *i.e.*, it is undefined or has no information content. In a pre-ordered set all least elements are isomorphic, in a partially ordered set there is a unique least element.

Two pre-ordered/partially ordered sets D and E are *isomorphic* ($D \cong E$) iff there is a bijection $f : D \to E$ such that for all $x, y \in D$, $x \sqsubseteq y$ iff $f(x) \sqsubseteq f(y)$. Two pre-ordered sets X and Y are *equivalent* ($X \simeq Y$) iff there exist monotone functions $f : X \to Y$ and $g : Y \to X$ such that for all $x \in X$ and $y \in Y$, $g(f(x)) \cong x$ and $f(g(y)) \cong y$. Note that isomorphic pre-orders are equivalent.

Given a pre-ordered/partially ordered set D, any collection $\{d_i : i < \omega\}$ such that for all i, $d_i \sqsubseteq d_{i+1}$. is called a *chain*. We denote such a chain as $(d_i)_i$. An element $d \in D$ such that $d_i \sqsubseteq d$ for all i is an *upperbound* for $(d_i)_i$. A *least upperbound* (lub) for a chain $(d_i)_i$ is an upperbound d such that for all other upperbounds d': $d \sqsubseteq d'$. It is easy to see that for two least upperbounds d and d' for some chain, we have $d \equiv d'$ and hence $d = d'$ in a partial order. That is, in a partial order, lubs are unique. The least upperbound for a chain $(d_i)_i$ (if it exists) is written as $\bigsqcup_i d_i$.

Definition 2.2 *1. A pre-ordered set D is called* complete *iff it has least upperbounds for all chains.*

2. A complete partially ordered set (cpo) *D is a partially ordered set D that is complete (as a pre-order).*

An element $d \in D$ is *finite* if for all chains $(d_i)_i$, if $d \sqsubseteq \bigsqcup_i d_i$, then $d \sqsubseteq d_k$ for some k. A finite element d has the property that any chain $(d_i)_i$ with lub d stabilizes. That is, there exists a k such that $d \sqsubseteq d_k \sqsubseteq d$. Hence $d_{k'} \equiv d$ for all $k' \geq k$. We denote the collection of all finite elements of D by $K(D)$. The elements of $D \setminus K(D)$ are called the *infinite* elements in D.

Definition 2.3 *1. A pre-ordered/partially ordered set D is* algebraic *iff for all $d \in D$ there exists a chain $(d_i)_i \subseteq K(D)$ such that $\bigsqcup_i d_i = d$.*

2. An algebraic pre-ordered/partially ordered set D is called ω-algebraic *iff $K(D)$ is countable.*

If D is algebraic, then $K(D)$ is called the *basis* of D. An (ω-)algebraic cpo is also called a *(ω-)domain*. The only difference between an ω-domain and a domain is the cardinality of the basis. Since none of our results depend on the cardinality of the basis, we will use the term 'domain' in both cases. Note that domains have a strong computational intuition, as the denotation of a (recursive) program is defined as the lub of its finite approximations, with each approximation corresponding to a finite depth of the recursion. Hence a denotational semantics will always map programs to elements of a domain.

The ordering relation on a pre-order X induces an ordering relation on chains.

Definition 2.4 *Let X be a pre-order. Let $(x_n)_n$ and $(y_m)_m$ be chains in X. We say that $(x_n)_n$* approximates *$(y_m)_m$, denoted as $(x_n)_n \precsim (y_m)_m$, iff $\forall n \exists m. x_n \sqsubseteq y_m$.*

Likewise we say that a chain $(x_n)_n$ is equivalent to a chain $(y_m)_m$, denoted as $(x_n)_n \sim (y_m)_m$, iff $(x_n)_n \precsim (y_m)_m$ and $(y_m)_m \precsim (x_n)_n$. We note the following simple but important result.

Proposition 2.5 *Let $(d_i)_i$ and $(e_j)_j$ be chains in a domain D such that $(d_i)_i, (e_j)_j \subseteq K(D)$. Then $(d_i)_i \precsim (e_j)_j$ iff $\bigsqcup_i d_i \sqsubseteq \bigsqcup_j e_j$.*

Proof For all i, we have $d_i \sqsubseteq \bigsqcup_j e_j$. Hence $\bigsqcup_i d_i \sqsubseteq \bigsqcup_j e_j$. Conversely, if $\bigsqcup_i d_i \sqsubseteq \bigsqcup_j e_j$, then for all i, $d_i \sqsubseteq \bigsqcup_j e_j$ hence $d_i \sqsubseteq e_j$ for some j, by finiteness of d_i. □

The proposition does not hold for chains in the whole of D: take a chain $(d_i)_i$ with lub $d \neq d_i$ for all i, and consider the constant chain $(d)_j$. Then $\bigsqcup_i d_i = d = \bigsqcup_j d$ but not $(d)_j \precsim (d_i)_i$ hence not $(d)_j \sim (d_i)_i$.

Corollary 2.6 *Let D be algebraic. Then for all $x, y \in D$,*

$$x \sqsubseteq y \text{ iff } \forall a \in K(D). a \sqsubseteq x \text{ implies } a \sqsubseteq y$$

Corollary 2.7 *Let D and E be domains. Then $D \cong E$ iff $K(D) \cong K(E)$.*

2.1 Continuous functions

Next we consider functions that preserve (some of) the structure present in a domain.

Definition 2.8 *Let D and E be domains and $f : D \to E$.*

1. *f is* monotone *iff $d \sqsubseteq d'$ implies $f(d) \sqsubseteq f(d')$ for all $d, d' \in D$.*
2. *f is* continuous *iff it is monotone and commutes with taking least upperbounds, that is, $f(\bigsqcup_i d_i) = \bigsqcup_i f(d_i)$.*

Note that we do *not* insist that functions should preserve the bottom element. Functions that do so are called *strict.*

Now we are able to define the following categories:

1. (ω-)**Dom** is the category of (ω-)domains and continuous functions
2. (ω-)**POrd** is the category of (countable) pre-orders and monotone functions.

Note that (ω-)**Dom** is a subcategory of **POrd**, but not a full subcategory since not every monotone function between between domains is also continuous.

Let D and E be domains. Every monotone function $f : K(D) \to E$ yields a function $\uparrow f : D \to E$ given by

$$\uparrow f(x) = \bigsqcup_{i<\omega} f(x_i)$$

where $(x_i)_i \subseteq K(D)$ is some chain such that $x = \bigsqcup_i x_i$.

Lemma 2.9 *Let D and E be domains. Let $f : K(D) \to E$ be monotone and let $g : D \to E$ be monotone and continuous. Then*

1. *$\uparrow f$ is well-defined and continuous.*
2. *$f = (\uparrow f) \upharpoonright K(D)$.*
3. *$g = \uparrow(g \upharpoonright K(D))$.*

Proof For the first claim, let $(y_i)_i$ be another chain with lub x. Then, by Proposition 2.5, $(y_i)_i \sim (x_i)_i$. Hence $(f(y_i))_i \sim (f(x_i))_i$ by monotonicity of f. Hence $\bigsqcup_i f(y_i) = \bigsqcup_i f(x_i)$. The second claim is obvious. For the third claim, write $\tilde{g} = g \upharpoonright K(D)$. Then

$$g(x) = g\left(\bigsqcup x_i\right) = \bigsqcup g(x_i) = \bigsqcup \tilde{g}(x_i) = \uparrow\tilde{g}(x)$$

□

2.2 Domain constructions

First of all, observe that we can turn each countable set X into a domain $X_\perp$ by adjoining a least element $\perp$ and stipulating that $\perp \sqsubseteq x$ and $x \sqsubseteq x$ for all $x \in X$. Cpo's of this form are called *flat*. Every set-theoretic function $f : X \to Y$ can be extended to a continuous function $f_\perp : X_\perp \to Y_\perp$ by defining $f_\perp(\perp) = \perp$ and $f_\perp(x) = f(x)$ for $x \in X$.

Let D and E be domains. We define the following constructs yielding new domains:

- $D_\perp$ is the 'lifted version' of D. The underlying set is $\{\langle 0, x\rangle : x \in D\} \cup \{\perp\}$. The order is given by $\perp \sqsubseteq \langle 0, x\rangle$ for all $x \in D$, and $\langle 0, x\rangle \sqsubseteq \langle 0, y\rangle$ iff $x \sqsubseteq y$.
- $D \times E$ is the cartesian product of D and E. The underlying set is
  $$\{\langle x, y\rangle : x \in D, y \in E\}$$
  The order is given by $\langle x_1, y_1\rangle \sqsubseteq \langle x_2, y_2\rangle$ iff $x_1 \sqsubseteq x_2$ and $y_1 \sqsubseteq y_2$. Its bottom element is $\langle \perp, \perp\rangle$.
- $D \otimes E$ as a tensor product of D and E. The underlying set is $\{\langle x, y\rangle : x \in D\backslash\{\perp\}, y \in E\backslash\{\perp\}\} \cup \{\perp\}$. For $x, y \in D \otimes E$ we define $x \sqsubseteq y$ iff $x \equiv \perp$ or $x \equiv \langle x_1, y_1\rangle, y \equiv \langle x_2, y_2\rangle$ and $x_1 \sqsubseteq x_2$ and $y_1 \sqsubseteq y_2$.
- Obviously we can generalize these constructions to arbitrary finite products. We will denote these by $\prod$ and $\bigotimes$, respectively.
- $D + E$ is the sum of D and E. The underlying set is
  $$\{\langle 0, x\rangle : x \in D \setminus \{\perp\}\} \cup \{\langle 1, y\rangle : y \in E \setminus \{\perp\}\} \cup \{\perp\}$$
  This is just the counterpart of the disjoint union of ordinary sets. For $x, y \in D + E$, the order is given by $x \sqsubseteq y$ iff $x \equiv \perp$ or $x \equiv \langle i, x'\rangle, y \equiv \langle i, y'\rangle$ and $x' \sqsubseteq y'$ $(i = 0, 1)$. Note that $D + E$ is the coproduct of D and E with respect to strict functions.
- One can generalize $+$ to arbitrary finite sums. We denote this by $D_1 + \cdots + D_n$.

Proposition 2.10 *If D and E are (ω-)domains, then so are $D \times E$, $D \otimes E$, $D + E$ and $D_\perp$.*

3 Chain completion

In this section we investigate how one can construct in a uniform way a domain out of a pre-ordered set. Intuitively, this construction adds (formal) limit points to the chains that exist in the pre-order. Such a situation often occurs in practice: we are capable of inductively defining a set of finite elements with an ordering relation (*e.g.* finite lists, finite trees) and then need to "complete" this set to a domain. We call the construction *chain completion*. Chain completion also extends to mappings: every monotone function between two pre-ordered sets induces a *continuous* function between the completions of those pre-orders. Thus we obtain a functor from (ω-)**POrd** to (ω-)**Dom**.

