The Secrets of Causality

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Abstract

In this paper a model for a relational calculus for distributed program design is introduced. Here, the distributed programs are compositions of so-called *processes*.

The construction of the model is guided by desired properties for several forms of composition of processes, such as sequential composition and feedback.

Functionality (determinism) and totality of processes are defined. After the observation that functionality and totality are not preserved by the feedback operator the class of *causal* processes is introduced. It is shown that causality guarantees the preservation by the feedback of functionality and totality.

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Contents

1 Introduction

The task of the axiomatic development of a calculus for program design is threefold: First, the model in which all relevant calculations can be made has to be determined. Second, the model has to be fixed by a set of axioms as clear and precise as possible. Third, a complete set of theorems has to be derived from the axioms, without having recourse to the model, such that designers of programs can perform their calculations in a clear and compact way.

We face the task of developing a *relational* calculus for *distributed* program design. In his pioneering work Kahn [7] gave semantics to deterministic processes interacting in a network. Since then, a number of people has tried to extend Kahn's work to non-deterministic processes. Unfortunately, it is the construction of feedback loops that introduces anomalies (undesired behaviour), as is shown by Brock and Ackerman [3]. Additional properties involving timing aspects circumvents these problems.

Our approach is different: instead of manipulating the model of Kahn to introduce non-determinism in the calculus, we start with a well-established relational algebra, and extend this algebra with axioms and definitions for constructs needed in a calculus for distributed program design. Before doing so, calculations in the model have to be performed to justify the axioms introducing the new constructs.

In the model, all processes calculate simultaneously throughout time, and communicate with other processes in a data-driven, asynchronous nature. Buffering of messages is not assumed: a buffer is just an ordinary process in some network of processes, and has to be mentioned explicitly. Consequently, if a process is not ready to accept a message, this message could be lost if there is no adequate buffering. One has to use protocol schemes to avoid this loss of information.

Processes can be connected in networks of processes using composition constructions. These compositions describe the topology of the network. The connection therefore represents the flow of data (which contrasts flow of control).

This paper is organised as follows. In Chapter 2 we present the basic relational algebra. Research in the field of relational calculi has been intensive, see for example Maddux [8], Schmidt & Ströhlein [12] and Tarski [13]. More specificly, the relational calculus we will use is the one presented in Backhouse *et al.* [2]. Consequently, we can use the theorems derived in this calculus. Next, important notions like functionality, totality and the Principle of Extensionality are introduced. Finally, the feedback construction is presented. In a (relational) calculus for distributed program design the construction of the feedback operator is vital. Components in a network, processes, are connected with arbitrary other components, communicating (asynchronously) with each other. The occurrence of *loops* is then of course inevitable.

In Chapter 3 we discuss the specific model we have in mind. The model we propose differs in several ways from the history model of Kahn: timing of messages plays an significant role in our model, whereas in the model of Kahn the order of messages is important. On the other hand, the assumptions on the time domain are very weak, and the assumptions on the message domain are even weaker.

Chapter 4 is the stepping stone in this paper: a first attempt is made to define a new notion dubbed *causality*. Its necessity emerged from preservation problems of the feedback operator.

In Chapters 5 and 6, preservation of functionality and totality by the feedback construction is discussed, and the preservation of causality itself by several important operators from the relational algebra.

The key chapter of this paper is Chapter 7: it presents the final definition of causality, after the preliminary work in the three proceeding chapters. Because a lot of the hard work has been done in the previous chapters, Chapter 7 is relatively short.

Finally, Chapter 8 presents a weaker variation of causality which is sometimes sufficient to solve preservation problems.

2 The relational algebra

This chapter summarises the algebraic framework in which the calculations are made. This framework can also be found in [2]. There are only some minor differences in notations and terminology. All the proofs given in this chapter can also be found in [2]; the calculations that are shown here merely serve as illustrations.

In Section 2.1, the relational algebra is presented. As the basic sequential composition we will encounter the angelic relational composition. However, in implementations the composition tends to be demonic. When comparing angelic composition with demonic composition, it turn out that the former 'delivers a result' whenever a sensible result exists. However, angelic composition cannot be implemented effectively. In contrast, the latter is undefined whenever the possibility of failure exists. Then why use the angelic composition? The reason is that demonic composition does not have calculational properties as nice as those for angelic composition. Some of the disadvantages are the non-distribution of demonic composition over disjunction and conjunction (from the left), due to its anti-monotonic behaviour. Moreover, there are typing problems in using demonic composition. On the other hand, angelic composition has nice distribution rules with respect to disjunction, and reasonable conditions for the distribution over conjunction. Furthermore, there is no need for typing considerations when composing two processes with the angelic composition. Last but not least, angelic composition is easily transformed into demonic composition by requiring totality of the processes involved.

After introducing the framework, useful concepts, such as functionality and totality, are defined in Section 2.2. The concept of functionality (or determinism) is an important one in our calculus: apart from the fact that most implementations are deterministic, functional processes obey important distribution laws not satisfied by processes in general. Totality is used to transform the angelic composition of the calculationally derived processes into the demonic composition of an implementation.

Finally, in Section 2.3 the principle of extensionality is explained and axiomatised.

2.1 The algebraic framework

The framework is a relational algebra ($\mathcal{A}$, $\sqsubseteq$, $\circ$, I, $^{\cup}$, $\ll$, $\gg$). It is introduced in four layers, connected by interfaces.

A model for the axiomatisation is the structure of the set-theoretical relations over some universe. That is, the objects of $\mathcal{A}$ are sets of pairs of elements from the universe. Together with the introduction of new objects in $\mathcal{A}$ and new operators for constructing new objects from old ones, interpretations (or justifications) in the model are given. In each interpretation the quantification is over all f, g, h and i from the universe. To express that a pair (f, g) is an element of relation R the notation $f\,\langle R\rangle\,g$ is used. In operational terms this says: for input g (describing a complete input history) a possible output of R is f (describing a complete output history). In Chapter 3 the elements of the universe are specified more detailed.

It is the objective that calculations are made only in the algebraic framework, that is, only using the axioms. However, one may prefer to do the calculations in the model, making use of the interpretations.

2.1.1 Lattice

The first layer is the structure of a lattice. Let $\mathcal{A}$ be a set, the elements of which are to be called *procs* (from processes). The identifiers P, Q, etc. range over $\mathcal{A}$. On $\mathcal{A}$ the structure of a complete, universally distributive lattice is imposed:

Axiom 1

$$(\mathcal{A}, \sqsubseteq)$$

is a complete, universally distributive lattice. Here, "lattice" means that the join operator, denoted by $\sqcup$, and the meet operator, denoted by $\sqcap$, are associative, commutative and idempotent, binary infix operators with unit elements $\perp\!\!\!\perp$ and $\top\!\!\!\top$, respectively.

"Complete" means that the extrema $\sqcup(P \ : \ P\in\mathcal{B} \ : \ P)$ and $\sqcap(P \ : \ P\in\mathcal{B} \ : \ P)$ exist for all bags $\mathcal{B}$ of procs.

"Universally distributive" means:

$$P \sqcap \sqcup(Q \ : \ Q\in\mathcal{B} \ : \ Q) \ = \ \sqcup(Q \ : \ Q\in\mathcal{B} \ : \ P\sqcap Q)$$

and

$$P \sqcup \sqcap(Q \ : \ Q\in\mathcal{B} \ : \ Q) \ = \ \sqcap(Q \ : \ Q\in\mathcal{B} \ : \ P\sqcup Q)$$

for all bags $\mathcal{B}$ of procs.

Also the ordering $\sqsupseteq$ will be used, meaning $P\sqsupseteq Q \ \equiv \ Q\sqsubseteq P$.

□

The distributivity properties in Axiom 1 are also referred to as $P\sqcap$ is universally *cupjunctive* and $P\sqcup$ is universally *capjunctive.*

Before continuing, some remarks are made on operator precedence. First, the connectives in the predicate calculus ($\equiv$, $\Leftarrow$, $\Rightarrow$, $\vee$ and $\wedge$) have lower precedence than all the operators in the relational algebra. Next, the operators in the lattice structure $=$, $\sqsubseteq$ and $\sqsupseteq$ all have equal precedence; so do $\sqcup$ and $\sqcap$; and, the former is lower than the latter.

The lattice structure is well-known from the literature. For example, the predicate calculus embeds the structure. It is called a *plat*, standing for power set lattice. Calculations in the structure are referred to by the hint 'plat calculus'.

The interpretations in the set-theoretical model are:

$$f\,\langle P \sqcup Q\rangle\,g \ \equiv \ f\,\langle P\rangle\,g \vee f\,\langle Q\rangle\,g$$

$$f\,\langle \perp\!\!\!\perp\rangle\,g \ \equiv \ \mathit{False}$$

$$f\,\langle P \sqcap Q\rangle\,g \ \equiv \ f\,\langle P\rangle\,g \wedge f\,\langle Q\rangle\,g$$

$$f\,\langle \top\!\!\!\top\rangle\,g \ \equiv \ \mathit{True}$$

$$P \sqsubseteq Q \ \equiv \ \forall(\,f,g \ : \ f\,\langle P\rangle\,g \ : \ f\,\langle Q\rangle\,g\,)$$

The lattice introduced in [2] is also complemented.

2.1.2 Sequential composition

The second layer is the monoid structure for sequential composition:

Axiom 2

$$(\mathcal{A}, \circ, I)$$

is a monoid, that is, $\circ$ is an associative binary infix operator with unit element I.

The interface with the plat structure is: $\circ$ is co-ordinatewise universally cupjunctive. That is, for bags $\mathcal{B}$ and $\mathcal{C}$ of procs,

$$(\sqcup \mathcal{B}) \circ (\sqcup \mathcal{C}) = \sqcup(P, Q \ : \ P \in \mathcal{B} \wedge Q \in \mathcal{C} \ : \ P \circ Q)$$

□

From this axiom, it follows that $\circ$ is monotonic with respect to $\sqsubseteq$ and has $\perp\!\!\perp$ as a left and right zero. To give an illustration of a calculation in the relational algebra the result $\top\!\!\top \circ \top\!\!\top = \top\!\!\top$ is proved.

Theorem 3

$$\top\!\!\top \circ \top\!\!\top = \top\!\!\top$$

Proof by mutual inclusion:

$$
\begin{array}{ll}
 & \top\!\!\top \\
\sqsupseteq & \quad \{ \ \top\!\!\top \text{ is top element of the lattice } \} \\
 & \top\!\!\top \circ \top\!\!\top \\
\sqsupseteq & \quad \{ \ \top\!\!\top \text{ is top element: } \top\!\!\top \sqsupseteq I\text{; monotonicity of composition } \} \\
 & I \circ \top\!\!\top \\
= & \quad \{ \ I \text{ is identity of composition } \} \\
 & \top\!\!\top
\end{array}
$$

□

Sequential composition has a higher precedence than the operators in the lattice structure. So, $P \sqcap Q \circ R$ is to be read as $P \sqcap (Q \circ R)$.

In the model, sequential composition is the angelic relational composition:

$$f \langle P \circ Q \rangle g \ \equiv \ \exists(h \ :: \ f \langle P \rangle h \wedge h \langle Q \rangle g)$$

$$f \langle I \rangle g \ \equiv \ f = g$$

Expressions constructed with sequential composition are to be read from right to left. However, it is only by defining what is meant by *functionality*, Subsection 2.2.2, that this direction is 'formalised'.

2.1.3 Reverse

The third layer is the reverse structure. It is axiomatised as follows:

Axiom 4

$$(\mathcal{A}, {}^{\cup})$$

introduces the unary postfix operator ${}^{\cup}$.

The interface with the plat structure is the following Galois connection:

$$P^{\cup} \sqsubseteq Q \;\equiv\; P \sqsubseteq Q^{\cup}$$

The interface with the monoid structure of sequential composition is:

$$(P \circ Q)^{\cup} = Q^{\cup} \circ P^{\cup}$$

□

Because reverse is an unary operator, it has the highest precedence of all the operators in the relational algebra. So, $P \sqcap Q \circ R^{\cup}$ is to be read as $P \sqcap (Q \circ (R^{\cup}))$.

From the interface with the plat structure it follows that $^{\cup}$ is its own inverse. Moreover, reverse is universally cupjunctive and capjunctive. This implies $\perp\!\!\!\perp^{\cup} = \perp\!\!\!\perp$, $\top\!\!\!\top^{\cup} = \top\!\!\!\top$ and monotonicity of reverse.

From the interface with the monoid structure of sequential composition it follows that $I^{\cup} = I$. This is stated and proved as follows:

Theorem 5

$$I^{\cup} = I$$

Proof:

$$\begin{array}{ll}
 & I^{\cup} \\
= & \quad \{\ I \text{ is identity of composition }\} \\
 & I \circ I^{\cup} \\
= & \quad \{\ \text{reverse is its own inverse }\} \\
 & I^{\cup\cup} \circ I^{\cup} \\
= & \quad \{\ \text{reverse through composition }\} \\
 & (I \circ I^{\cup})^{\cup} \\
= & \quad \{\ I \text{ is identity of composition }\} \\
 & I^{\cup\cup} \\
= & \quad \{\ \text{reverse is its own inverse }\} \\
 & I
\end{array}$$

□

In the model reverse is defined by:

$$f\,\langle P^{\cup}\rangle\, g \;\equiv\; g\,\langle P\rangle\, f$$

2.1.4 Split and parallel

The fourth layer is the layer of the projections.

Axiom 6

$$(\mathcal{A}, \ll, \gg)$$

postulates the existence of two procs $\ll$ and $\gg$. The proc $\ll$ is called "first"; the proc $\gg$ is called "second". Before continuing with Axiom 6, two binary operators on procs, $\vartriangle$ and $\|$, referred to by "split" and "parallel", are defined as follows:

Definition 7 *split and parallel*

(1) $P \vartriangle Q \triangleq \ll^{\cup} \circ P \sqcap \gg^{\cup} \circ Q$

(2) $P \| Q \triangleq (P \circ \ll) \vartriangle (Q \circ \gg)$

(End of Definition 7)

The axiomatisation of the layer is completed as follows:

(1) $I \sqsupseteq I \| I$

(2) $(P \vartriangle Q)^{\cup} \circ R \vartriangle S = P^{\cup} \circ R \sqcap Q^{\cup} \circ S$

(3) $\top\!\!\top \circ \ll = \top\!\!\top \circ \gg$

□

(End of Axiom 6)

The binary operators $\vartriangle$ and $\|$ have equal precedence, and higher than all the other binary operators in the algebra. Parentheses will be used to disambiguate expressions where this is necessary.

The operators split and parallel are monotonic with respect to $\sqsubseteq$ because they are built using the monotonic operators $\circ$, $\sqcap$ and $^{\cup}$.

In the model the projections are defined by:

$$f \langle \ll \rangle g \equiv \exists(h :: g = (f, h))$$

$$f \langle \gg \rangle g \equiv \exists(h :: g = (h, f))$$

A less formal definition is:

$$f \langle \ll \rangle (g, h) \equiv f = g$$

$$f \langle \gg \rangle (g, h) \equiv f = h$$

Notice that this last definition does not cover the complete set of arguments: nothing has been said about forms like $f \langle \ll \rangle g$ where g cannot be split in two. When it is clear that the argument of the projections is a pair, the less formal definition is taken.

With the (less formal) definition in the model one can derive the interpretations of $\vartriangle$ and $\|$ in the model:

$$(f, g) \langle P \vartriangle Q \rangle h \equiv f \langle P \rangle h \wedge g \langle Q \rangle h$$

$$(f, g) \langle P \| Q \rangle (h, i) \equiv f \langle P \rangle h \wedge g \langle Q \rangle i$$

Later in this paper, a slightly more formal notation $f \blacktriangle g$ is used for pairs (f, g). Things would get too complicated, though, to explain the precise motivation for the formal notation already here.

A warning should be made. The parallel composition used in this paper is not similar to operators used in CSP or in dataflow networks. In those theories the parallel operator expresses (synchronised) *communication* of the operands. In this paper the parallel operator expresses *computation* in parallel of the operands without (direct) *communication.*

In fact, all composition constructions presented in this paper express simultaneous computation: all procs in the system compute in parallel. The kind of composition denotes the way how these procs are *connected.*

The following results can be derived from the axioms. They are known as "parallel-split fusion", "parallel-parallel fusion" and "computation rules". The derivation involves elementary relational calculus.

Theorem 8 *parallel-split fusion, parallel-parallel fusion*

(1) $P \parallel Q \circ R \vartriangle S = (P \circ R) \vartriangle (Q \circ S)$

(2) $P \parallel Q \circ R \parallel S = (P \circ R) \parallel (Q \circ S)$

Proof:

First notice that reverse distributes over parallel, that is:

$$
\begin{array}{ll}
 & (P \parallel Q)^{\cup} \\
= & \quad \{ \text{ definition parallel and split 7 } \} \\
 & (\ll^{\cup} \circ P \circ \ll \sqcap \gg^{\cup} \circ Q \circ \gg)^{\cup} \\
= & \quad \{ \text{ reverse through cap 4 } \} \\
 & (\ll^{\cup} \circ P \circ \ll)^{\cup} \sqcap (\gg^{\cup} \circ Q \circ \gg)^{\cup} \\
= & \quad \{ \text{ reverse through composition 4 } \} \\
 & \ll^{\cup} \circ P^{\cup} \circ \ll^{\cup\cup} \sqcap \gg^{\cup} \circ Q^{\cup} \circ \gg^{\cup\cup} \\
= & \quad \{ \text{ reverse is its own inverse 4 } \} \\
 & \ll^{\cup} \circ P^{\cup} \circ \ll \sqcap \gg^{\cup} \circ Q^{\cup} \circ \gg \\
= & \quad \{ \text{ definition split and parallel 7 } \} \\
 & P^{\cup} \parallel Q^{\cup}
\end{array}
$$

Then:

$$
\begin{array}{ll}
 & P \parallel Q \circ R \vartriangle S \\
= & \quad \{ \text{ reverse is its own inverse 4 } \} \\
 & (P \parallel Q)^{\cup\cup} \circ R \vartriangle S \\
= & \quad \{ \text{ above: reverse through parallel } \} \\
 & (P^{\cup} \parallel Q^{\cup})^{\cup} \circ R \vartriangle S \\
= & \quad \{ \text{ definition parallel 7 } \} \\
 & ((P^{\cup} \circ \ll) \vartriangle (Q^{\cup} \circ \gg))^{\cup} \circ R \vartriangle S \\
= & \quad \{ \text{ Axiom 6 } \} \\
 & (P^{\cup} \circ \ll)^{\cup} \circ R \sqcap (Q^{\cup} \circ \gg)^{\cup} \circ S \\
= & \quad \{ \text{ reverse through composition and is its own inverse 4 } \} \\
 & \ll^{\cup} \circ P \circ R \sqcap \gg^{\cup} \circ Q \circ S \\
= & \quad \{ \text{ definition split 7 } \} \\
 & (P \circ R) \vartriangle (Q \circ S)
\end{array}
$$

□

The computation rules are:

Theorem 9 *computation rules split*

(1) $\ll \circ P \vartriangle Q = P \sqcap \top\!\!\top \circ Q$

(2) $\gg \circ P \vartriangle Q = Q \sqcap \top\!\!\top \circ P$

(3) $\ll \circ P \vartriangle Q = P \Leftarrow \top\!\!\top \circ P \sqsubseteq \top\!\!\top \circ Q$

(4) $\gg \circ P \vartriangle Q = Q \Leftarrow \top\!\!\top \circ P \sqsupseteq \top\!\!\top \circ Q$

Proof:

First the result $(I \vartriangle \top\!\!\top)^{\cup} = \ll$ is derived:

$$
\begin{array}{ll}
 & (I \vartriangle \top\!\!\top)^{\cup} \\
= & \quad \{ \text{ definition split 7 } \} \\
 & (\ll^{\cup} \circ I \sqcap \gg^{\cup} \circ \top\!\!\top)^{\cup}
\end{array}
$$

$=$ { I is identity of composition 2; reverse through cap 4 }

$\ll^{\cup\cup} \sqcap (\gg^{\cup} \circ \top\!\!\top)^{\cup}$

$=$ { reverse through composition and is its own inverse 4 }

$\ll \sqcap \top\!\!\top^{\cup} \circ \gg$

$=$ { Axiom 4: $\top\!\!\top^{\cup} = \top\!\!\top$; Axiom 6 }

$\ll \sqcap \top\!\!\top \circ \ll$

$=$ { monotonicity of composition: $I \sqsubseteq \top\!\!\top \Rightarrow \ll \sqsubseteq \top\!\!\top \circ \ll$; plat calculus }

$\ll$

Then, the main result is:

$\ll \circ P \vartriangle Q$

$=$ { above result }

$(I \vartriangle \top\!\!\top)^{\cup} \circ P \vartriangle Q$

$=$ { Axiom 6 }

$I^{\cup} \circ P \sqcap \top\!\!\top^{\cup} \circ Q$

$=$ { Theorem 5; Axiom 4: $\top\!\!\top^{\cup} = \top\!\!\top$ }

$P \sqcap \top\!\!\top \circ Q$

Continuing with the assumption $\top\!\!\top \circ P \sqsubseteq \top\!\!\top \circ Q$:

$P \sqcap \top\!\!\top \circ Q$

$=$ { monotonicity of composition: $I \sqsubseteq \top\!\!\top \Rightarrow P \sqsubseteq \top\!\!\top \circ P$; plat calculus }

$P \sqcap \top\!\!\top \circ P \sqcap \top\!\!\top \circ Q$

$=$ { assumption; plat calculus }

$P \sqcap \top\!\!\top \circ P$

$=$ { $P \sqsubseteq \top\!\!\top \circ P$; plat calculus }

P

□

In Subsection 2.2.1, the first two expressions of Theorem 9 are further transformed to other, nicer expressions. Also the computation rules for parallel are given in that subsection.