3.1 Completing pre-orders

Since we want to complete a pre-ordered set to a domain, Proposition 2.5 suggests the following construction. Given a pre-order X, form the collection $C(X)$ of all chains in X.

Lemma 3.1 *$C(X)$ with $\lesssim$ is a pre-ordered set.*

Let $\mathcal{C}X = C(X)/\!\sim$ be the corresponding partial order induced by the pre-order.

Theorem 3.2 *Let X be a (countable) pre-order. Then $\mathcal{C}X$ is a (ω-)domain. Moreover, $K(\mathcal{C}X) \simeq X$.*

Proof First we check all requirements. Obviously, $(\bot)_i$ is the least element and $\lesssim$ is a partial order on the equivalence classes induced by $\sim$. We calculate with representatives of these equivalence classes. Let

$$(x_i^1)_i \lesssim (x_i^2)_i \lesssim \cdots \lesssim (x_i^n)_i \lesssim \cdots$$

be a chain. Define the following collection $\{y_n : n < \omega\}$ inductively:

- $y_1 = x_1^1$;
- $y_{n+1} = x_k^{n+1}$ where k is the least index such that $y_n \sqsubseteq x_k^{n+1}$ and for all $m \leq n+1$ and $i \leq n+1$, we have that $x_i^m \sqsubseteq x_k^{n+1}$.

By assumption, this collection $\{y_n : n < \omega\}$ is well-defined and has the following properties.

1. $\{y_n : n < \omega\}$ is a chain;
2. for all n there exists a $k \geq n$ such that $y_n = x_k^n$;
3. for all n and k, we have that $x_k^n \sqsubseteq y_{\max\{n,k\}}$.

We call $(y_n)_n$ the *diagonal* of the chain $((x_i^n)_i)_n$ and sometimes denote it by $\bigvee_n (x_i^n)_i$.

We have to check that the construction of this diagonal is a congruence with respect to the equivalence relation $\sim$. So let

$$(z_i^1)_i \lesssim (z_i^2)_i \lesssim \cdots (z_i^n)_i \lesssim \cdots$$

be another chain such that $(x_i^n)_i \sim (z_i^n)_i$ for all n. Let $(y'_n)_n$ be the diagonal of this chain. We have

$$y'_n = z_k^n \sqsubseteq x_l^n \sqsubseteq y_{\max\{n,l\}}$$

for some $l \geq k$. Hence $(y_i)_i \sim (y'_n)_n$.

$(y_n)_n$ is a least upperbound for the chain: for all n and i, $x_i^n \sqsubseteq y_{\max\{n,i\}}$. This shows that $(y_n)_n$ is an upperbound. Let $(z_n)_n$ be another upperbound. Then, for all n, $y_n = x_k^n \sqsubseteq z_m$ for some m. Hence $(y_n)_n$ is least.

Let $(a)_i$ be a constant chain for some $a \in X$ such that $(a)_i \precsim (y_n)_n$. Then $a \sqsubseteq y_n$ for some n. Hence $(a)_i \precsim (x_i^n)_i$. Hence, for all $a \in X$, $(a)_i$ is a finite element. Moreover, each chain $(x_i)_i$ is the lub of the chain of chains $(x_1)_i \precsim (x_2)_i \precsim \cdots$ Let $(e_i)_i$ be another finite element. Then $(e_i)_i \sim \bigvee_n (e_n)_i$. Hence $(e_i)_i \precsim (e_k)_i$ for some k. Conversely, $(e_k)_i \precsim (e_i)_i$. Hence all finite element are equivalent to a constant chain $(a)_i$ for some $a \in X$. This shows that $\mathcal{C}X$ is a domain.

If X is countable, we have (at most) countably many of such chains $(a)_i$. Hence $\mathcal{C}X$ is an ω-domain. □

There is an interesting subcase in the proof of the preceding theorem, namely when for all n, $x_i^n \sqsubseteq x_i^{n+1}$ for all i. In this case the diagonal is indeed $(x_n^n)_n$. Moreover, it is not hard to see that any chain (with respect to $\precsim$) in $\mathcal{C}X$ can be brought into this form. It is interesting to see what happens if we apply the construction to a domain D, that is, form the structure $\mathcal{C}D$. In this case, let $(x_i)_i$ be a chain of finite elements with (infinite) lub x. Then $(x_i)_i \precsim (x)_i$ but not conversely. So we lose isomorphism between D and $\mathcal{C}D$ since all infinite elements $x \in D$ appear, in a sense, twice in $\mathcal{C}D$: once as an infinite element $(x_i)_i$ and once as a finite (!) element $(x)_i$.

Corollary 3.3 *Let X, Y be pre-ordered sets. Then $\mathcal{C}X \cong \mathcal{C}Y$ iff $X \simeq Y$.*

3.2 Completing monotone functions

In the previous section we saw that one can map the objects of (ω)-**POrd** onto the objects of (ω)-**Dom**. In this section we look at the arrows in the categories and show that each monotone function $f : X \to Y$ induces a continuous function $\mathcal{C}f : \mathcal{C}X \to \mathcal{C}Y$.

We can generalize Proposition 2.9 as follows. Let X and Y be pre-ordered sets and let $f : X \to Y$ be monotonic. Then f can be extended to $\mathcal{C}f : \mathcal{C}X \to \mathcal{C}Y$ given by

$$\mathcal{C}f([(x_n)_n)]) = [(f(x_n))_n]$$

where $[(y_m)_m]$ denotes the equivalence class of the chain $(y_m)_m$ with respect to $\sim$. Below we will work with representatives of these equivalence classes.

Lemma 3.4 *$\mathcal{C}f$ is well-defined, monotone and continuous.*

Proof By monotonicity of f, $\mathcal{C}f$ is clearly monotonic and well-defined. $\mathcal{C}f$ is also continuous: Let $((x_i^n)_i)_n$ be a chain in $\mathcal{C}X$, that is, $(x_i^n)_i \precsim (x_i^{n+1})_i$ for all n. Let $\bigvee_n((x_i^n)_i)_n = (y_n)_n$. By definition, $\mathcal{C}f(\bigvee_n((x_i^n)_i)_n) = (f(y_n))_n$.

On the other hand, there exists a chain $(\mathcal{C}f((x_i^n)_i))_n = ((f(x_i^n))_i)_n$ in $\mathcal{C}Y$ by monotonicity of f. This chain gives rise to the diagonal $\bigvee_n(\mathcal{C}f((x_i^n)_i))_n = (y'_n)_n$.

We have to prove that $(f(y_n))_n \sim (y'_n)_n$. Consider $y_n = x_k^n$ for some k. Then $f(y_n) = f(x_k^n)$ is an element of the chain $(f(x_i^n))_i$. Hence $f(y_n) \sqsubseteq y'_{\max\{n,k\}}$. Hence $(f(y_n))_n \lesssim (y'_n)_n$. The converse is proved similarly. □

Theorem 3.5 *$\mathcal{C}$: (ω-)***POrd** *→ (ω-)***Dom** *is a functor.*

Proof Well-definedness of $\mathcal{C}$ follows from lemma 3.4, and it is easy to see that $\mathcal{C}1 = 1$ and $\mathcal{C}(f \circ g) = \mathcal{C}f \circ \mathcal{C}g$. □

We can ask whether K can be extended to a functor. Ideally, the resulting functor $\mathcal{K}$ should be in some sense "inverse" to $\mathcal{C}$. That is, we want $\mathcal{C}$ and $\mathcal{K}$ to form an adjunction [Lan71]. This is impossible, for take $f : D \to E$ as $f(x) = e$ for some $e \notin K(E)$. Some further reflection shows that this essentially the only case that goes wrong. A continuous function f is called *finitely continuous* if $f(K(D)) \subseteq K(E)$. Let (ω-)**FDom** be the category of (ω-)domains with finitely continuous functions. Then $\mathcal{K}$: (ω-)**FDom** → (ω-)**POrd** given by $\mathcal{K}(D) = K(D)$ and $\mathcal{K}(f) = f \restriction K(D)$ is a functor. In this case, we have that $\mathcal{C} \circ \mathcal{K} \cong 1$ and $\mathcal{K} \circ \mathcal{C}$ corresponds to dividing out the equivalence induced by the pre-order.

Proposition 3.6 *The functor $\mathcal{K}$: (ω-)***FDom** *→ (ω-)***POrd** *is right adjoint to $\mathcal{C}$.*

Proof We already have observed the bijection of hom-sets

$$\frac{X \to \mathcal{K}D}{\overline{\mathcal{C}X \to D}}$$

for any pre-ordered set X and domain D. We leave it as an exercise to show that this bijection is *natural:*

$$\phi(f \circ h) = \phi(f) \circ \mathcal{C}h \qquad \phi(\mathcal{K}k \circ f) = k \circ \phi(f)$$

for all monotone functions $f : X \to \mathcal{K}D$ and all arrows $k : D \to D'$ and $h : X' \to X$, where ϕ denotes the bijection. □

In the following corollary, (ω-)**Ord** denotes the category of (countable) partially ordered sets and monotone functions.

Corollary 3.7 *The adjunction $\mathcal{C} \dashv \mathcal{K}$ restricts to an equivalence of categories [Lan71] (ω-)***Ord** *$\simeq$ (ω-)***FDom***.*

3.3 Ideal completion

In this section a short introduction is given to a completion method that is used frequently in the literature. It is based on a slightly different notion of complete partially ordered set.

Definition 3.8 *Let D be a partially ordered set.*

- *A subset $S \subseteq D$ is* directed *if $\forall a, b \in S \exists c \in S. a \sqsubseteq c \wedge b \sqsubseteq c$ and $S \neq \emptyset$.*
- *D is a* directed complete partially ordered set (dcpo) *if D has lubs of all directed subsets.*

For example, every chain in a cpo is a directed set. The next proposition shows that the difference between a cpo and a dcpo is only a matter of cardinality.

Lemma 3.9 *Let S be a countable directed subset of some partially ordered set D. Then S contains a chain $(a_n)_n$ such that $\bigsqcup S = \bigsqcup_n a_n$ (if both lubs exist).*

Proof Let $S \subseteq D$ be countable and directed. Let $s_0, s_1, s_2, \ldots$ be a enumeration without repetitions of S. Consider the following collection

- $a_1 \sqsupseteq s_0, s_1$
- $a_{n+1} \sqsupseteq a_n, s_{n+1}$

Since S is directed, we may choose some a_n for all n such that $a_n \in S$ and the conditions are satisfied. By construction, $(a_n)_n$ forms a chain and it is easy to see that $\bigsqcup_n a_n = \bigsqcup S$ and one exists iff the other does. □

It follows from the lemma that directed subsets are a natural generalization of chains and indeed they seem to be more convenient to work with, at least on a theoretical level. On the level of constructing a semantics for a programming language, however, chains will be the things one encounters mostly. From this point of view the the following definitons are the natural generalizations of the previous ones. An element a in a dcpo D is called *finite* if whenever $a \sqsubseteq \bigsqcup S$ for a directed set S, then there exists an element $s \in S$ such that $a \sqsubseteq s$. The collection of finite elements of D is denoted by $K(D)$. A dcpo D is called *algebraic* if for every $d \in D$ the collection of finite elements below d is a directed set with lub d. D is called ω-algebraic if moreover $K(D)$ is countable.