A thorough exploration of the properties for split and parallel can be found in [2]. There are some minor differences in the notation, the main one being that in this paper the notation $\|$ is used whereas in [2] the notation $\times$ is used.

2.1.5 The Cone Rule and Dedekind's Rule

An axiom that guarantees, among other consequences, that I and $\top\!\!\top$ differ from $\perp\!\!\perp$ is the Cone Rule. In general, the Cone Rule gives a condition to conclude that a proc is not equal to the uninteresting proc $\perp\!\!\perp$.

Axiom 10 *Cone Rule*

$$\top\!\!\top \circ P \circ \top\!\!\top = \top\!\!\top \;\equiv\; P \neq \perp\!\!\perp$$

□

Then, the claimed consequences can be proved:

Theorem 11

(1) $I \neq \perp\!\!\perp$

(2) $\top\!\!\top \neq \perp\!\!\perp$

(3) $\ll \neq \perp\!\!\perp$

(4) $\gg \neq \perp\!\!\perp$

Proof:

$$
\begin{array}{ll}
 & I \neq \perp\!\!\perp \\
= & \quad \{ \text{ Cone Rule 10 } \} \\
 & \top\!\!\top \circ I \circ \top\!\!\top \;=\; \top\!\!\top \\
= & \quad \{ \text{ Axiom 2: } I \text{ is identity of composition } \} \\
 & \top\!\!\top \circ \top\!\!\top \;=\; \top\!\!\top \\
= & \quad \{ \text{ Theorem 3 } \} \\
 & \mathit{True}
\end{array}
$$

And, for all P, by contraposition:

$$
\begin{array}{ll}
 & \ll = \perp\!\!\perp \\
\Rightarrow & \quad \{ \perp\!\!\perp \text{ is zero of composition } \} \\
 & \ll \circ\, P \vartriangle I \;=\; \perp\!\!\perp \\
= & \quad \{ \text{ Theorem 9 (1) } \} \\
 & P \sqcap \top\!\!\top \circ I \;=\; \perp\!\!\perp \\
= & \quad \{ I \text{ is unit of composition; } \top\!\!\top \text{ is unit of cap } \} \\
 & P = \perp\!\!\perp \\
= & \quad \{ \text{ Theorem 11 (1) } \} \\
 & \mathit{false}
\end{array}
$$

□

A second axiom that connects the first three layers of the relational framework is Dedekind's Rule. Although this rule looks terrifying (there are five free variables), it turns out that this rule is very useful in calculations.

Axiom 12 *Dedekind's Rule*

$$
\begin{array}{ll}
 & P \circ Q \;\sqsupseteq\; R \circ S \sqcap T \\
\Leftarrow & \\
 & P \;\sqsupseteq\; R \sqcap T \circ S^{\cup} \;\wedge\; Q \;\sqsupseteq\; S \sqcap R^{\cup} \circ T
\end{array}
$$

□

To remember this rule one has to be able to reconstruct its syntactic shape. For example, the first conjunct of the antecedent is obtained as follows: take the consequent $P \circ Q \sqsupseteq R \circ S \sqcap T$, forget about the *right* arguments of the compositions (Q and S) and add $S^{\cup}$ on the *right* side of T. The second conjunct is obtained in a symmetrical way.

In calculations it is often the case that one of the conjuncts becomes trivially true, for example if $Q = S$. So applying Dedekind does not really stretch the proofs.

A first formulation of this rule can be found in Riguet [11]. Riguet attributes the rule to Dedekind who suggested a slightly weaker variation. Although the axiom only contains three free variables, it is less useful for our calculations, because it strongly tends to widen the proofs. It looks like:

$$
(R \sqcap T \circ S^{\cup}) \;\circ\; (S \sqcap R^{\cup} \circ T) \;\sqsupseteq\; R \circ S \sqcap T
$$

The interpretation in the model of Dedekind's rule, Axiom 12, boils down to the following statement in predicate calculus:

$$
\begin{array}{ll}
 & \forall(\, f, g, h \;:\; f\,\langle R\rangle\, h \wedge h\,\langle S\rangle\, g \wedge f\,\langle T\rangle\, g \;:\; \exists(\, h' \;::\; f\,\langle P\rangle\, h' \wedge h'\,\langle Q\rangle\, g\,)) \\
\Leftarrow & \\
 & \forall(\, f, g, h \;:\; f\,\langle R\rangle\, h \wedge h\,\langle S\rangle\, g \wedge f\,\langle T\rangle\, g \;:\; f\,\langle P\rangle\, h \wedge h\,\langle Q\rangle\, g\,)
\end{array}
$$

To illustrate the use of Dedekind, an important theorem dealing with distribution of sequential composition over cap is proved and some corollaries are stated. Notice that until now nothing has been said about such distributions.

Theorem 13 *distribution over cap*

(1) $(P \sqcap Q) \circ R = P \circ R \sqcap Q \circ R \;\Leftarrow\; Q \sqsupseteq Q \circ R \circ R^{\cup}$

(2) $(P \sqcap Q \circ \top\!\!\top) \circ R \;=\; P \circ R \sqcap Q \circ \top\!\!\top$

Proof by mutual inclusion:

First: notice that the consequent is symmetric in P and Q. So in the antecedent, Q could equally well have been replaced by P. Then:

$$
\begin{array}{ll}
 & (P \sqcap Q) \circ R \sqsubseteq P \circ R \sqcap Q \circ R \\
= & \quad \{ \text{ plat calculus } \} \\
 & (P \sqcap Q) \circ R \sqsubseteq P \circ R \;\wedge\; (P \sqcap Q) \circ R \sqsubseteq Q \circ R \\
\Leftarrow & \quad \{ \text{ monotonicity of composition } \} \\
 & P \sqcap Q \sqsubseteq P \;\wedge\; P \sqcap Q \sqsubseteq Q \\
= & \quad \{ \text{ plat calculus } \} \\
 & \mathit{True}
\end{array}
$$

And:

$$
\begin{array}{ll}
 & (P \sqcap Q) \circ R \sqsupseteq P \circ R \sqcap Q \circ R \\
\Leftarrow & \quad \{ \text{ Dedekind 12 } \} \\
 & P \sqcap Q \sqsupseteq P \sqcap Q \circ R \circ R^{\cup} \;\wedge\; R \sqsupseteq R \sqcap P^{\cup} \circ Q \circ R \\
\Leftarrow & \quad \{ \text{ plat calculus } \} \\
 & Q \sqsupseteq Q \circ R \circ R^{\cup}
\end{array}
$$

The proof of the second statement follows the same structure.

□

Try to prove this result with the alternative formulation of Dedekind. The proof gets wider now[1].

Some important corollaries with respect to distribution over split are:

Corollary 14

(1) $P \vartriangle Q \circ R = (P \circ R) \vartriangle (Q \circ R) \;\Leftarrow\; P \sqsupseteq P \circ R \circ R^{\cup} \vee Q \sqsupseteq Q \circ R \circ R^{\cup}$

(2) $P \vartriangle (Q \circ \top\!\!\top) \circ R = (P \circ R) \vartriangle (Q \circ \top\!\!\top)$

□

This concludes almost the axiomatisation of the algebraic framework. In Section 2.3 a last axiom called Extensionality will be added to the set of axioms for relational algebra $\mathcal{A}$. But first, other important constructions have to be presented.

2.2 Definitions and properties

In this section, additional constructions such as domains, functionality and feedback, are defined and some of their most important properties are stated.

[1] It could also be the case that we switched to 'religion-mode'.

2.2.1 Interfaces, domains and typing

In the sequel, procs below I are heavily used. They act like restricted identities, and are called *interfaces*:

Definition 15 *interfaces*

$$
\begin{array}{ll}
 & P \text{ is an interface} \\
\triangleq & \\
 & P \sqsubseteq I
\end{array}
$$

□

Two trivial interfaces are the procs I and $\perp\!\!\!\perp$. Identifiers A and B are used to denote interfaces, that is, $A \sqsubseteq I$ and $B \sqsubseteq I$. In the model, interfaces can be interpreted by:

$$f \langle A \rangle g \Rightarrow f = g$$

These interfaces play the role of restricted identities. A basic and useful theorem about interfaces is:

Theorem 16

(1) $\quad A = A \sqcap I$

(2) $\quad A = A^{\cup}$

(3) $\quad A \circ (P \sqcap Q) \;=\; A \circ P \sqcap Q$

□

The proof is omitted, though it is another good example of the usefulness of Dedekind's Rule 12. Notice that the third statement of Theorem 16 actually *equivales* the defining property of interfaces. This follows by instantiating P and Q both to I. Some straightforward corollaries of the theorem are:

Corollary 17

(1) $\quad A = A \circ A$

(2) $\quad A \circ B = B \circ A = A \sqcap B$

(3) $\quad A \circ (P \sqcap Q) \;=\; A \circ P \sqcap A \circ Q$

□

The 'least' left and right interfaces of a proc occur frequently in calculations. They are called left and right domain and are denoted by $P{<}$ and $P{>}$, respectively. The domains are defined via the following Galois connection:

Definition 18 *domains*

Left domain:

$$P{<} \sqsubseteq Q \;\equiv\; P \sqsubseteq (Q \sqcap I) \circ \top\!\!\!\top$$

Right domain:

$$P{>} = (P^{\cup}){<}$$

□

From the defining Galois connection it follows that the domain operators are universally cupjunctive, which implies $\perp\!\!\perp< = \perp\!\!\perp> = \perp\!\!\perp$ and monotonicity of the operators.

Moreover, the left and right domain functions have a closed form. The following theorem states these forms. The proof is omitted.

Theorem 19 *domains explicitly*

$$P< \;=\; P \circ \top\!\!\top \sqcap I \quad \text{and} \quad P> \;=\; I \sqcap \top\!\!\top \circ P$$

□

Notice that domains are interfaces. This is, among other useful properties, formally stated as follows:

Theorem 20

(1) $P< \sqsubseteq I$ and $P> \sqsubseteq I$

(2) $P< \circ Q \;=\; P \circ \top\!\!\top \sqcap Q$ and $P \circ Q> \;=\; P \sqcap \top\!\!\top \circ Q$

(3) $P< \circ P = P$ and $P = P \circ P>$

(4) $A< = A> = A$, for all A

□

It is a straightforward exercise to derive the interpretation of domains in the model. From Theorem 19 one can conclude for the left domain:

$$f\,\langle P< \rangle\, g \;\equiv\; \exists(h :: f\,\langle P \rangle\, h\,) \wedge f = g$$

To introduce a more familiar way to express the type of a proc the following definition is given:

Definition 21 *typing*

$$P \in A \sim B \;\equiv\; A \circ P = P \wedge P = P \circ B$$

□

In the model, the conjunct $A \circ P = P$ reads:

$$f\,\langle P \rangle\, g \Rightarrow f\,\langle A \rangle\, f$$

An appealing example of typing is:

$$P \in P< \sim P>$$

which follows from Theorem 20 (3). $P<$ and $P>$ are the smallest interfaces such that typing is correct. Other ways of typing could have been:

$$A \circ P = A \circ P \circ B = P \circ B$$

which is a weaker formulation than Definition 21. The disadvantage is that this definition only gives limited typing information: only in the presence of interface A one can remove B from $A \circ P \circ B$. An even weaker formulation is:

$$A \circ P \circ B = P \circ B$$

This alternative definition comes very close to conventional typing. It is, however, a highly asymmetric one, which is inconvenient in a framework which is closed under reverse, Axiom 4.

There are a number of equivalent formulations for typing a proc. The only reason for choosing the formulation given in Definition 21 is that it allows the designer to apply the rule of Leibniz, which is easier to use than, for example, monotonicity.

Theorem 22 *other formulations for typing*

The following four statements are all equivalent:

(1) $P = A \circ P \ \wedge \ P = P \circ B$

(2) $P \sqsubseteq A \circ \top\!\!\top \circ B$

(3) $P = A \circ P \circ B$

(4) $P{<} \sqsubseteq A \ \wedge \ P{>} \sqsubseteq B$

Proof by mutual implication:

$$
\begin{array}{ll}
 & P = A \circ P \ \wedge \ P = P \circ B \\
= & \quad \{ \text{ plat calculus } \} \\
 & P = A \circ (P \sqcap \top\!\!\top) \ \wedge \ P = (P \sqcap \top\!\!\top) \circ B \\
= & \quad \{ \text{ Corollary 17 (2) } \} \\
 & P = P \sqcap A \circ \top\!\!\top \ \wedge \ P = P \sqcap \top\!\!\top \circ B \\
= & \quad \{ \text{ plat calculus } \} \\
 & P \sqsubseteq A \circ \top\!\!\top \ \wedge \ P \sqsubseteq \top\!\!\top \circ B \\
\Leftarrow & \quad \{ \ \top\!\!\top \text{ is top element; monotonicity of composition } \} \\
 & P \sqsubseteq A \circ \top\!\!\top \circ B \\
\Leftarrow & \quad \{ \ P \sqsubseteq \top\!\!\top\text{; monotonicity of composition } \} \\
 & P = A \circ P \circ B \\
\Leftarrow & \quad \{ \text{ Leibniz } \} \\
 & P = A \circ P \ \wedge \ P = P \circ B
\end{array}
$$

And finally:

$$
\begin{array}{ll}
 & P \sqsubseteq A \circ \top\!\!\top \ \wedge \ P \sqsubseteq \top\!\!\top \circ B \\
= & \quad \{ \text{ Theorem 16 (1): } A = A \sqcap I\text{; Definition 18 } \} \\
 & P{<} \sqsubseteq A \ \wedge \ P{>} \sqsubseteq B
\end{array}
$$

□

For a more intensive exploration of properties about domains and typing used in this paper one is referred to [2]. In that paper the term *monotypes* is used instead of interfaces.

With the notion of domains, the computation rules for split, Theorem 9, can be rewritten, making use of Theorem 20. Also the computation rules for parallel are given.

Theorem 23 *computation rules split and parallel, revised*

(1) $\ll \circ\, P \vartriangle Q \ = \ P \circ Q{>}$

(2) $\gg \circ\, P \vartriangle Q \ = \ Q \circ P{>}$

(3) $\ll \circ\, P \,\|\, Q \ = \ P \circ \ll \circ\, I \,\|\, Q{>}$

(4) $\gg \circ\, P \,\|\, Q \ = \ Q \circ \gg \circ\, P{>} \,\|\, I$

□

2.2.2 Functionality and totality

Until now, nothing has been said about functions, functionality or determinism. All operators introduced thus far apply to arbitrary procs. These procs can be interpreted as (non-deterministic) relations. In this subsection it is specified what is meant by a functional (or deterministic) proc.

Definition 24 *functionality*

$$P \text{ is functional}$$
$$\triangleq$$
$$P \circ P^{\cup} \sqsubseteq I$$

□

In expressions the identifiers F and G are used to denote functional procs. In typing a proc, the notation "$\leftarrow$" is used to denote that the proc is functional. So, by definition:

Definition 25

$$F \in A \leftarrow B$$
$$\triangleq$$
$$F \in A \sim B \ \wedge \ F \text{ is functional}$$

□

In the model the definition of functionality reads:

$$f\,\langle P \rangle\, h \wedge g\,\langle P \rangle\, h \ \Rightarrow \ f = g$$

For a function F, the formulation $f\,\langle F \rangle\, g$ can be interpreted as $f = F.g$, thus showing the embedding of functions in the algebra of relations. Notice that the direction of a proc "from right to left" is implied. If one reads from left to right, the notion of functionality will become the notion of injectivity. Injectivity is dual to functionality. Proc P is said to be injective if and only if $P^{\cup}$ is functional.

Definition 26 *injectivity*

$$P \text{ is injective}$$
$$\triangleq$$
$$P^{\cup} \text{ is functional}$$

□

The notation "$\rightarrow$" is used for injectivity:

Definition 27

$$P \in A \rightarrow B$$
$$\triangleq$$
$$P \in A \sim B \ \wedge \ P \text{ is injective}$$

□

One of the main properties of functions is that they distribute from the right over cap and split. This follows from Theorem 13 and Corollary 14.

Theorem 28 *function distribution*

(1) For non-empty bag $\mathcal{B}$: $\sqcap(P : P \in \mathcal{B} : P) \circ F = \sqcap(P : P \in \mathcal{B} : P \circ F)$

(2) $P \vartriangle Q \circ F = (P \circ F) \vartriangle (Q \circ F)$

Proof:

First: one inclusion is trivialy true by monotonicity. For the other inclusion:

$$
\begin{array}{ll}
 & \sqcap(P : P \in \mathcal{B} : P) \circ F \sqsupseteq \sqcap(P : P \in \mathcal{B} : P \circ F) \\
= & \qquad \{ \text{ assume } Q \text{ is in non-empty bag } \mathcal{B} \ \} \\
 & \sqcap(P : P \in \mathcal{B} : P) \circ F \sqsupseteq Q \circ F \sqcap \sqcap(P : P \in \mathcal{B} : P \circ F) \\
\Leftarrow & \qquad \{ \text{ Dedekind 12 } \} \\
 & \sqcap(P : P \in \mathcal{B} : P) \sqsupseteq Q \sqcap \sqcap(P : P \in \mathcal{B} : P \circ F) \circ F^{\cup} \\
\Leftarrow & \qquad \{ \text{ monotonicity } \} \\
 & \sqcap(P : P \in \mathcal{B} : P) \sqsupseteq Q \sqcap \sqcap(P : P \in \mathcal{B} : P \circ F \circ F^{\cup}) \\
\Leftarrow & \qquad \{ \ F \text{ ranges over functions: } I \sqsupseteq F \circ F^{\cup} \ \} \\
 & \sqcap(P : P \in \mathcal{B} : P) \sqsupseteq Q \sqcap \sqcap(P : P \in \mathcal{B} : P) \\
= & \qquad \{ \text{ plat calculus } \} \\
 & \mathit{true}
\end{array}
$$

And second:

$$
\begin{array}{ll}
 & P \triangle Q \circ F = (P \circ F) \triangle (Q \circ F) \\
\Leftarrow & \qquad \{ \text{ Corollary 14 (1) } \} \\
 & P \sqsupseteq P \circ F \circ F^{\cup} \\
\Leftarrow & \qquad \{ \text{ monotonicity of composition; } I \text{ is identity of composition } \} \\
 & I \sqsupseteq F \circ F^{\cup} \\
= & \qquad \{ \ F \text{ ranges over functions: Definition 24 } \} \\
 & \mathit{True}
\end{array}
$$

□

Sequential composition, split and parallel composition all preserve functionality. Notice the simplicity of the proofs.

Theorem 29 *functionality preservation*

(1) $\quad F \circ G$ is functional

(2) $\quad F \triangle G$ is functional

(3) $\quad F \parallel G$ is functional

Proof:

First:

$$
\begin{array}{ll}
 & F \circ G \circ (F \circ G)^{\cup} \\
= & \qquad \{ \text{ reverse through composition } \} \\
 & F \circ G \circ G^{\cup} \circ F^{\cup} \\
\sqsubseteq & \qquad \{ \text{ assumption } G \text{ is functional: } G \circ G^{\cup} \sqsubseteq I \text{; monotonicity of composition } \} \\
 & F \circ F^{\cup} \\
\sqsubseteq & \qquad \{ \text{ assumption } F \text{ is functional } \} \\
 & I
\end{array}
$$

Third:

$$
\begin{array}{ll}
 & F \parallel G \circ (F \parallel G)^{\cup} \\
= & \qquad \{ \text{ reverse distributes over parallel } \} \\
 & F \parallel G \circ F^{\cup} \parallel G^{\cup} \\
= & \qquad \{ \text{ parallel-parallel fusion, Theorem 8 (2) } \} \\
 & (F \circ F^{\cup}) \parallel (G \circ G^{\cup}) \\
\sqsubseteq & \qquad \{ \ F \text{ and } G \text{ are functional: Definition 24 and monotonicity of parallel } \}
\end{array}
$$

$$I \parallel I$$

$\sqsubseteq$ { Axiom 6 }

$$I$$

And second: because of the functionality preservation properties of parallel and sequential composition and making use of parallel-split fusion 8 (1), the proof obligation boils down to functionality of $I \vartriangle I$:

$$I \vartriangle I \circ (I \vartriangle I)^{\cup}$$

$=$ { definition split 7 }

$$(\ll^{\cup} \sqcap \gg^{\cup}) \circ (\ll^{\cup} \sqcap \gg^{\cup})^{\cup}$$

$=$ { reverse distributes over cap and is its own inverse }

$$(\ll^{\cup} \sqcap \gg^{\cup}) \circ (\ll \sqcap \gg)$$

$\sqsubseteq$ { monotonicity }

$$\ll^{\cup} \circ \ll \sqcap \gg^{\cup} \circ \gg$$

$=$ { definition parallel 7 }

$$I \parallel I$$

$\sqsubseteq$ { Axiom 6 }

$$I$$

□

This concludes the short discussion of functionality. Next, the notion of totality on some interface is defined:

Definition 30 *totality*

$$P \text{ is total on } A$$

$$\triangleq$$

$$\top\!\!\top \circ A \sqsubseteq \top\!\!\top \circ P$$

□

For example, all procs are total on the interface $\perp\!\!\!\perp$. There are numerous equivalent formulations of totality. Two examples of formulations equivalent to "P is total on A" are $A \sqsubseteq P>$ and $A \sqsubseteq P^{\cup} \circ P$. The notation "$\rightsquigarrow$" is used to denote that the proc is total:

Definition 31

$$P \in A \rightsquigarrow B$$

$$\triangleq$$

$$P \in A \sim B \ \wedge\ P \text{ is total on } B$$

□

The notation "$\leftarrowtail$" means in addition that the proc is functional:

Definition 32

$$P \in A \leftarrowtail B$$

$$\triangleq$$

$$P \in A \rightsquigarrow B \ \wedge\ P \text{ is functional}$$

□

Note that "$\leftarrowtail$" could equally well have been defined by:

$$P \in A \leftarrowtail B$$

$$\triangleq$$

$$P \in A \leftarrow B \ \wedge\ P \text{ is total on } B$$

In the model totality of P on A reads:

$$f\langle A\rangle f \;\Rightarrow\; \exists(h \;::\; h\langle P\rangle f)$$

The dual of totality is surjectivity. Proc P is said to be surjective on A if and only if $P^{\cup}$ is total on A. The notation "$\curvearrowright$" is used for surjectivity:

Definition 33 *surjectivity*

$$P \text{ is surjective on } A$$
$$\triangleq$$
$$P^{\cup} \text{ is total on } A$$

□

and consequently the notation

Definition 34

$$P \in A \curvearrowright B$$
$$\triangleq$$
$$P \in A \sim B \;\wedge\; P \text{ is surjective on } A$$

□

The typings $\rightsquigarrow$ and $\mapsfrom$ are used very often. For example, $F \in \mathbb{N} \mapsfrom \mathbb{Z}$ expresses that F is a partial function, mapping *each* element in $\mathbb{Z}$ to a result in $\mathbb{N}$. For convenience, their precise meaning is written out in full:

Theorem 35

(1) $\quad P \in A \rightsquigarrow B \;\equiv\; A \circ P = P \;\wedge\; \top\!\!\top \circ B = \top\!\!\top \circ P$

(2) $\quad P \in A \mapsfrom B \;\equiv\; A \circ P = P \;\wedge\; \top\!\!\top \circ B = \top\!\!\top \circ P \;\wedge\; P \circ P^{\cup} \sqsubseteq I$

□

Next, the totality preservation of several basic constructions is shown. A result that will prove useful is:

Theorem 36

$$\top\!\!\top \circ P \vartriangle Q \;=\; \top\!\!\top \circ P \sqcap \top\!\!\top \circ Q$$

□

The following theorem summarises totality preservation properties of fundamental composition operators.

Theorem 37 *totality preservation*

(1) $\quad P \circ Q \in I \rightsquigarrow B \;\Leftarrow\; P \in I \rightsquigarrow A \;\wedge\; Q \in A \rightsquigarrow B$

(2) $\quad P \vartriangle Q \;\in\; I \rightsquigarrow A \circ B \;\Leftarrow\; P \in I \rightsquigarrow A \;\wedge\; Q \in I \rightsquigarrow B$

(3) $\quad P \,\|\, Q \;\in\; I \rightsquigarrow A \,\|\, B \;\Leftarrow\; P \in I \rightsquigarrow A \;\wedge\; Q \in I \rightsquigarrow B$

Proof:

Notice that the I in the typing $P \in I \rightsquigarrow A$ is vacuous: according to Theorem 35 (1) this typing equivales $\top\!\!\top \circ A = \top\!\!\top \circ P$.

First:

$$
\begin{array}{ll}
 & P \circ Q \in I \rightsquigarrow B \\
= & \qquad \{ \text{ Theorem 35 (1) } \} \\
 & \top\!\!\top \circ B = \top\!\!\top \circ P \circ Q \\
= & \qquad \{ \text{ assumption on } P\text{: } \top\!\!\top \circ A = \top\!\!\top \circ P \ \} \\
 & \top\!\!\top \circ B = \top\!\!\top \circ A \circ Q \\
= & \qquad \{ \text{ assumption on } Q\text{: } A \circ Q = Q \ \} \\
 & \top\!\!\top \circ B = \top\!\!\top \circ Q \\
= & \qquad \{ \text{ assumption on } Q \ \} \\
 & \mathit{True}
\end{array}
$$

Second:

$$
\begin{array}{ll}
 & P \vartriangle Q \ \in \ I \rightsquigarrow A \circ B \\
= & \qquad \{ \text{ Theorem 35 (1) } \} \\
 & \top\!\!\top \circ A \circ B \ = \ \top\!\!\top \circ P \vartriangle Q \\
= & \qquad \{ \ \top\!\!\top \text{ is unit of cap; Theorem 36 } \} \\
 & (\top\!\!\top \circ A \sqcap \top\!\!\top) \circ B \ = \ \top\!\!\top \circ P \sqcap \top\!\!\top \circ Q \\
= & \qquad \{ \text{ Theorem 16 (3) } \} \\
 & \top\!\!\top \circ A \sqcap \top\!\!\top \circ B \ = \ \top\!\!\top \circ P \sqcap \top\!\!\top \circ Q \\
\Leftarrow & \qquad \{ \text{ Leibniz } \} \\
 & \top\!\!\top \circ A = \top\!\!\top \circ P \ \wedge \ \top\!\!\top \circ B = \top\!\!\top \circ Q \\
= & \qquad \{ \text{ Theorem 35 (1) } \} \\
 & P \in I \rightsquigarrow A \ \wedge \ Q \in I \rightsquigarrow B
\end{array}
$$

And third:

$$
\begin{array}{ll}
 & P \| Q \ \in \ I \rightsquigarrow A \| B \\
= & \qquad \{ \text{ Theorem 35 (1) } \} \\
 & \top\!\!\top \circ A \| B \ = \ \top\!\!\top \circ P \| Q \\
= & \qquad \{ \text{ definition parallel 7 (2) } \} \\
 & \top\!\!\top \circ (A \circ \ll) \vartriangle (B \circ \gg) \ = \ \top\!\!\top \circ (P \circ \ll) \vartriangle (Q \circ \gg) \\
= & \qquad \{ \text{ Theorem 36 } \} \\
 & \top\!\!\top \circ A \circ \ll \sqcap \top\!\!\top \circ B \circ \gg \ = \ \top\!\!\top \circ P \circ \ll \sqcap \top\!\!\top \circ Q \circ \gg \\
\Leftarrow & \qquad \{ \text{ Leibniz } \} \\
 & \top\!\!\top \circ A = \top\!\!\top \circ P \ \wedge \ \top\!\!\top \circ B = \top\!\!\top \circ Q \\
= & \qquad \{ \text{ Theorem 35 (1) } \} \\
 & P \in I \rightsquigarrow A \ \wedge \ Q \in I \rightsquigarrow B
\end{array}
$$

□

With these definitions of domains, functionality and totality one can list some typing properties of the basic procs. For example, the property that all interfaces are functions, total on 'themselves' is stated:

Theorem 38 *typing of basic processes*

(1) $\quad A \in A \leftarrowtail A$, for all A

in particular

(2) $\quad \bot\!\!\bot \in \bot\!\!\bot \leftarrowtail \bot\!\!\bot$

and

(3) $\quad I \in I \leftarrowtail I$

Moreover

(4) $\quad \top\!\!\top \in I \rightsquigarrow I$

,

(5) $\quad \ll \ \in \ I \leftarrowtail I \| I$

and

(6) $\quad \gg \ \in \ I \leftarrowtail I \| I$

□

This concludes the limited discussion on giving type information about procs.

2.2.3 Feedback

The rich structure of the algebra introduced in earlier subsections makes it possible to define the feedback operator. No new axioms are assumed to capture the notion. A lot of the desired properties of feedback, such as the computation rules, follows from its definition. It would be naive, though, to think that all the problems other researchers in the field encountered are circumvented.

The following definition of the feedback operator is proposed:

Definition 39 *Feedback*

$$P^{\sigma} \triangleq (P \sqcap \gg) \circ I \vartriangle \top\!\!\top$$

□

Notice that there are no type restrictions on P.

Using interpretations in the model given in earlier subsections the following interpretation for the feedback construction is derived:

$$f\,\langle P^{\sigma}\rangle\, g \;\equiv\; f\,\langle P\rangle\,(g, f)$$

Because feedback is constructed from universally cupjunctive functions, it follows that feedback itself is universally cupjunctive. This is stated and proved in the next theorem:

Theorem 40 *universal cupjunctivity*

$$(\sqcup(P : P \in \mathcal{B} : P))^{\sigma} = \sqcup(P : P \in \mathcal{B} : P^{\sigma})$$

Proof:

$$
\begin{array}{cl}
 & (\sqcup(P : P \in \mathcal{B} : P))^{\sigma} \\
= & \qquad \{ \text{ Definition 39 } \} \\
 & (\sqcup(P : P \in \mathcal{B} : P) \sqcap \gg) \circ I \vartriangle \top\!\!\top \\
= & \qquad \{ \text{ universal cupjunctivity of } \sqcap\gg \text{ and of } \circ I \vartriangle \top\!\!\top \ \} \\
 & \sqcup(P : P \in \mathcal{B} : (P \sqcap \gg) \circ I \vartriangle \top\!\!\top) \\
= & \qquad \{ \text{ Definition 39 } \} \\
 & \sqcup(P : P \in \mathcal{B} : P^{\sigma})
\end{array}
$$

□

From the above it follows that the feedback operator $^{\sigma}$ is monotonic and $\perp\!\!\perp$-strict, i.e., $\perp\!\!\perp^{\sigma} = \perp\!\!\perp$. Other properties are the computation rules for feedback:

Theorem 41 *computation rules*

(1) $$F^{\sigma} = F \circ I \vartriangle F^{\sigma}$$

(2) $$P^{\sigma} = (P \sqcap \gg) \circ I \vartriangle P^{\sigma}$$

□

Notice the extra "$\sqcap\gg$" in the second statement.

A healthyness condition of a relational calculus for distributed program design should be that it contains the results of Kahn for deterministic processes, [7]. One property of the feedback operator should be its preservation of functionality. This is, in general, a false statement in the calculus introduced thus far.

$$\neg\forall(\, F \;::\; F^{\sigma} \text{ is a function}\,)$$

For example, take the function $\gg$. Then,

$$\begin{array}{ll} & \gg^{\sigma} \\ = & \quad \{ \text{ definition feedback 39 } \} \\ & (\gg \sqcap \gg) \circ I \vartriangle \top\!\top \\ = & \quad \{ \text{ cap is idempotent } \} \\ & \gg \circ\, I \vartriangle \top\!\top \\ = & \quad \{ \text{ split computation 9 } \} \\ & \top\!\top \sqcap \top\!\top \circ I \\ = & \quad \{ \; I \text{ is identity of composition; cap is idempotent } \} \\ & \top\!\top \end{array}$$

But, $\top\!\top$ is not a function, at least not in the interesting models. Another problem is that, in general, feedback does not preserve totality of its argument. In Section 4.1 an example of this undesired behaviour is given.