Proposition 3.10 *Let D be a partially ordered set. Then D is an ω-algebraic dcpo iff D is an ω-algebraic cpo.*

Proof In order to avoid confusion, we write $K^*(D)$ for the finite elements in a dcpo and $K(D)$ for the the finite elements in a cpo, the difference being definitional.

Let D be an ω-algebraic dcpo. Then D is obviously a cpo and $K^*(D) \subseteq K(D)$. That is, every element that is finite according to the dcpo definition is also finite according to the cpo definition. For $d \in D$, write $F(d) \subseteq K^*(D)$ for the collection of finite elements below d. For a directed set S, write $F(S)$ for $\bigcup\{F(a) : a \in S\}$. It is easy to show that $F(S)$ is also directed and $\bigsqcup S = \bigsqcup F(S)$. Moreover, $F(S)$ is countable. Let $a \in K(D)$. Let S be directed such that $a \sqsubseteq \bigsqcup S$. Then $a \sqsubseteq \bigsqcup F(S)$ and hence $a \sqsubseteq \bigsqcup_n a_n$ for the chain $(a_n)_n \subseteq K^*(D) \subseteq K(D)$ as given by Lemma 3.9. By assumption, $a \sqsubseteq a_n$ for some n and hence $a \in K^*(D)$. This shows that $K^*(D) = K(D)$ which therefore is countable.

Let $d \in D$. Then $d = \bigsqcup F(d)$. Since $F(d)$ is countable we have by Lemma 3.9 a chain $(a_n)_n \subseteq K(D)$ such that $d = \bigsqcup_n a_n$. This shows that D is algebraic with basis $K^*(D)$.

Conversely, assume that D is an ω-algebraic cpo. First we show that D is a dcpo as well. Let $S \subseteq D$ be directed. Let $F(S) \subseteq K(D)$ be the set of finite elements below some $s \in S$. We show that $F(S)$ is a directed set. First of all, since S in non-empty, $F(S)$ is non-empty. Let $a, d \in F(S)$. Then there are elements $s_1, s_2 \in S$ such that $a \sqsubseteq s_1$ and $b \sqsubseteq s_2$. Since S is directed, there exists an element s_3 such that $s_1, s_2 \sqsubseteq s_3$. Let $(e_n)_n \subseteq K(D)$ be a chain with lub s_3. Then for some m, $a, b \sqsubseteq e_m$ and $e_m \in F(S)$. $F(S)$ is countable and hence there exists a chain $(a_n)_n$ such that $\bigsqcup_n a_n = \bigsqcup F(S) = \bigsqcup S$. Hence D is a dcpo. Analogously to the previous case we can show that $K^*(D) = K(D)$. Finally, let $d \in D$. Then $F(d)$ is directed and $d = \bigsqcup F(d)$. Hence each $d \in D$ is the lub of the directed set of finite elements below it. □

A notion closely related to directed sets, is that of an *ideal*. For a pre-ordered set X, a subset $I \subseteq X$ is an ideal if it is a downwards closed directed set. Downwards closed means that if $x \in I$ and $y \sqsubseteq x$ then $y \in I$ for all $x, y \in X$. The *ideal completion* $\mathcal{I}X$ of a pre-ordered set X is defined as follows: the underlying set of $\mathcal{I}X$ is the collection of all ideals of X, and the order is $\subseteq$.

Proposition 3.11 *$\mathcal{I}X$ is an algebraic dcpo.*

Proof First of all, $\mathcal{I}X$ is a dcpo: The least element of $\mathcal{I}X$ is $\{\bot\}$ and the lub of a directed set S of ideals is given by $\bigcup S$. Next, for $x \in X$, write

$$\downarrow x = \{y \in X : y \sqsubseteq x\}$$

for the prinipal ideal generated by x. Then $\downarrow x$ is a finite element in $\mathcal{I}X$ and for any ideal I, I is the lub of the directed set $\{\downarrow x : x \in I\}$. □

Corollary 3.12 *Let X be a countable pre-ordered set. Then $\mathcal{I}X \cong \mathcal{C}X$.*

Proof First, by the assumption on X and Proposition 3.11, $\mathcal{I}X$ is an ω-algebraic dcpo and hence an ω-algebraic cpo by Proposition 3.10. Next, for every $x, y \in X$, $x \sqsubseteq y$ iff $\downarrow x \subseteq \downarrow y$. Hence there exists an order preserving bijection between the basis of $\mathcal{I}X$ and the basis of $\mathcal{C}X$. The claim now follows from Corollary 2.7. □

The reason why we prefer to work with chain completion instead of ideal completion will become apparent in the next section: chain completion seems to give a firmer grip on least upperbounds than ideal completion does. Nevertheless, as the proof of Proposition 3.11 indicates, for most theoretical matters ideal completion is easier to handle than chain completion.

4 The Powerdomain Construction

Let D be a ω-algebraic cpo and $K(D)$ its countable set of finite elements. We want to construct a domain $\mathcal{P}^*(D)$ that is the "computational" powerset of D.

First of all, we need to determine the elements of the powerdomain $\mathcal{P}^*(D)$. We go back to the original intuition for the set of outcomes of a finitely nondeterministic program (*c.f.* [Plo79, Smy78]). In these programs all choices that can be made are between finitely many alternatives. The possible executions of a program, then, can be seen as a finitely branching, possibly infinite tree. The actual runs of the program are the paths in this tree. Every branching point in the tree corresponds to a nondeterministic choice made by the program.

We can label the tree with the finite approximations to the final outcomes: each branching point is labeled with the finite element computed along the unique path to that point. The collection of outcomes of the program is the set of least upperbounds of chains of finite elements that label the paths in the tree. Thus we are led to the following definition. The finite elements of the powerdomain correspond to cross-sections of a computation tree and hence the basis of the powerdomain $\mathcal{P}^*(D)$ is the collection $\mathcal{F}(D)$ of finite, nonempty subsets of finite elements from D. If D is ω-algebraic, then $\mathcal{F}(D)$ is countable. Note that different computation trees may have the same cross-sections. There is, however, no a priori reason why the computation trees should be operationally equivalent. The powerdomain cannot model the differences and hence may indeed identify two operationally non-equivalent programs. We will come back to this later.

Next, we need a notion of approximation between these sets of elements from D. Since we have "destroyed" the information of how the elements in some finite set $X \subseteq K(D)$ were obtained, we can only refer to the elements in X. Intuitively, we can imagine three such orders.

1. The Hoare order: which has the reading *everything X can do, Y can do better.*

$$X \sqsubseteq_H Y \text{ iff } \forall x \in X \exists y \in Y . x \sqsubseteq y$$

2. The Smyth order: which reads *everything Y can do can be approximated by X.*

$$X \sqsubseteq_S Y \text{ iff } \forall y \in Y \exists x \in X . x \sqsubseteq y$$

3. The Egli-Milner order which consists of the conjunction of the previous two:

$$X \sqsubseteq_{EM} Y \text{ iff } X \sqsubseteq_H Y \text{ and } X \sqsubseteq_S Y$$

Obviously, these are only pre-orders. In the sequel we restrict attention to the Egli-Milner order since this seems to be the most natural choice. The powerdomain $\mathcal{P}^*(D)$ is defined as the chain completion of the pre-ordered set $(\mathcal{F}(X), \sqsubseteq_{EM})$. More precisely, it is the *Plotkin powerdomain* [Plo79]. The name seems a bit confusing since the ordering on the finite elements in $\mathcal{P}^*(D)$ is called "Egli-Milner". We come back to this later.

In order to make $\mathcal{P}^*$ into a functor on (ω-)**Alg**, we need to define its action on continuous functions. That is, for a continuous function $f : D \to E$, we want to define a continuous function $\mathcal{P}^*f : \mathcal{P}^*(D) \to \mathcal{P}^*(E)$. The theory in section 3.2 shows that it is sufficient to define a monotone function $\hat{f} : \mathcal{F}(D) \to \mathcal{P}^*(E)$. We proceed as follows: Let $X \in \mathcal{F}(D)$. Consider the set

$$F_X = \{f(x) : x \in X\}$$

which is essentially the desired image of X under $\mathcal{P}^*f$. This is a finite set since X is finite, hence we may write it as

$$F_X = \{e_1, \ldots, e_n\}$$

F_X need not be a finite set of *finite* elements however, thus we have to be careful about how we define $\mathcal{P}^*f$ which must map chains in $\mathcal{F}(D)$ to chains in $\mathcal{F}(E)$. Observe that, as E is a domain, for every $e_i \in F_X$ there is a chain $(e^i_k)_k$ in $K(E)$ with lub e_i. Now define

$$\hat{f}(X) = (F^k_X)_k \equiv (\{e^i_k : 0 < i \leq n\})_k$$

Obviously, for every k, F^k_X is a finite set of finite elements. Furthermore, $F^k_X \sqsubseteq_{EM} F^{k+1}_X$ for all k. So $(F^k_X)_k$ indeed forms a chain in $\mathcal{F}(E)$. Also, $F^k_X \sqsubseteq_{EM} F_X$ for all k. Now define $\mathcal{P}^*f : \mathcal{P}^*(D) \to \mathcal{P}^*(E)$ as $\uparrow\hat{f}$, that is,

$$\mathcal{P}^*f((X_i)_i) = \bigvee_{i<\omega} \hat{f}(X_i)$$

Lemma 4.1 *For any continuous function $f : D \to E$, $\hat{f} : \mathcal{F}(D) \to \mathcal{P}^*(E)$ is well-defined and monotone.*

Proof This follows immediately from Lemma 5.3 below. □

Theorem 4.2 *Let D, E be domains. Let $f : D \to E$ be monotone and continuous. Then $\mathcal{P}^*f : \mathcal{P}^*(D) \to \mathcal{P}^*(E)$ is monotone and continuous.*

Proof Immediate from Lemma 4.1 and Theorem 3.5. □

Theorem 4.3 *$\mathcal{P}^*(\cdot)$ is an endofunctor on (ω-)***Alg***.*

Proof It is easy to check that $\mathcal{P}^*(1_D) = 1_{\mathcal{P}^*(D)}$ and $\mathcal{P}^*(f \circ g) = \mathcal{P}^*f \circ \mathcal{P}^*g$. □

5 Interpreting the construction

At this point the question remains: how should one interpret the equivalence classes of chains that arise in the above construction? We would like to view them as sets of elements from D. This section is devoted to establishing exactly which sets arise in the construction. More formally, we want to define a subset of $\mathcal{P}(D)$, the full powerset of D, with a suitable partial order such that this structure is isomorphic to $(\mathcal{P}^*(D), \precsim)$.