2.3 Points and extensionality

In this last section *points* are introduced. Points are constant total functional procs. They represent the elements f of the model.

Definition 42 *points*

$$\begin{array}{ll} & point.x \\ \triangleq & \\ & x = x \circ \top\!\top \;\wedge\; x \in I \leftarrow\!\dashv I \end{array}$$

□

The identifiers x and y range over points.

In the model the conjunct $x = x \circ \top\!\top$ is interpreted as:

$$f\,\langle x\rangle\, g \;\equiv\; f\,\langle x\rangle\, h$$

It represents the "constant" part of points. In words it says: no matter what the input is (g or h), the output is always the same (f).

Some properties for points used in this paper are:

Theorem 43

(1) $\quad x \circ x^{\cup} = x{<}$

(2) $\quad x \vartriangle I \circ P \;=\; x \vartriangle P \;=\; I \,\|\, P \circ x \vartriangle I$

(3) $\quad x \vartriangle I \circ (y \vartriangle I)^{\cup} \;=\; (x \circ y^{\cup}) \,\|\, I$

(4) $\quad \top\!\top \circ x = \top\!\top$

□

The first statement follows from the functionality of points and is in fact a property that is valid for all functions ($P \circ P^{\cup} = P<$ is even equivalent to stating that P is a function). The second and the third statement follow from the property that points are constant. The last property describes totality on I.

With points it is possible to formulate and axiomatise the principle of extensionality. In functional programming the mechanism of extensionality is used to prove that two functions are equal:

$$\forall(f,g \;::\; f=g \;\equiv\; \forall(x \;::\; f.x=g.x))$$

Moreover, the above equivalence *is* extensionality. In the relational algebra the above translates to the axiom:

Axiom 44 *Extensionality*

$$\forall(P,Q \;::\; P=Q \;\equiv\; \forall(x \;::\; P\circ x = Q\circ x))$$

□

This is the last axiom for the relational algebra $\mathcal{A}$. No more axioms on the algebra are needed to prove the results presented in this paper. There are many other possible formulations of extensionality, some of which are:

(1) $\qquad I = \sqcup(x \;::\; x \circ x^{\cup})$

(2) $\qquad \top\!\!\top = \sqcup(x \;::\; x)$

For an intensive exploration of points and extensionality in the relational calculus the reader is referred to [10]. It is the objective to try to avoid the principle of extensionality as much as possible, and do the calculations 'point-free'. Sometimes, proofs get shorter then. Moreover, the use of extensionality and points sometimes clouds the important aspects of a calculation. Though, we do not want to get 'religious' about using extensionality and points.

Brock and Ackerman point out in their paper [3] that the history model of deterministic networks, where processes are functions between input and output histories, cannot be extended to a history model of non-deterministic networks, where processes are relations. To solve the problems they introduce a network model they called *scenarios*. This model is the history model of non-deterministic networks restricted by a causality requirement on the input and output histories. This causality requirement captures information of timing aspects between input and output.

In Subsection 2.2.3 the problem of preservation of functionality by feedback emerged. In the spirit of Brock and Ackerman these problems are solved by adding time information. The introduction of time in the calculus is the main subject of the following chapter.

3 Back to the model

To extend the relational algebra presented in the previous chapter with constructions, the notion of *time* is introduced to solve the problems about functionality preservation of feedback. In fact, more details are imposed on the universe of elements.

To be more specific: the new universe is the set of functions $\mathcal{M} \leftarrow\!\!\dashv \mathcal{T}$. These functions are called *chronicles.* Here, $\mathcal{M}$ is some message domain which is only restricted in a very weak way: it is closed under cartesian product. $\mathcal{T}$ is some time domain, and will be the main subject of discussion in this chapter.

Now, the interpretation of $f\,\langle P\rangle\, g$ is: for some input chronicle g, which describes the input for every moment in time, a possible output chronicle is f, describing the output in time. There could be another output chronicle h for input chronicle g because nothing has been said about (non-)functionality of P.

The time domain will be the main subject of discussion in the next sections. The input and output chronicles are functional relations of another relational algebra $\mathcal{B}$. The reason for this choice is that it forces the person who makes calculations in the model to be precise and formal.

Firstly, in Section 3.1 the time domain is axiomatised. Secondly, the notion of archimedean functions is introduced in Section 3.2. Those functions will serve as a key to solving functionality and totality preservation of feedback. Finally, new procs in $\mathcal{A}$ are defined in Section 3.3 using the time constructions.

3.1 Time domain

Assume a relational algebra $(\mathcal{B}, \subseteq, \bullet, \mathcal{I}, ^{-1}, \ll, \gg)$ to satisfy the axioms of a relational algebra as introduced in the Chapter 2. The elements of $\mathcal{B}$ are called relations; the identifiers r and s range over $\mathcal{B}$. It is assumed that $(\mathcal{B}, \subseteq)$ is also complemented. The operator $\neg$ is used to denote a complemented relation. The join and meet operators in $\mathcal{B}$ are denoted by $\cup$ and $\cap$, respectively, with unit elements $\bot$ and $\top$. Additional axioms like De Morgan and Law of Contradiction hold in $\mathcal{B}$.

For split and parallel in $\mathcal{B}$ the notations $\blacktriangle$ and $\times$ are used; for left and right domains in $\mathcal{B}$ the notations $\prec$ and $\succ$ are used. Typing of relations in $\mathcal{B}$ will be the same as the typing of procs in $\mathcal{A}$. Relations below $\mathcal{I}$ are called *identities.* The identifiers a and b range over these identity relations. Without further comments, all results for $\mathcal{A}$ stated in the previous chapter will be used for $\mathcal{B}$ also.

In the algebra $\mathcal{B}$, the principle of extensionality will be used. Extensionality was introduced in Section 2.3.

The time domain $\mathcal{T}$ is axiomatised as follows. First, it is assumed that $\mathcal{T}$ is a (restricted) identity:

Axiom 45 *identity*

$\mathcal{T}$ is an identity relation: $\mathcal{T} \subseteq \mathcal{I}$

□

On this type $\mathcal{T}$ an order $<$ is imposed. For convenience the notation $>$ is used as an abbreviation for $<^{-1}$. The first axiom expressing transitivity of $<$ is:

Axiom 46 *transitivity*

$$< \bullet < \subseteq <$$

□

The second axiom on $<$ is one expressing irreflexivity:

Axiom 47 *irreflexivity*

$$< \cap \mathcal{I} = \bot$$

□

It follows trivially that $>$ is also transitive and irreflexivity:

Theorem 48 *transitivity*

(1) $\quad > \bullet > \subseteq >$

(2) $\quad > \cap \mathcal{I} = \bot$

□

The third axiom establishes the property that $<$ induces a chain on $\mathcal{T}$:

Axiom 49 *chain*

$$< \cup \mathcal{T} \cup > = \mathcal{T} \bullet \top \bullet \mathcal{T}$$

□

Alternative names for $\mathcal{T}$ being a chain are 'linearly ordered' and 'totally ordered'. In literature, the triple of Axioms 46, 47 and 49 is sometimes called a trichotomy. In the model the three axioms read:

(1) $\quad t\,\langle<\rangle\,t' \;\wedge\; t'\,\langle<\rangle\,t'' \;\Rightarrow\; t\,\langle<\rangle\,t''$

(2) $\quad t\,\langle<\rangle\,t' \;\Rightarrow\; t\neq t'$

(3) $\quad t\,\langle<\rangle\,t' \;\vee\; t=t' \;\vee\; t\,\langle>\rangle\,t'$ for all t, t in $\mathcal{T}$

A result of Axiom 49 is the typing of the orders:

Theorem 50 *typing*

(1) $\quad < \in \mathcal{T}\sim\mathcal{T}$

(2) $\quad > \in \mathcal{T}\sim\mathcal{T}$

Proof:

$$
\begin{array}{ll}
 & < \in \mathcal{T}\sim\mathcal{T} \\
= & \quad \{\ \text{Definition 21 and Theorem 22}\ \} \\
 & < \subseteq \mathcal{T}\bullet\top\bullet\mathcal{T} \\
= & \quad \{\ \text{Axiom 49}\ \} \\
 & < \subseteq <\cup\,\mathcal{T}\cup > \\
= & \quad \{\ \text{plat calculus}\ \} \\
 & \mathit{True}
\end{array}
$$

□

A model for these axioms is $\mathbb{Z}$ and the ordering less-then "<". A few other useful definitions are:

Definition 51

(1) $\quad \leqslant \;\triangleq\; <\cup\,\mathcal{T}$

(2) $\quad \geqslant \;\triangleq\; \leqslant^{-1}$

□

Next, it is proved that $\leqslant$ cancels out if it is composed with $<$:

Theorem 52

$$< \;=\; <\bullet\leqslant$$

Proof:

$$
\begin{array}{ll}
 & <\bullet\leqslant \\
= & \quad \{\ \text{Definition 51}\ \} \\
 & <\bullet(<\cup\,\mathcal{T}) \\
= & \quad \{\ \text{cupjunctivity of composition}\ \} \\
 & <\bullet< \;\cup\; <\bullet\mathcal{T} \\
= & \quad \{\ \text{Theorem 50 and Definition 21}\ \} \\
 & <\bullet< \;\cup\; < \\
= & \quad \{\ \text{Axiom 46; plat calculus}\ \} \\
 & <
\end{array}
$$

□

Notice that it is still possible that $\mathcal{T}$ is empty. That is, taking for $\mathcal{T}$ and $<$ the empty relation $\bot$ fulfills the Axioms 45, 46, 47 and 49.

Next, it is axiomatised that there are non-empty relations below $\mathcal{T}$, or more operational, there are moments of time in the time domain. These moments are denoted by t and t'. Their properties are defined as follows:

Definition 53 *moments in time*

$$t \text{ is a moment in time}$$
$$\triangleq$$
$$t \neq \bot \;\wedge\; t \bullet \top \bullet t \subseteq \mathcal{T}$$

□

Some properties for these moments in time are:

Theorem 54

(1) $\quad t \subseteq \mathcal{T}$

(2) $\quad t \in t \leftarrow\!\!\dashv t$

(3) $\quad t \bullet \top \bullet t = t$

□

It is assumed that there is a specific moment t_0 below $\mathcal{T}$. This relation t_0 is the minimal moment in time. It is the start of "everything", that is, there has been no input or input processing prior to t_0:

Axiom 55 *Big Bang*

There exists a moment in time, denoted by t_0, such that

$$< \bullet\, t_0 = \bot$$

□

For an extensive exploration of relations satisfying Definition 53, one is referred to [2]. In that paper the relations are dubbed "unit types" and are below I.

A corollary of Definition 53 is that $t \bullet \top$ is a point below $\mathcal{T}$. This allows extensional arguments over moments in time, for example $\mathcal{T} = \cup(\, t \;::\; t\,)$.

In the next section one more axiom will be added to the set of axioms defining $\mathcal{T}$.

3.2 Archimedean functions

To capture the notion of *progress in time* of a proc archimedean functions are introduced. They will be used to formalise the progress of a proc. First, it is defined what is meant by the transitive and reflexive closure of a relation:

Definition 56 *transitive and reflexive closure*

Let r^n, for all $n \geqslant 0$, be defined by:

$$r^0 \triangleq \mathcal{I}$$

$$r^{n+1} \triangleq r \bullet r^n$$

Then, the transitive and reflexive closure of r is defined as:

$$r* \;=\; \cup(\, n \;:\; n \geqslant 0 \;:\; r^n \,)$$

□

There are several equivalent definitions for the transitive and reflexive closure of relation r. For example, it could be defined as the least solution of the fixpoint equation in X: $X = \mathcal{I} \cup r \bullet X$. The definition given above is probably the most familiar one. The following definition for archimedeans is taken:

Definition 57 *archimedeans*

$$arch.\alpha$$

$$\triangleq$$

$$\alpha \in \mathcal{T} \leftarrow\!\!\dashv \mathcal{T} \;\wedge\; \mathcal{T} \subseteq {<} \bullet \alpha \;\wedge\; \alpha \bullet {<} \subseteq {<} \bullet \alpha \;\wedge\; {<} \bullet \alpha * = \mathcal{T} \bullet \top \bullet \mathcal{T}$$

□

Identifiers α and β range over archimedeans. The conjunct $\mathcal{T} \subseteq {<} \bullet \alpha$ can be interpreted as:

$$\forall(\, t \;::\; t < \alpha.t \,)$$

This expresses the progress.

To guarantee, for example, that archimedeans are closed under sequential composition, the property "increasing" is needed. This is exactly described by the conjunct $\alpha \bullet {<} \subseteq {<} \bullet \alpha$:

$$\forall(\, t, t' \;::\; t < t' \Rightarrow \alpha.t < \alpha.t' \,)$$

It is still possible, though, that there are accumulation points. To avoid this undesired behaviour the fourth conjunct is added. It can be interpreted as α being unbounded:

$$\forall(\, t, t' \;::\; \exists(\, n \;:\; n \geqslant 0 \;:\; t < \alpha^n.t' \,)\,)$$

From the axioms in the previous section and from the definition of archimedeans it does not follow that there really *exist* such archimedean functions in relational algebra $\mathcal{B}$. This has to be axiomatised:

Axiom 58 *existence of archimedeans*

$$\exists(\, \alpha \;::\; \mathit{True} \,)$$

□

Notice that the existence of archimedean functions implies that there exist no maximal elements in the time domain $\mathcal{T}$. This is formally described by the typing properties (keeping in mind that $<$ is irreflexive):

Theorem 59

(1) $\quad {<} \in \mathcal{T} \curvearrowleft\!\!\sim \mathcal{T}$

(2) $\quad {>} \in \mathcal{T} \sim\!\!\curvearrowright \mathcal{T}$

Proof:

$$\begin{array}{ll} & {<} \in \mathcal{T} \curvearrowleft\!\!\sim \mathcal{T} \\ = & \quad \{\ \text{Theorem 35 (1); reverse}\ \} \\ & {<} \in \mathcal{T} \sim \mathcal{T} \;\wedge\; {<} \bullet \top = \mathcal{T} \bullet \top \\ = & \quad \{\ \text{Theorem 50}\ \} \end{array}$$

$$
\begin{array}{ll}
 & < \bullet \top = \mathcal{T} \bullet \top \\
= & \quad \{\ \mathcal{I} \subseteq \alpha* \text{ so: } \top = \alpha* \bullet \top\ \} \\
 & < \bullet \alpha* \bullet \top = \mathcal{T} \bullet \top \\
\Leftarrow & \quad \{\ \text{compose with } \top;\ \mathcal{T} \neq \bot\text{: Cone Rule}\ \} \\
 & < \bullet \alpha* = \mathcal{T} \bullet \top \bullet \mathcal{T} \\
= & \quad \{\ \text{Definition 57}\ \} \\
 & \mathit{True}
\end{array}
$$

□

This concludes the discussion about archimedeans.

3.3 Equal up-to

In this section two new procs are defined; the definition of the first one, "pre-compose", is a stepping stone for the definition of the second proc, "equal up-to". The definitions will be at the point level, thereby diverging from the objective to reason at a point-free level. However, one should keep in mind that the following definitions will not be part of the final calculus. Only the results obtained in the last four chapters of this paper are to be axiomatised. The definitions and calculations in this section then merely serve as justifications for these axioms.

The set $\mathcal{A}$, introduced in Chapter 2, is assumed to be a set of relations, where the relations are sets of pairs of functions $\in \mathcal{B}$, total on $\mathcal{T}$. To be more precise: the set of relations $\mathcal{A}$ is $\mathcal{P}(\mathcal{D} \times \mathcal{D})$, where

$$\mathcal{D} \triangleq \mathcal{I} \hookleftarrow \mathcal{T}$$

The identifiers f, g and h range over $\mathcal{D}$.

A remark has to be made. In Subsection 2.1.4 the parallel operator was axiomatised and the point-wise definitions of the projections where given in the model. For example, $\ll$ was to be interpreted as $f\,\langle\ll\rangle\,(g,h) \;\equiv\; f = g$. Now, a problem emerges: the pair (g,h) does not have type $\mathcal{I} \hookleftarrow \mathcal{T}$. It is just an element of the product of relational algebras; it is of the type $(\mathcal{I} \hookleftarrow \mathcal{T}) \times (\mathcal{I} \hookleftarrow \mathcal{T})$. To avoid this problem an isomorphic mapping from $(\mathcal{I} \hookleftarrow \mathcal{T}) \times (\mathcal{I} \hookleftarrow \mathcal{T})$ to $(\mathcal{I} \times \mathcal{I}) \hookleftarrow \mathcal{T}$ is needed. Fortunately, this mapping is present. It is the split operator in $\mathcal{B}$: $\blacktriangle \in (\mathcal{I} \times \mathcal{I}) \hookleftarrow \mathcal{T} \simeq (\mathcal{I} \hookleftarrow \mathcal{T}) \times (\mathcal{I} \hookleftarrow \mathcal{T})$.

From now on, pairs in the model will be written using $\blacktriangle$. For example, the definition in the model of parallel now reads:

$$f \blacktriangle g\,\langle P \parallel Q \rangle\, h \blacktriangle i \;\equiv\; f\,\langle P \rangle\, h \wedge g\,\langle Q \rangle\, i$$

3.3.1 Pre-compose

At the moment, the set of instruments to compare two chronicles is very limited: I, $\ll$ and $\gg$. To be able to compare two chronicles, for example only on some interval of $\mathcal{T}$, the function "pre-compose", mapping relations in $\mathcal{B}$ to procs in $\mathcal{A}$, is defined.

All calculations in this subsection will be entirely on the level of the model. In the next subsection, where the main proc is defined, no calculations in the model will be made, because all the necessary calculations will have been done in this subsection. Pre-compose is defined as follows:

Definition 60 *pre-compose*

$$f\,\langle {}^{\circ}r \rangle\, g \;\equiv\; f \supseteq g \bullet r$$

□

Before continuing, we derive the following theorem, obtained when pre-compose is applied to an identity:

Theorem 61

$$f\,\langle{}^\circ a\rangle\,g \;\equiv\; f\bullet a = g\bullet a$$

Proof:

$$\begin{array}{ll} & f\,\langle{}^\circ a\rangle\,g \\ = & \quad\{\ \text{Definition 60}\ \} \\ & f \supseteq g\bullet a \\ = & \quad\{\ \Rightarrow\text{: compose with } a\text{, Corollary 17 (1); } \Leftarrow\text{: } \mathcal{I}\supseteq a\text{, monotonicity}\ \} \\ & f\bullet a \supseteq g\bullet a \\ = & \quad\{\ f \text{ and } g \text{ are total functions; } a\subseteq\mathcal{I}\ \} \\ & f\bullet a = g\bullet a \end{array}$$

□

As an example of this new operator take some moment in time $t\subseteq\mathcal{T}$ and consider the proc ${}^\circ t$. In the model the proc ${}^\circ t$ relates every two chronicles that are equal on moment t: $f\,\langle{}^\circ t\rangle\,g \;\equiv\; f\bullet t = g\bullet t$.

The list of properties that are needed in the sequel is long. Part of the verification is left out, but the reader should not hesitate in trying to prove the results.

Theorem 62

(1) $\quad {}^\circ\bot = \top\!\!\top$

(2) $\quad I \sqsubseteq {}^\circ a$

(3) $\quad {}^\circ\mathcal{T} = I$

(4) $\quad {}^\circ(a\bullet b) = {}^\circ a \circ {}^\circ b$

(5) $\quad ({}^\circ a)^\cup = {}^\circ a$

(6) $\quad a\subseteq b \;\Rightarrow\; {}^\circ b \sqsubseteq {}^\circ a$

Proof:

Only the first, fourth and sixth statement are proved. Because the fourth property is the most difficult one to prove, it is postponed. First:

$$\begin{array}{ll} & {}^\circ\bot = \top\!\!\top \\ = & \quad\{\ \text{definition of} = \text{in the model}\ \} \\ & \forall(f,g :: f\,\langle{}^\circ\bot\rangle\,g \;\equiv\; f\,\langle\top\!\!\top\rangle\,g) \\ = & \quad\{\ \text{Definition 60; definition of } \top\!\!\top \text{ in the model}\ \} \\ & \forall(f,g :: f\supseteq g\bullet\bot \;\equiv\; \mathit{True}) \\ = & \quad\{\ \text{predicate calculus; } \bot \text{ is zero of composition}\ \} \\ & \forall(f,g :: f\supseteq\bot) \\ = & \quad\{\ \bot \text{ is bottom element}\ \} \\ & \mathit{True} \end{array}$$

Sixth: assume $a\subseteq b$. Then:

$$\begin{array}{ll} & f\,\langle{}^\circ b\rangle\,g \\ = & \quad\{\ \text{Definition 60}\ \} \\ & f\supseteq g\bullet b \end{array}$$

$$
\begin{array}{ll}
\Rightarrow & \quad \{ \text{ assumption; monotonicity } \} \\
& f \supseteq g \bullet a \\
= & \quad \{ \text{ Definition 60 } \} \\
& f \langle {}^\circ a \rangle g
\end{array}
$$