If a chain $(X_i)_i$ becomes stable, that is, $X_k \equiv X_{k+1}$ for all k larger than some N, the answer is provided by the next lemma. In the lemma, *Con* is the convex closure operation:

$$Con(X) = \{y : \exists x_1, x_2 \in X. x_1 \sqsubseteq y \sqsubseteq x_2\}$$

Lemma 5.1 *For all $X, Y \in \mathcal{P}(D)$,*

1. *$X \sqsubseteq_{EM} Y$ iff $Con(X) \sqsubseteq_{EM} Con(Y)$.*

2. $X \equiv_{EM} Y$ *iff* $Con(X) = Con(Y)$.

From this lemma it follows that the finite elements of $\mathcal{P}^*(D)$ are in a one-to-one correspondence with the convex closures of finite sets of finite elements from D. Note, however, that these convex closures themselves need not be finite, as the following example shows.

Example 5.2 *Consider the ordinal* $2\omega + 1$, *ordered in the usual way,*

$$0 \leq 1 \leq 2 \leq \cdots \leq \omega \leq \omega + 1 \leq \cdots \leq 2\omega$$

The only infinite elements are ω *and* 2ω. *One easily shows*

$$Con(\{1, \omega + 1\}) = \{1, 2, \ldots, \omega, \omega + 1\}$$

Hence this convex closure not only has an infinite number of elements, but contains an infinite element as well.

The situation is less clear, however, when a chain does not become stable. Chains with this property correspond to the infinite elements in the powerdomain. Given a chain $(X_i)_i$, intuitively this chain arises as the cross-sections of some infinite tree. Hence we define the following collection for a chain $(X_i)_i \subseteq \mathcal{F}(D)$

$$Up(X_i)_i = \{\bigsqcup_i x_i : x_i \in X_i \ \&\ (x_i)_i \text{ chain}\}$$

The next proposition shows that we are on the right track.

Lemma 5.3 *Let* $(X_i)_i \subseteq \mathcal{F}(D)$ *be a chain. Then for all* $A \in \mathcal{F}(D)$,

$$A \sqsubseteq_{EM} Up(X_i)_i \textit{ iff for some } i\ A \sqsubseteq_{EM} X_i$$

Proof Assume that $A \sqsubseteq_{EM} Up(X_i)_i$. Let $a \in A$. Then there is a $\bigsqcup_i x_i \in Up(X_i)_i$ such that $a \sqsubseteq \bigsqcup_i x_i$. Hence, by finiteness of a, there is an n such that $a \sqsubseteq x_n \in X_n$. For each $a \in A$, let m_a be the minimum of such n's. Since there are only finitely many $a \in A$, there is a maximum m for the set of all those m_a. Hence $A \sqsubseteq_H X_m$.

We now show that $A \sqsubseteq_S X_{m'}$ for some $m' \geq m$. Assume towards a contradiction that such m' does not exist. This means that for all $n > m$, there exists a $x_n \in X_n$ such that $a \not\sqsubseteq x_n$ for all $a \in A$. Then for each such $x_n \in X_n$, there exists a finite chain

$$c_m \sqsubseteq c_{m+1} \sqsubseteq \cdots \sqsubseteq x_n$$

such that $a \not\sqsubseteq c_k$ for all $a \in A$ and $c_k \in X_k$ for all $m \leq k < n$. Since each X_k is finite, we get a finitely branching, infinite tree, and hence, by König's lemma, an infinite path

$$d_n \sqsubseteq d_{n+1} \sqsubseteq \cdots \sqsubseteq d_{n+i} \sqsubseteq \cdots$$

exists in this tree. This path determines the element $\bigsqcup_i d_{n+i} \in Up(X_i)_i$. Since by assumption $A \sqsubseteq_{EM} Up(X_i)_i$, there exists an $a \in A$ such that $a \sqsubseteq \bigsqcup_i d_{n+i}$. Hence $a \sqsubseteq d_{n+i'}$ for some i'. Contradiction.

The other direction is trivial. □

Proposition 5.4 *Let* $(X_i)_i, (Y_k)_k \subseteq \mathcal{F}(D)$ *be chains. Then*

$$(X_i)_i \precsim (Y_k)_k \text{ iff } \forall A \in \mathcal{F}(D)[A \sqsubseteq_{EM} Up(X_i)_i \Longrightarrow A \sqsubseteq_{EM} Up(Y_k)_k]$$

Proof Immediate from Lemma 5.3. □

We now define another pre-order on $\mathcal{P}(D)$. For $X, Y \in \mathcal{P}(D)$, let $X \trianglelefteq Y$ iff

- for all $y \in Y$, there exists some $x \in X$ such that $x \sqsubseteq y$, and
- for all chains $(a_i)_i \subseteq K(D)$, if $\bigsqcup_i a_i \in X$, then for all i there exists some $y \in Y$ such that $a_i \sqsubseteq y$.

$\trianglelefteq$ clearly is a pre-order. We denote the induced equivalence relation by $\triangleq$.

Lemma 5.5 *Let* $(X_i)_i, (Y_k)_k \subseteq \mathcal{F}(D)$ *be chains. Then*

$$(X_i)_i \precsim (Y_k)_k \text{ iff } up(X_i)_i \trianglelefteq Up(Y_k)_k$$

Proof Let $(X_i)_i \precsim (Y_k)_k$. Let $y \in Up(Y_k)_k$. Then, for all i, there is some element $x_i \in X_i$ such that $x_i \sqsubseteq y$. By König's lemma, there is an element $x \in Up(X_i)_i$ such that $x \sqsubseteq y$. Next, let $(a_n)_n \subseteq K(D)$ be a chain such that $\bigsqcup_n a_n \in Up(X_i)_i$. Then, for each n, there exists an element $x_i \in X_i$ for some i, such that $a_n \sqsubseteq x_i$, by finiteness of a_n. Hence for some k there exists an element $y_k \in Y_k$ such that $a_n \sqsubseteq y_k$ by definition of $\precsim$. Hence there is an element $y \in Up(Y_k)_k$ such that $a_n \sqsubseteq y$.

Conversely, let $A \in \mathcal{F}(D)$ such that $A \sqsubseteq_{EM} Up(X_i)_i$. We want to prove that $A \sqsubseteq_{EM} Up(Y_k)_k$ which is sufficient in view of Lemma 5.4. Let $a \in A$. Then there is some $x \in Up(X_i)_i$ such that $a \sqsubseteq x$. Hence there exists an element $x_i \in X_i$ for some i, such that $a \sqsubseteq x_i$. Hence there is some $y \in Up(Y_k)_k$ such that $a \sqsubseteq x_i \sqsubseteq y$, by assumption. Let $y \in Up(Y_k)_k$. Then there is $x \in Up(X_i)_i$ such that $x \sqsubseteq y$. Hence there is $a \in A$ such that $a \sqsubseteq x \sqsubseteq y$. Hence $A \sqsubseteq_{EM} Up(Y_k)_k$. □

Obviously one can give chains $(X_i)_i$ and $(Y_k)_k$ such that $(X_i)_i \sim (Y_k)_k$ but $Up(X_i)_i \neq Up(Y_k)_k$. We therefore seek a closure operator Cl such that $(X_i)_i \sim (Y_k)_k$ iff $Cl(Up(X_i)_i) = Cl(Up(Y_k)_k)$. The following example shows that the convex closure of such sets does not work.

Example 5.6 *Let A be a (countable) alphabet and define $\bar{A} = \{\bar{a} : a \in A\}$. Let $D = (A \cup \bar{A})^+ \cup (A \cup \bar{A})^\omega \cup (A \cup \bar{A})^* \cdot \{\bot\}$. We define the following order on D. First define an order on $A \cup \bar{A}$ by putting $a \sqsubseteq a$, $\bar{a} \sqsubseteq \bar{a}$ and $a \sqsubseteq \bar{a}$ for all $a \in A$. Then define, for $x, y \in D$, $x \sqsubseteq y$ iff*

- $|x| = |y|$ *and* $(x)_i \sqsubseteq (y)_i$ *for all* $0 < i \leq |x|$, *or*
- $x = x'\bot$ *and* $y = y'y''$ *such that* $|x'| = |y'|$ *and* $(x')_i \sqsubseteq (y')_i$ *for all* $0 < i \leq |x'|$.

Here $|x|$ denotes the number of symbols in x and $(x)_i$ the i^{th} symbol in x. It is easy to see that D with the given order is a domain. Consider the following sets, for some $a \in A$,

$$X_n = \{\bar{a}, \ldots, \bar{a}^n, a^n\bot\}$$

$$Y_n = \{\bar{a}, \ldots, \bar{a}^n, \bar{a}^n\bot, a^n\bot\}$$

Then $X_n \sqsubseteq_{EM} X_{n+1}$ and $Y_n \sqsubseteq_{EM} Y_{n+1}$ for all n. We have

$$Con(Up(X_n)_n) = \{\bar{a}\}^* \cup \{a^\omega\}$$

$$Con(Up(Y_n)_n) = \{\bar{a}\}^* \cup \{\bar{a}^\omega, a^2\bar{a}^\omega, a^3\bar{a}^\omega, \ldots, a^\omega\}$$

But for all n, $X_n \sqsubseteq_{EM} Y_n$ and $Y_n \sqsubseteq_{EM} X_{n+1}$. Hence $(X_n)_n \sim (Y_n)_n$ but $Con(Up(X_n)_n) \neq Con(Up(Y_n)_n)$.

Plotkin [Plo79] defined a closure operator $(\cdot)^*$ based on topological considerations. Later on, he defined a closure operator more directly in terms of the *Up*-sets [Plo81]. The following definition is a slight variant of this closure operation. It is derived from Lemma 5.5.

Definition 5.7 *Let $X \in \mathcal{P}(D)$. Then*

$$Cl(X) = \{\bigsqcup y_i : \exists x \in X. x \sqsubseteq \bigsqcup y_i \ \&\ \forall i \exists x_i \in X. y_i \sqsubseteq x_i\}$$

where $(y_i)_i \subseteq K(D)$.

Obviously, $Con(X) \subseteq Cl(X)$ for all $X \in \mathcal{P}(D)$. Also, $Con(X) = Cl(X)$ for all finite $X \in \mathcal{P}(D)$. Note that *Cl* does not enjoy all properties of *Con*. In particular, Lemma 5.1 cannot be adapted: $X \sqsubseteq_{EM} Y$ does not necessarily imply $Cl(X) \sqsubseteq_{EM} Cl(Y)$.

In the sequel the closure operator *Cl* is only used in conjunction with *Up*. It is therefore natural to ask if both operators can be merged into one. The following proposition, due to Frank Nordemann, shows this can be done. The proof is straightforward and therefore omitted.

Proposition 5.8 *Let D be ω-algebraic. Let $(X_i)_i$ be a chain in $\mathcal{F}(D)$. Define Cl' by*

$$Cl'((X_i)_i) = \{\bigsqcup_i y_i : \forall i \exists j \geq i. y_i \in Con(X_i \cup X_j)\}$$

Then $Cl(Up(X_i)_i) = Cl'((X_i)_i)$.

We first give a few technical lemmas.

Lemma 5.9 *For all $X, Y \in \mathcal{P}(D)$,*

1. *$X \subseteq Cl(X)$;*
2. *$X \subseteq Y$ implies $Cl(X) \subseteq Cl(Y)$;*
3. *$Cl(Cl(X)) = Cl(X)$.*

Proof

1. $X \subseteq Con(X) \subseteq Cl(X)$.

2. Obvious.

3. Let $x \equiv \bigsqcup_i x_i \in Cl(Cl(X))$, all x_i finite. Then there is a $y \in Cl(X)$ such that $y \sqsubseteq x$. Hence there is a $y' \in X$ such that $y' \sqsubseteq y \sqsubseteq x$. For all i there exists a $y_i \in Cl(X)$ such that $x_i \sqsubseteq y_i$. Let $y_i = \bigsqcup_k y_{ik}$, all y_{ik} finite. Then $x_i \sqsubseteq y_{ik'}$ for some k'. Hence there exists a $y' \in X$ such that $x_i \sqsubseteq y'$. Hence $x \in Cl(X)$. The converse inclusion is trivial. □

Cl is a (set theoretic) closure operator that extends *Con*, but differs on the infinite sets. It also follows from the lemma that in order to prove $Cl(X) \subseteq Cl(Y)$, it is sufficient to prove that $X \subseteq Cl(Y)$.

Lemma 5.10 *For all $X, Y \in \mathcal{P}(D)$,*

1. $X \triangleq Cl(X)$.
2. $Cl(X) \triangleq Cl(Y)$ *iff* $Cl(X) = Cl(Y)$.

Proof

1. Immediate from the definition of *Cl* and $\trianglelefteq$.
2. Assume $Cl(X) \triangleq Cl(Y)$. Let $x \in Cl(X)$. Then there is a $y \in Cl(Y)$ such that $y \sqsubseteq x$. Let $(x_i)_i \subseteq K(D)$ be such that $\bigsqcup_i x_i = x$. Then for all i, there exists some $y_i \in Cl(Y)$ such that $x_i \sqsubseteq y_i$. Hence $x \in Cl(Y)$ from which it follows that $Cl(X) \subseteq Cl(Y)$. Likewise, $Cl(Y) \subseteq Cl(X)$. The other direction is trivial. □

Theorem 5.11 *Let $(X_i)_i, (Y_k)_k \subseteq \mathcal{F}(D)$ be chains. Then*

$$(X_i)_i \sim (Y_k)_k \text{ iff } Cl(Up(X_i)_i) = Cl(Up(Y_k)_k)$$

Proof

$$\begin{aligned}(X_i)_i \sim (Y_k)_k \quad &\text{iff} \quad Up(X_i)_i \triangleq Up(Y_k)_k \\ &\text{iff} \quad Cl(Up(X_i)_i) \triangleq Cl(Up(Y_k)_k) \\ &\text{iff} \quad Cl(Up(X_i)_i) = Cl(Up(Y_k)_k)\end{aligned}$$

□

Summarizing, we arrive at the following characterization of the powerdomain over D:

- its elements are the sets $Cl(Up(X_i)_i)$ where $(X_i)_i \subseteq \mathcal{F}(D)$ is a chain;
- these sets are ordered by $\trianglelefteq$;
- least upperbounds are given by $\bigsqcup_{n<\omega} Cl(Up(X_i^n)_i)_n = Cl(Up(\bigvee_{n<\omega}((X_i^n)_i)_n))$.

We still use $\mathcal{P}^*(D)$ to denote this structure.

There is another characterization of the order $\trianglelefteq$. First, define the domain $\mathbf{O} = \{\bot, \top\}$ ordered by $\bot \sqsubseteq \top$. Then consider the collection $[D \to \mathbf{O}]$ of all continuous functions from D to $\mathbf{O}$. Such a function can be viewed as an *experiment* on D [Plo79, Plo81]. Define the following pre-order on $\mathcal{P}(D)$

$$X \sqsubseteq_P Y \text{ iff } \forall f \in [D \to \mathbf{O}].f(X) \sqsubseteq_{EM} f(Y)$$

where $f(X)$ denotes the direct image of X under f. $\sqsubseteq_P$ is the so-called *Plotkin order*.

We are going to relate $\sqsubseteq_P$ to $\trianglelefteq$. First we need a lemma.

Lemma 5.12 *For all $X, Y \in \mathcal{P}(\mathbf{O})$, $X \sqsubseteq_{EM} Y$ iff*

- *$\bot \in Y$ implies $\bot \in X$;*
- *$\top \in X$ implies $\top \in Y$.*

Proposition 5.13 *Let $X, Y \in \mathcal{P}(D)$. Then $X \sqsubseteq_P Y$ iff $X \trianglelefteq Y$.*

Proof

($\Longleftarrow$) Let $f : D \to \mathbf{O}$. Then $\bot \in f(Y)$ implies $\exists y \in Y.f(y) = \bot$. Hence $\exists x \in X.f(x) = \bot$ by monotonicity of f. $\top \in f(X)$ implies $\exists x \in X.f(x) = \top$. Hence $f(x_i) = \top$ for some finite $x_i \sqsubseteq x$. Hence $\exists y \in Y.f(y) = \top$.

($\Longrightarrow$) Let $y \in Y$. Define $f : D \to \mathbf{O}$ by $f(d) = \top$ iff $d \not\sqsubseteq y$. f is clearly monotone and continuous. Then $\bot \in f(Y)$, hence $\bot \in f(X)$ which means that there exists some $x \in X$ such that $x \sqsubseteq y$. Let $x \in X$ and assume $x = \bigsqcup_i x_i$. Consider, for any i, the step function $(x_i, \top)$ which is defined as

$$(x_i, \top)(d) = \begin{cases} \top & \text{if } x_i \sqsubseteq d \\ \bot & \text{otherwise} \end{cases}$$

Then, for all i, $\top \in (x_i, \top)(X)$. Hence, for all i, $\top \in (x_i, \top)(Y)$ which means $\forall i \exists y \in Y.x_i \sqsubseteq y$. □

Corollary 5.14 *For all $X, Y \in \mathcal{P}(D)$, $X \equiv_P Y$ iff $Cl(X) = Cl(Y)$.*

This gives us the next description of the powerdomain over D: it is the collection of sets $Cl(Up(X_i)_i)$ ordered by $\sqsubseteq_P$.

Remarks

- The first question one might ask is whether or not the collection of all *closed* subsets of $\mathcal{P}(D)$ equals the collection of sets in $\mathcal{P}^*(D)$. The answer is "no" as we now show. Consider a countable alphabet A and define A^{str}, the streamset over A, as

$$A^{str} = A^+ \cup A^\omega \cup A^* \cdot \{\bot\}$$

where $\perp \notin A$ is a new special symbol. Here, as usual, A^+ denotes the collection of all finite sequences and A^ω the collection of all infinite sequences over A.

We order A^{str} by $x \sqsubseteq y$ iff $x = y$, or $x \equiv x'\perp$ and $y \equiv x'y'$ for some $y' \in A^{str}$.

Obviously A^{str} is a domain. It differs slightly from the well-known domain of finite and infinite sequences over A, ordered by the prefix ordering. This difference turns out to be crucial for this example.

Now consider the sets

$$X = \{a\}^+ \qquad Y = \{a\}^+ \cup \{a^\omega\}$$

for some $a \in A$. Then we have $Cl(X) = X$ and $Cl(Y) = Y$. However, X can never arise as $Up(X_i)_i$ for $X_i \in \mathcal{F}(D)$; it then should contain a^ω as a simple application of König's lemma shows. That is, Y is the lub of such a chain. Note that we do have $Y \sqsubseteq_P X$, but not $X \sqsubseteq_P Y$. Note also that this example does not contradict [Plo81], since X is not Scott-compact.

- Our second remark uses the streamset A^{str} as the domain for the denotational semantics of a simple programming language. Given the alphabet A and a countable set P of procedure variables or names, the statements of the language $\mathcal{L}$ are inductively defined as

$$s ::= a \mid s;s \mid s+s \mid x$$

where $a \in A$ and $x \in P$. ; denotes sequential composition and + denotes nondeterministic choice. A declaration d is a mapping $d : P \to \mathcal{L}$.

The denotational semantics $[\![\cdot]\!] : \mathcal{L} \to A^{str}$ is straightforward and the precise definition is omitted.

Now consider the declaration $d(x) = (x;a) + b$. In establishing $[\![x]\!]$ we need to compute the lub of the following chain

$$X_0 = \{\perp\}, X_1 = \{b, \perp\}, X_2 = \{b, ab, \perp\}$$

$$X_{n+1} = \{b, ab, \ldots, a^n b, \perp\}, \ldots$$

We then have $Up(X_i)_i = \{b\} \cdot \{a\}^* \cup \{\perp\}$ and we see that $ba^\omega \in Cl(Up(X_i)_i)$. But ba^ω is an infinite (diverging) stream that does not correspond to any possible computation of x. Hence the denotational semantics $[\![\cdot]\!]$ is not correct with respect to our intuitive operational model.

Actually, this is quite a serious situation. Even for the simple language $\mathcal{L}$, a straightforward denotational semantics fails to be correct. Analyzing what goes awry, we see that the problem is essentially caused by the dual role of $\perp$: it denotes both an "unspecified", or "least informative" element and as such the starting point for the iteration by which we determine the value of a recursive procedure, and at the same time it denotes "divergence" which should be some maximal element. In the context of deterministic programs, this blurring of both roles is quite harmless. Here it is not.

One way out is by insisting that a declaration such as $d(x)$ above is not well-formed: we require that every recursive call to a procedure is preceded by (at least one) real action. This requirement is called *guarded recursion*, and is adopted in most process algebras (*c.f.* [dBK90]). In the context of concurrent logic languages, at least, this requirement comes quite naturally: before we can make a recursive call we have to unify an atom with the head of a clause, and this unification step delivers guarded recursion.

In the context of parallel imperative languages, however, we see no reason why we should have guarded recursion: x seems perfectly reasonable defined. The way out here seems to be to make the procedure call visible in the denotation. For instance, we could define our semantics such that we would obtain

$$X_0 = \{\perp\}, X_1 = \{b, \pi\perp\}, X_2 = \{b, \pi ab, \pi\pi\perp\}, \ldots$$

where $\pi \notin A$ denotes the "act of calling". This option has some intuitive justification since in a real computer we have to make an incarnation record before can execute a (new) instance of a procedure.

Another way out consists of defining an altogether new notion of powerdomains using multisets, as advocated by Abramsky [Abr90]. His notion of powerdomain is capable of describing unguarded recursion. It is moreover able to capture *unbounded nondeterminism* where a choice-construct may choose between infinitely many alternatives. Unbounded nondeterminism is very hard to deal with in the setting of domains (see [AP86] and references therein).

Next we study the classes of continuous functions on powerdomains. For the rest of this section, fix some continuous function $f : D \to E$. We need a few technical lemmas.

Lemma 5.15 *For all $X \subseteq D$, $Cl\{f(x) : x \in Cl(X)\} = Cl\{f(x) : x \in X\}$.*

Proof Let $\bigsqcup_i y_i = y \in LHS$ where all y_i are finite. Then there is an $x \in Cl(X)$ such that $f(x) \sqsubseteq y$. Also there is an $x' \in X$ such that $x' \sqsubseteq x$. Hence $f(x') \sqsubseteq y$. Furthermore, there are $x_i \in Cl(X)$ such that $y_i \sqsubseteq f(x_i)$. Let $x_i = \bigsqcup_k x_{ik}$. Then $y_i \sqsubseteq f(x_{ik'})$ for some k'. Now, for all i and k, there are $x'_{ik} \in X$ such that $x_{ik} \sqsubseteq x'_{ik}$. Hence $y_i \sqsubseteq f(x_{ik'}) \sqsubseteq f(x'_{ik'})$. Hence $y \in RHS$. The converse inclusion is trivial. □

Lemma 5.16 *Let $Y \subseteq D$ be finite. Then $X \sqsubseteq_P Y$ iff $X \sqsubseteq_{EM} Y$.*

Proof It is straightforward to show that $X \sqsubseteq_{EM} Y$ implies $X \sqsubseteq_P Y$. For the other direction, we have $\forall y \in Y \exists x \in X. x \sqsubseteq y$. For the second conjunct, let $x = \bigsqcup_i x_i \in X$. We have, for all i, there is some $y_i \in Y$ such that $x_i \sqsubseteq y_i$. Since Y is finite, infinitely many of those y_i are the same. Hence there is a $y \in Y$ such that $x_i \sqsubseteq y$ for all i, or $x \sqsubseteq y$. □

Lemma 5.17 *Suppose $\{(X_i^k)_i : k < \omega\}$ is a collection of chains such that $(X_i^k)_i \precsim (X_i^{k+1})_i$ for all k. Suppose furthermore that $Up(X_i^k)_i$ is finite for all k. Then*

$$Cl(Up \bigvee (X_i^k)_i) = Cl(\{\bigsqcup a_k : a_k \in Cl(Up(X_i^k)_i)\})$$

Proof By Lemma 5.16, $Up(X_i^k)_i \sqsubseteq_{EM} Up(X_i^{k+1})_i$ for all k. From this it follows that $Cl(\{\bigsqcup a_k : a_k \in Cl(Up(X_i^k)_i)\}) = Cl(\{\bigsqcup a_k : a_k \in Up(X_i^k)_i\})$. Assume w.l.o.g. that for all i and k, $X_i^k \sqsubseteq_{EM} X_i^{k+1}$. We prove that $\{\bigsqcup a_k : a_k \in Up(X_i^k)_i\} \equiv_P Up\bigvee_k(X_i^k)_i$ from which the lemma follows.

We first prove that $\{\bigsqcup a_k : a_k \in Up(X_i^k)_i\} \sqsubseteq_P Up\bigvee_k(X_i^k)_i$. Assume $a \in Up\bigvee_k(X_i^k)_i$, where $a = \bigsqcup a_k$ with $a_k \in X_k^k$. For each k and $i \leq k$, the set $Y_i^k = \{x \in X_i^k : x \sqsubseteq a_k\}$ is non-empty. By König's lemma we can choose for each k a chain $(y_i^k)_i$ where $y_i^k \in X_i^k$ such that $\bigsqcup y_i^k \sqsubseteq a$. Since by assumption every set $Up(X_i^k)_i$ is finite, we can apply König's lemma again to find a chain $(y_k)_k$ with $y_k \in Up(X_i^k)_i$ such that $\bigsqcup y_k \sqsubseteq a$. Let $(b_k)_k$ be a chain with lub b such that $b_k \in Up(X_i^k)_i$. Suppose furthermore that $b = \bigsqcup y_n$ where $(y_n)_n \subseteq K(D)$. Then for all n, $y_n \sqsubseteq b_k$ for some k. Hence $y_n \sqsubseteq x$ for some $x \in X_i^k$ for some i. Hence $y_n \sqsubseteq x'$ for some $x' \in X_m^m$ where $m = \max\{k, i\}$. From this it follows that there exists some element $y \in \bigvee_k(X_i^k)_i$ such that $y_n \sqsubseteq y$.

We now prove that $Up\bigvee_k(X_i^k)_i \sqsubseteq_P \{\bigsqcup a_k : a_k \in Up(X_i^k)_i\}$. Let $(a_k)_k$ be a chain with lub a such that $a_k \in Up(X_i^k)_i$. Each a_k determines an element $x_k \in X_k^k$. By König's lemma, there exists an element $x \in Up(X_k^k)_k$ such that $x \sqsubseteq a$. Let $(b_k)_k$ be a chain with lub b such that $b_k \in X_k^k$. If $(z_n)_n \subseteq K(D)$ is a chain such that $\bigsqcup z_n = b$, then for each n, $z_n \sqsubseteq b_k$ for some k. Hence $z_n \sqsubseteq x$ for some $x \in Up(X_i^k)_i$. Hence there exists an element $y \in \{\bigsqcup a_k : a_k \in Up(X_i^k)_i\}$ such that $z_n \sqsubseteq y$. □

Define $\tilde{f} : \mathcal{P}(D) \to \mathcal{P}(E)$ by

$$\tilde{f}(X) = Cl\{f(x) : x \in X\}$$

We are going to show that, for every f, $\tilde{f}$ is indeed the desired extension of f to the powerdomain.

Lemma 5.18 *$\tilde{f}(X) = Cl(Up(\hat{f}(X))$ for all $X \in \mathcal{F}(D)$, where $\hat{f}$ is defined in section 4.*

Proof In the notation of section 4, $\tilde{f}(X) = Cl(F_X)$. Furthermore, $Up(\hat{f}(X)) = F_X$ since X is finite. □

Theorem 5.19 *$\tilde{f}(Cl(Up(X_i)_i)) = Cl(Up(\mathcal{P}^*f((X_i)_i)))$ for all chains $(X_i)_i \subseteq \mathcal{F}(D)$.*

Proof By Lemmas 5.15, 5.17 and 5.18, we have

$$\begin{aligned}
\tilde{f}(Cl(Up(X_i)_i)) &= Cl\{f(x) : x \in Up(X_i)_i\} \\
Cl(Up(\mathcal{P}^*f((X_i)_i)) &= Cl(Up(\bigvee \hat{f}(X_i))) \\
&= Cl\{\bigsqcup e_i : e_i \in Cl(Up(\hat{f}(X_i)))\} \\
&= Cl\{\bigsqcup e_i : e_i \in \tilde{f}(X_i)\}
\end{aligned}$$

Let $(x_i)_i$ be a chain with $x_i \in X_i$. Then $(f(x_i))_i$ is a chain with $f(x_i) \in \tilde{f}(X_i)$. Hence

$$\{f(x) : x \in Up(X_i)_i\} \subseteq Cl\{\bigsqcup e_i : e_i \in \tilde{f}(X_i)\}$$

Let, for all i, $y_i \in \tilde{f}(X_i)$ such that $(y_i)_i$ is a chain. Then each y_i determines the set

$$X_i' = \{x \in X_i : f(x) \sqsubseteq y_i\}$$

For all i, X_i' is non-empty. Hence, by König's lemma, we find a chain $(x_i)_i$ through $(X_i)_i$ such that $\bigsqcup_i f(x_i) = f(\bigsqcup_i x_i) \sqsubseteq \bigsqcup_i y_i$. Since X_i is a finite set, $y_i \sqsubseteq f(x)$ for some $x \in X_i$. Hence $y_i \sqsubseteq f(x')$ for some $x' \in Up(X_i)_i$. Hence

$$\{\bigsqcup e_i : e_i \in \tilde{f}(X_i)\} \subseteq Cl\{f(x) : x \in Up(X_i)_i\}$$

□

6 Compatible domains

It is quite surprising that the Egli-Milner order 'becomes' the Plotkin order on infinite sets. The purpose of this section is to explore some of their relationships. In particular, we want to establish under what conditions (on a domain D) the Egli-Milner order and the Plotkin order coincide. Example 6.1 appears to provide a prime case for the difference between both orders.

Example 6.1 *Let D be a cpo given by the following data*

- *the elements are* $\{\bot\} \cup \{(a,n) : n \leq \omega\} \cup \{(b,n) : n \leq \omega\} \cup \{(\top,\bar{n}) : n < \omega\}$*;*
- *the order is given by* $(a,n) \sqsubseteq (a,m)$ *iff* $n \leq m$*;* $(b,n) \sqsubseteq (b,m)$ *iff* $n \leq m$*;* $(a,n),(b,n) \sqsubseteq (\top,\bar{m})$ *iff* $n \leq m$ *and* $\bot \sqsubseteq x$ *for all* $x \in D$*.*

See figure 1.

Consider the following sets

$$X_i = \{\bot, (a,0), \ldots, (a,i)\}$$
$$Y_i = \{(b,i), (\top,\bar{0}), \ldots, (\top,\bar{i})\}$$

We have $X_i \sqsubseteq_{EM} Y_i$ *for all* i*. Hence* $(X_i)_i \precsim (Y_i)_i$*. But*

$$Cl(Up(X_i)_i) = \{\bot, (a,0), \ldots, (a,\omega)\} \not\sqsubseteq_{EM}$$

$$Cl(Up(Y_i)_i) = \{(b,\omega), (\top,\bar{0}), \ldots, (\top,\bar{n}), \ldots\}$$

We readily see what goes wrong in the previous example: the element a^ω does not approximate anything in $Cl(Up(Y_i)_i)$. Therefore we impose the following restriction on the underlying domain D. It effectively states that a situation as depicted in the example cannot occur.

Definition 6.2 *Let D be a (ω-)domain.*

1. *Let* $(x_i)_i, (y_i)_i \subseteq K(D)$ *be chains with lubs* x *and* y*, respectively. Assume that* $\{z_i : i < \omega\}$ *is a collection of elements such that for all* i*,* $x_i, y_i \sqsubseteq z_i$*. Then* $(x_i)_i$ *and* $(y_i)_i$ *are* compatible *iff there is a chain* $(c_i)_i \subseteq K(D)$ *(with lub* c*) such that* $\forall i \exists k \geq i. c_i \sqsubseteq z_k$ *and* $x, y \sqsubseteq c$*.*
2. *D is* compatible *if all chains in* $K(D)$ *are pairwise compatible.*

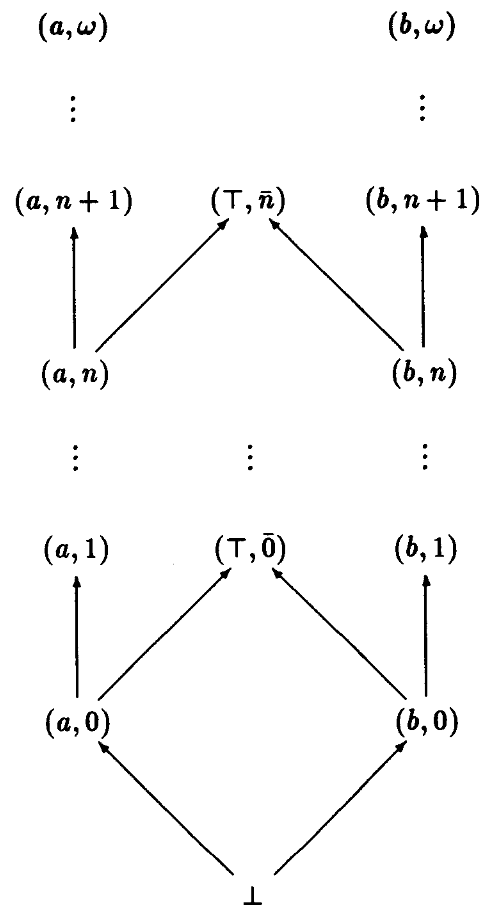


Figure 1: A non-compatible domain

For some examples of compatible domains:

- all flat domains are compatible;
- if D has all (binary) joins, then D is compatible. Hence all (ω-algebraic) (semi-) lattices are compatible.
- if D has bounded (binary) joins, then D is compatible. By bounded binary join we mean that if there is z such that $x, y \sqsubseteq z$ then $x \vee y$ exists. From this it follows that all Scott-domains (ω-algebraic bounded complete cpo's) are compatible.

We now give another description of compatible ω-domains. We need to introduce some notation and a lemma.

For $X \subseteq D$, let $mub(X)$ denote the set of minimal upperbounds of X. That is, elements d such that $x \sqsubseteq d$ for all $x \in X$ and if d' is any upperbound of X such that moreover $d' \sqsubseteq d$, then $d' = d$. We say that $mub(X)$ is *complete* if for any upperbound d' of X there is a $d \in mub(X)$ such that $d \sqsubseteq d'$.

Definition 6.3 *Let D be a ω-domain. Then*

1. *D has property M if for each finite $A \subseteq K(D)$, $mub(A)$ is finite and complete.*
2. *D is SFP if D has property M, and moreover for each finite $A \subseteq K(D)$, the set $U^\infty(A)$ is finite, where*

$$\begin{aligned} U^0(A) &= A \\ U^{n+1}(A) &= mub(U^n(A)) \\ U^\infty(A) &= \bigcup_{n<\omega} U^n(A) \end{aligned}$$

Note that if a, b are finite, and d is a minimal upperbound for $\{a, b\}$, then d is finite too: Let $d = \bigsqcup_i d_i$ for some chain $(d_i)_i \subseteq K(D)$, then $a, b \sqsubseteq d_i$ for some i, hence $d = d_i \in K(D)$. Note also that D has property M iff for all $\{a, b\}$ where $a \neq b \in K(D)$, $mub(\{a, b\})$ is finite and complete.

We need a lemma concerning greatest lower bounds.

Lemma 6.4 *Let D be a compatible ω-domain. Let $\{x_i : i \leq \omega\}$ be an anti-chain, that is, $x_i \sqsupseteq x_{i+1}$ for all i. Then $\bigwedge_i x_i$, the greatest lowerbound for the anti-chain, exists.*

Proof Consider the set

$$X = \{a \in K(D) : \forall i. a \sqsubseteq x_i\}$$

Since $\bot \in X$, $X \neq \emptyset$. Let $a_0, a_1, a_2, \ldots$ be an enumeration of X. First we claim that X is directed. Let $a, b \in X$ and consider the constant chains $(a)_i$ and $(b)_i$. Let the i^{th} occurrence of a and b be majorated by x_i. Since D is assumed to be compatible, there exists a chain $(c_i)_i$ with lub c such that $a, b \sqsubseteq c$ and for all n, $c_n \sqsubseteq x_i$ for all i. Hence $c \sqsubseteq x_i$ for all i. Since a and b are finite, there exists an n such that $a, b \sqsubseteq c_n$. Now $c_n \in X$. Hence X is directed.

By Proposition 3.10, X has a lub a. We claim that a is the desired glb. Obviously, $a \sqsubseteq x_i$ for all i and hence is a lowerbound. Let $b = \bigsqcup_k b_k$ be another lowerbound. Since, for all k, $b_k \sqsubseteq x_i$ for all i, $b_k \in X$. Hence $b_k \sqsubseteq a$ and $b \sqsubseteq a$. □

Proposition 6.5 *Let D be ω-algebraic. Then D is compatible iff D has property M.*

Proof Assuming that D has property M, an easy application of König's lemma shows that D is compatible. Conversely, let $a, b \in K(D)$ and let $a, b \sqsubseteq x$ for some $x \in D$. Consider the set

$$C = \{c \in K(D) : a, b \sqsubseteq c \sqsubseteq x\}$$

By finiteness of a, b, $C \neq \emptyset$. Let $c_0, c_1, c_2, \ldots$ be an enumeration of C.

Let $f : \mathbf{N} \to \mathbf{N}$ be the following increasing function: $f(0) = 0$ and $f(n+1)$ is the minimal $m > f(n)$ such that $c_m \sqsubseteq c_{f(n)}$, if such an m exists, and $f(n+1) = f(n)$ otherwise.

Then $c = \bigwedge_n c_{f(n)}$ exists, by Lemma 6.4, and $a, b \sqsubseteq c \sqsubseteq x$. We have to show that c is minimal. Suppose that there exists a (finite) c' such that $a, b \sqsubseteq c' \sqsubseteq c$. Then $c' \in C$ and $c' \sqsubseteq c_{f(n)}$ for all n. Since $c' \in C$, $c' \equiv c_m$ for some m. By the construction of f, we must have $m = f(n)$ for some n. Hence $c \sqsubseteq c'$ which means $c = c'$ This shows that c is minimal. Hence $mub(\{a, b\})$ is complete.

Assume by way of contradiction that $mub(\{a, b\})$ is infinite. Let $d_0, d_1, d_2, \ldots$ be an enumeration *without repetitions* of $mub(\{a, b\})$. By compatibility of D, there exists a chain $(c_i)_i$ with lub c such that $a, b \sqsubseteq c$ and $\forall i \exists k \geq i. c_i \sqsubseteq d_k$. By minimality of the d_k, $\forall i \exists k \geq i. c_i = d_k$. Fix some i and let $k \geq i$ be such that $c_i = d_k$. Then for $i' > k$,

$$d_k = c_i \sqsubseteq c_{i'} = d_{k'}$$

for some $k' \geq i' > k$. By minimality of $d_{k'}$, $d_k = d_{k'}$. Contradiction. □

Corollary 6.6 *If D is SFP, then*

1. *D is compatible.*
2. *$\mathcal{P}^*(D)$ is compatible.*

The importance of this Corollary lies in the fact that the category **SFP** is closed under all (standard) domain constructions, and that one can solve reflexive domain equations involving these contructors in **SFP** [SP82]. This means in particular that the solution of such an equation involving the powerdomain constructor, may use the Egli-Milner order.

Remark From the 2/3 *SFP* Theorem [Plo81], it follows that compatible ω-domains are precisely the Lawson-compact ω-domains. Note, however, that the latter notion is topologically defined. Here we have only used familiar order-theoretic notions.

From Example 6.1 it is clear that compatibility is a necessary condition on D for the order to be $\sqsubseteq_{EM}$: if a domain D is not compatible then there exist chains $(x_i)_i$ and $(y_i)_i$ such that for all i, $x_i, y_i \sqsubseteq z_i$ for some z_i but no chain $(c_i)_i$ with the required properties. Hence we can build a chain in $\mathcal{F}(D)$ just like in Example 6.1, that gives the required contradiction. That the condition is sufficient is the content of the next theorem.

Theorem 6.7 *Let D be compatible. Let $(X_i)_i$ and $(Y_k)_k$ be chains in $\mathcal{F}(D)$. Then*

$$(X_i)_i \precsim (Y_k)_k \text{ iff } Cl(Up(X_i)_i) \sqsubseteq_{EM} Cl(Up(Y_k)_k)$$

Proof Assume $(X_i)_i \precsim (Y_k)_k$. By Proposition 5.14 we know $Cl(Up(X_i)_i) \sqsubseteq_P Cl(Up(Y_k)_k)$ hence

$$\forall y \in Cl(Up(Y_k)_k) \exists x \in Cl(Up(X_i)_i). x \sqsubseteq y$$

To prove the other conjunct, assume, without loss of generality, that $X_i \sqsubseteq_{EM} Y_i$ for all i. Let $x \in Cl(Up(X_i)_i)$ and assume $x = \bigsqcup_n a_n$ for some chain $(a_n)_n \subseteq K(D)$. Assume $(a_n)_n$ does not stabilize. For each a_n, there is some $x_n \in Up(X_i)_i$ such that $a_n \sqsubseteq x_n$. Hence for each n, there is a $x'_{k_n} \in X_{k_n}$ such that $a_n \sqsubseteq x'_{k_n}$ for some minimal k_n. Each x'_{k_n} gives rise to a set $Y'_{k_n} \subseteq Y_{k_n}$:

$$Y'_{k_n} = \{y \in Y_{k_n} : x'_{k_n} \sqsubseteq y\}$$

By assumption, all Y'_{k_n} are non-empty. Hence they determine a finitely branching infinite tree through $(Y_k)_k$. By König's lemma, there exists an infinite chain $(y_i)_i$, with lub y. Note that x and y may be incomparable. However we have that for all i, there exists a $y'_k \in Y'_k$ for some $k \geq i$ such that $y_i \sqsubseteq y'_k$; also, $a_n \sqsubseteq y'_k$ by definition of the sets Y'_k. Hence, by assumption, there exists a chain $(c_i)_i$ such that $x, y \sqsubseteq c \equiv \bigsqcup_i c_i$ and, for all i, $c_i \sqsubseteq y'_{k'}$ for some k'. Hence there exists a $y' \in Up(Y_k)_k$ such that $c_i \sqsubseteq y'$. Hence $c \in Cl(Up(Y_k)_k)$. Hence

$$\forall x \in Cl(Up(X_i)_i) \exists y \in Cl(Up(Y_k)_k). x \sqsubseteq y$$

The other direction is immediate from Lemma 5.5. □

We know that one can form a domain from the closures of upperbound sets, but we do not yet fully know what the least upperbounds in this domain are. The following theorem shows that they are exactly like we would expect them to be, at least when D is compatible.

Theorem 6.8 *Let D be compatible. Let, for all n, $(X_i^n)_i \subseteq \mathcal{F}(D)$ be a chain such that $(X_i^n)_i \precsim (X_i^{n+1})_i$. Then $(Cl(Up(X_i^n)_i))_n$ forms a chain with lub $Cl(X)$ where*

$$X = \{\bigsqcup x_n : x_n \in Cl(Up(X_i^n)_i)\}$$

Proof Theorem 6.7 implies that $(Cl(Up(X_i^n)_i))_n$ forms a chain with respect to $\sqsubseteq_{EM}$. We already know what the least upperbound of such a chain is: it is the element defined by the diagonal as defined in Theorem 3.2. So we show that this diagonal delivers the set from the theorem. We write $(D_n)_n$ for this diagonal. Since we may assume for all n that $X_i^n \sqsubseteq_{EM} X_i^{n+1}$ for all i, $D_n = X_n^n$.

We are going to show $Up(D_n)_n \equiv_P X$. It is easy to see that $Up(D_n)_n \sqsubseteq_P X$. For the other direction, let $x \in X$ and $x = \bigsqcup_n x_n (\in Cl(Up(X_i^n)_i))$. Also, $x = \bigsqcup_i e_i$ for some chain $(e_i)_i \subseteq K(D)$. Hence, for all i, $e_i \sqsubseteq x_n$ for some n. Hence there exists a $x'_n \in X_k^n$ for some k, such that $e_i \sqsubseteq x'_n$. Hence there exists some $d \in Up(D_n)_n$ such that $e_i \sqsubseteq d$. For the other conjunct, let $(y_n)_n$ be a chain through $(D_n)_n$. Then each y_n gives rise to the following sets

$$Y_n^n = \{y_n\}$$

and for $1 \leq i < n$,

$$Y_n^i = \{x \in X_n^i : \exists y \in Y_n^{i+1}. x \sqsubseteq y\}$$

We can view the formation of these sets as "pulling y_n back" over the approximation relation. By construction we have that $y \sqsubseteq y_n$ for all $y \in Y_n^i$, for $i < n$.

All these sets are finite, hence we can apply König's lemma in the following way. First determine a chain $(c_i)_i$ through X_i^1, by considering the finitely branching infinite tree generated by all finite chains leading up to some element from Y_i^1. Let $\bigsqcup_i c_i = c_1$. We have that $c_1 \sqsubseteq \bigsqcup_n y_n$. The chain $(c_i)_i$ determines a set

$$Z_n^1 = \{x \in Y_n^1 : \exists i. c_i \sqsubseteq x\}$$

Infinitely many of these Z_n^1 are non-empty. They select sets $Z_n^2 \subseteq Y_n^2$:

$$Z_n^2 = \{x' \in Y_n^2 : \exists x \in Z_n^1. x \sqsubseteq x'\}$$

We use König's lemma again to find a chain $(x_i')_i$ through the X_n^2 as previously. Since D is compatible, there exists a chain $(c_i')_i$ such that $\bigsqcup_i c_i, \bigsqcup_i x_i' \sqsubseteq \bigsqcup_i c_i' = c_2$. Again we have that $c_2 \sqsubseteq \bigsqcup_n y_n$. Since for all i, $c_i' \sqsubseteq y \in Y_n^2$ for some $n \geq i$, $c_2 \in Cl(Up(X_i^2)_i)$. We continue this way, using $(c_i')_i$ as starting chain. Thus we get a chain $(c_n)_n$ through $Cl(Up(X_i^n)_i)$, such that $\bigsqcup_n c_n \sqsubseteq \bigsqcup_n y_n$. Hence $X \sqsubseteq_P D$. □

Remark. Note that we have proved the previous theorem only for *compatible* domains. The question arises whether or not it holds for *arbitrary* domains. Intuitively, we would expect so, but we were not able to prove it. In order to prove such a theorem, we have to prove that for all d in the diagonal there is an $x \in X$ such that $x \sqsubseteq d$. We needed compatibility to do this.

7 More about functions

In this section we study continuous functions that are closely related to the notion of powerdomain in more detail.

Singleton There exists a function $\{\!|\cdot|\!\} : D \to \mathcal{P}^*(D)$ given by $\{\!|x|\!\} = (\{x\})_i$ for $x \in K(D)$. Then for $x = \bigsqcup_i x_i$ where $(x_i)_i \subseteq K(D)$, $\{\!|x|\!\} = (\{x_i\})_i$. We have $Cl(Up\{\!|x|\!\}) = \{x\}$.

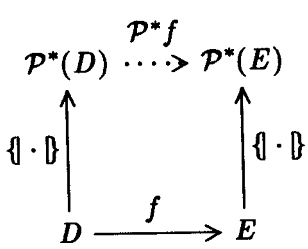
Union We define $\uplus : \mathcal{P}^*(D) \times \mathcal{P}^*(D) \to \mathcal{P}^*(D)$ by

$$(X)_i \uplus (Y)_i = (X \cup Y)_i$$

for finite elements $X, Y \in \mathcal{F}(D)$. $\uplus$ lifts in the standard way to the whole of $\mathcal{P}^*(D)$. We have

$$Cl(Up((X_i)_i \uplus (Y_i)_i) = Cl(\{\bigsqcup d_i : d_i \in X_i \cup Y_i\})$$

These two constructions give us another way of characterizing $\mathcal{P}^*f$. A function $f : \mathcal{P}^*(D) \to \mathcal{P}^*(E)$ is called *linear* if $f((X_i)_i \uplus (Y_i)_i) = f((X_i)_i) \uplus f((Y_i)_i)$. We can prove: $\mathcal{P}^*f$ is the unique linear function such that the following diagram commutes.



Note first of all that, by definition, $\mathcal{P}^*f$ is linear and obeys the equation. For uniqueness, assume g obeys the equation. Then, for finite $(X)_i = (\{x_1, \ldots, x_n\})_i$, we have

$$\begin{aligned} g((X)_i) &= g(\{\!|x_1|\!\} \uplus \cdots \uplus \{\!|x_n|\!\}) \\ &= g(\{\!|x_1|\!\}) \uplus \cdots \uplus g(\{\!|x_n|\!\}) \\ &= \{\!|f(x_1)|\!\} \uplus \cdots \uplus \{\!|f(x_n)|\!\} \\ &= \mathcal{P}^*f((X)_i) \end{aligned}$$

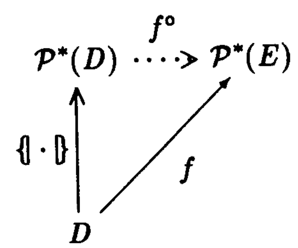
Big union We define $\biguplus : \mathcal{P}^*(\mathcal{P}^*(D)) \rightarrow \mathcal{P}^*(D)$ by

$$\biguplus(\{(X_1)_i, \ldots, (X_n)_i\})_k = (X_1 \cup \cdots \cup X_n)_i$$

$\biguplus$ is obviously monotone and hence lifts to a continuous function on $\mathcal{P}^*(\mathcal{P}^*(D))$. Given a function $f : D \rightarrow \mathcal{P}^*(E)$ we can form $f^\circ : \mathcal{P}^*(D) \rightarrow \mathcal{P}^*(E)$ by

$$f^\circ = \biguplus \circ \mathcal{P}^* f$$

Note that f° is the unique function making the following diagram commute.



In the preceding sections we saw that, given a function $f : D \rightarrow E$, we can construct $\mathcal{P}^*f : \mathcal{P}^*(D) \rightarrow \mathcal{P}^*(E)$. We now show that $\mathcal{P}^*(\cdot) : [D \rightarrow E] \rightarrow [\mathcal{P}^*(D) \rightarrow \mathcal{P}^*(E)]$ is a continuous function itself.

Proposition 7.1 *$\mathcal{P}^*$ is monotone and continuous.*

Proof We prove the proposition in two steps.

- Let $f, g \in [D \rightarrow E]$ such that $f \sqsubseteq g$. This means that $\forall x \in D. f(x) \sqsubseteq g(x)$. Consider $\tilde{f}$ and $\tilde{g}$. Take $X \in \mathcal{F}(D)$. It is easy to see that $\{f(x) : x \in X\} \sqsubseteq_{EM} \{g(x) : x \in X\}$. Hence $\mathcal{P}^*f \sqsubseteq \mathcal{P}^*g$.
- Let f_i be a chain in $[D \rightarrow E]$ with lub f which is defined as $f(x) = \bigsqcup_i f_i(x)$. Let $X \in \mathcal{F}(D)$ and consider

$$X_1 = \{f(x) : x \in X\}$$
$$X_2 = \{\bigsqcup y_i : y_i \in f_i(X)\}$$

  Obviously, $X_1 \subseteq X_2$. For the other direction, we first note that the cardinality of each $f_i(X)$ is bounded by the cardinality of X. Now consider some $\bigsqcup_i y_i \in X_2$. There exists an $x \in X$ such that $y_i = f_i(x)$ infinitely often. Hence $X_2 \subseteq X_1$. This shows that $\mathcal{P}^*(\bigsqcup f_i) = \bigsqcup \mathcal{P}^* f_i$. □

Corollary 7.2 $\mathcal{P}^* : (\omega\text{-})\mathbf{Alg} \to (\omega\text{-})\mathbf{Alg}$ *is a continuous functor.*

Proof The preceding Proposition states that $\mathcal{P}^*$ is a locally continuous functor, and these functors are continuous [Plo81, SP82]. □

Given $f : A \times B \to C$ we can define $f^\dagger : \mathcal{P}^*(A) \times \mathcal{P}^*(B) \to \mathcal{P}^*(C)$ by $f^\dagger((X)_i, (X')_i) = (Y_k)_k$ where $Y_k = \{e_{1k}, \ldots, e_{nk}\}$ such that $\bigsqcup_k e_{ik} = f(x, x')$ for some $x \in X$ and $x' \in X'$. Then we have

$$Cl(Up(f^\dagger(X, X'))) = Cl\{f(x, x') : x \in X, x' \in X'\}$$

Likewise, for $f : A \times B \to C$ we can define $f^\ddagger : \mathcal{P}^*(A) \times B \to \mathcal{P}^*(C)$ such that

$$f^\ddagger(X, b) = Cl\{f(x, b) : x \in X\}$$

Like before we can prove that

$$(\cdot)^\dagger : [A \times B \to C] \to [\mathcal{P}^*(A) \times \mathcal{P}^*(B) \to \mathcal{P}^*(C)]$$

and

$$(\cdot)^\ddagger : [A \times B \to C] \to [\mathcal{P}^*(A) \times B \to \mathcal{P}^*(C)]$$

are continuous functions.

8 Summary

In this paper we have described a completion procedure that constructs an (ω-) algebraic cpo from a (countable) pre-ordered set. The procedure is called chain completion. It induces an equivalence of categories (ω-)**Ord** $\simeq$ (ω-)**Dom**. Chain completion is used to define several notions of powerdomains on a given domain. In particular, we have described the Plotkin powerdomain construction in detail. This domain is isomorphic to a certain collection of subsets of the underlying domain, ordered by the Plotkin order. This identification is relevant for the purpose of defining denotational semantics for a programming language. We have given an order theoretic description of the condition on the underlying domain that ensures that the Plotkin order and the Egli-Milner order (which is used in the defintion of the Plotkin powerdomain) coincide.

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