And fourth:

$$
\begin{array}{ll}
& f \langle {}^\circ a \circ {}^\circ b \rangle g \\
= & \quad \{ \text{ definition } \circ \text{ in the model } \} \\
& \exists(h :: f \langle {}^\circ a \rangle h \wedge h \langle {}^\circ b \rangle g) \\
= & \quad \{ \text{ Definition 60 } \} \\
& \exists(h :: f \supseteq h \bullet a \wedge h \supseteq g \bullet b) \\
\Rightarrow & \quad \{ \text{ monotonicity } \} \\
& f \supseteq g \bullet b \bullet a \\
= & \quad \{ \text{ Corollary 17 (2); Definition 60 } \} \\
& f \langle {}^\circ (a \bullet b) \rangle g
\end{array}
$$

To get the other implication assume $f \supseteq g \bullet b \bullet a$:

$$
\begin{array}{ll}
& \exists(h :: f \supseteq h \bullet a \wedge h \supseteq g \bullet b) \\
= & \quad \{ \text{ Theorem 61 twice } \} \\
& \exists(h :: h \supseteq f \bullet a \wedge h \supseteq g \bullet b) \\
= & \quad \{ \text{ plat calculus } \} \\
& \exists(h :: h \supseteq f \bullet a \cup g \bullet b) \\
\Leftarrow & \quad \{ \text{ extend to some } h \in \mathcal{D} \text{ } \} \\
& f \bullet a \cup g \bullet b \text{ is a function} \\
= & \quad \{ \text{ relational calculus } \} \\
& g \bullet b \bullet a \bullet f^{-1} \subseteq \mathcal{I} \\
\Leftarrow & \quad \{ \text{ assumption; monotonicity } \} \\
& f \bullet f^{-1} \subseteq \mathcal{I} \\
= & \quad \{ \text{ Definition 24; } f \text{ is a function } \} \\
& \mathit{True}
\end{array}
$$

□

The function ${}^\circ\_$ satisfies a kind of junctivity property. It reflects the antimonotonic behaviour of ${}^\circ\_$ which already emerged in statement six of Theorem 62.

Theorem 63

$$
\sqcap(x : P.x : {}^\circ(e.x)) = {}^\circ(\cup(x : P.x : e.x))
$$

Proof:

$$
\begin{array}{ll}
& f \langle {}^\circ(\cup(x : P.x : e.x)) \rangle g \\
= & \quad \{ \text{ Definition 60 } \} \\
& f \supseteq g \bullet \cup(x : P.x : e.x) \\
= & \quad \{ \text{ cupjunctivity of composition } \} \\
& f \supseteq \cup(x : P.x : g \bullet (e.x)) \\
= & \quad \{ \text{ plat calculus } \} \\
& \forall(x : P.x : f \supseteq g \bullet (e.x)) \\
= & \quad \{ \text{ Definition 60 } \} \\
& \forall(x : P.x : f \langle {}^\circ(e.x) \rangle g) \\
= & \quad \{ \text{ definition } \sqcap \text{ in the model } \} \\
& f \langle \sqcap(x : P.x : {}^\circ(e.x)) \rangle g
\end{array}
$$

□

As a last useful property, the distribution of $^{\circ}a$ over the parallel operator $\|$ is mentioned:

Theorem 64

$$I \| I \circ {}^{\circ}a \circ I \| I \;=\; {}^{\circ}a \| {}^{\circ}a$$

Proof:
From the type restrictions $I \| I$ it follows that the input and output chronicles are pairs:

$$
\begin{array}{ll}
 & f \blacktriangle g \,\langle I \| I \circ {}^{\circ}a \circ I \| I \rangle\, h \blacktriangle i \\
= & \quad \{ \text{ definition } P \| Q \text{ in the model; } I \text{ is the identity } \} \\
 & f \blacktriangle g \,\langle {}^{\circ}a \rangle\, h \blacktriangle i \\
= & \quad \{ \text{ Definition 60 } \} \\
 & f \blacktriangle g \;\supseteq\; h \blacktriangle i \bullet a \\
= & \quad \{ \; a \text{ is an identity, so it is function: Theorem 28 (2) } \} \\
 & f \blacktriangle g \;\supseteq\; (h \bullet a) \blacktriangle (i \bullet a) \\
= & \quad \{ \; \Leftarrow\text{: monotonicity; } \Rightarrow\text{: Theorem 9 (3,4), } \top \bullet f = \top \bullet g \; \} \\
 & f \supseteq h \bullet a \;\wedge\; g \supseteq i \bullet a \\
= & \quad \{ \text{ Definition 60 } \} \\
 & f \,\langle {}^{\circ}a \rangle\, h \;\wedge\; g \,\langle {}^{\circ}a \rangle\, i \\
= & \quad \{ \text{ definition } P \| Q \text{ in the model } \} \\
 & f \blacktriangle g \,\langle {}^{\circ}a \| {}^{\circ}a \rangle\, h \blacktriangle i
\end{array}
$$

□

The stepping stone is in place. In the next subsection it is used to define the proc "equal up-to".

3.3.2 Equal up-to defined

To express that two chronicles are equal up-to a moment in time t the function equal up-to is defined, taking a relation in $\mathcal{B}$ to a proc in $\mathcal{A}$.

Definition 65 *equal up-to*

$${}^{\approx}r \triangleq {}^{\circ}((< \bullet r \bullet \top)_{\prec})$$

□

The function will mostly be applied to a moment in time. In the model it can be interpreted as:

$$f \,\langle {}^{\approx}t \rangle\, g \;\equiv\; \forall(\,t' \;:\; t' < t \;:\; f \bullet t' = g \bullet t'\,)$$

Notice that $(< \bullet r \bullet \top)_{\prec} \subseteq \mathcal{I}$. This makes it possible to translate all the results of the previous subsection about pre-compose to equal up-to. The following theorem states that ${}^{\approx}r$ is an equivalence relation (reflexive, idempotent and symmetric), satisfies a junctivity property, and distributes over the parallel operator.

Theorem 66

(1) $\quad I \sqsubseteq {}^{\approx}r$

(2) $\quad {}^{\approx}r \circ {}^{\approx}r = {}^{\approx}r$

(3) $\quad ({}^{\approx}r)^{\cup} = {}^{\approx}r$

(4) $\quad \sqcap(\,x \;:\; P.x \;:\; {}^{\approx}(E.x)\,) \;=\; {}^{\approx}(\cup(\,x \;:\; P.x \;:\; E.x\,))$

(5) $\quad I \| I \circ {}^{\approx}r \circ I \| I \;=\; {}^{\approx}r \| {}^{\approx}r$

(6) $\quad {}^{\approx}r \sqsupseteq {}^{\approx}r \| {}^{\approx}r$

Proof:

We only prove the second statement. The other results are direct translations of similar results for pre-compose, Theorems 62 and 64.

$$
\begin{array}{ll}
 & {}^{\approx}r \circ {}^{\approx}r \\
= & \quad \{ \text{ Definition 65 } \} \\
 & {}^{\circ}((< \bullet r \bullet \top)_{\prec}) \circ {}^{\circ}((< \bullet r \bullet \top)_{\prec}) \\
= & \quad \{ \ (< \bullet r \bullet \top)_{\prec} \subseteq \mathcal{I}: \text{ Theorem 62 (4) } \} \\
 & {}^{\circ}((< \bullet r \bullet \top)_{\prec} \bullet (< \bullet r \bullet \top)_{\prec}) \\
= & \quad \{ \ (< \bullet r \bullet \top)_{\prec} \subseteq \mathcal{I}: \text{ Corollary 17 (1) } \} \\
 & {}^{\circ}((< \bullet r \bullet \top)_{\prec}) \\
= & \quad \{ \text{ Definition 65 } \} \\
 & {}^{\approx}r
\end{array}
$$

□

The theorems that follow now are more specific applications of equal up-to on moments in time. The first statement is, of course, about the start of everything: t_0. It says that all chronicles are equal before Big Bang:

Theorem 67

$$
{}^{\approx}t_0 = \top\!\!\top
$$

Proof:

$$
\begin{array}{ll}
 & {}^{\approx}t_0 \\
= & \quad \{ \text{ Definition 65 } \} \\
 & {}^{\circ}((< \bullet t_0 \bullet \top)_{\prec}) \\
= & \quad \{ \text{ Axiom 55 } \} \\
 & {}^{\circ}((\bot \bullet \top)_{\prec}) \\
= & \quad \{ \ \bot \text{ is zero of composition } \} \\
 & {}^{\circ}(\bot_{\prec}) \\
= & \quad \{ \text{ domains are bottom strict } \} \\
 & {}^{\circ}\bot \\
= & \quad \{ \text{ Theorem 62 (1) } \} \\
 & \top\!\!\top
\end{array}
$$

□

Taking the expression ${}^{\approx}t \circ {}^{\approx}t'$ and interpreting it in the model, it seems to be equivalent to ${}^{\approx}t \sqcup {}^{\approx}t'$. To prove this without having recourse to the model two lemmas are used. The first lemma gives a sufficient condition to make sequential composition and cup coincide:

Lemma 68

$$
\begin{array}{ll}
 & {}^{\approx}r \circ {}^{\approx}s = {}^{\approx}r \sqcup {}^{\approx}s \\
\Leftarrow & \\
 & {}^{\approx}r \sqsubseteq {}^{\approx}s \ \vee \ {}^{\approx}s \sqsubseteq {}^{\approx}r
\end{array}
$$

Proof:
A theorem from relational calculus that will be used is:

$$
\begin{array}{ll}
 & P \circ Q = P \sqcup Q \\
\Leftarrow & \\
 & (I \sqsubseteq P \sqsubseteq Q \ \wedge \ Q \circ Q \sqsubseteq Q) \ \vee \ (I \sqsubseteq Q \sqsubseteq P \ \wedge \ P \circ P \sqsubseteq P)
\end{array}
$$

Then:

$$
\begin{array}{ll}
 & {}^{\approx}r \circ {}^{\approx}s = {}^{\approx}r \sqcup {}^{\approx}s \\
\Leftarrow & \qquad \{ \text{ above result from relational calculus } \} \\
 & (I \sqsubseteq {}^{\approx}r \sqsubseteq {}^{\approx}s \ \wedge\ {}^{\approx}s \circ {}^{\approx}s \sqsubseteq {}^{\approx}s) \ \vee\ (I \sqsubseteq {}^{\approx}s \sqsubseteq {}^{\approx}r \ \wedge\ {}^{\approx}r \circ {}^{\approx}r \sqsubseteq {}^{\approx}r) \\
= & \qquad \{ \text{ Theorem 66 (1,2) } \} \\
 & {}^{\approx}r \sqsubseteq {}^{\approx}s \vee {}^{\approx}s \sqsubseteq {}^{\approx}r
\end{array}
$$

□

The second lemma gives a sufficient condition when an equal up-to structure is included in another:

Lemma 69

$$
\begin{array}{ll}
 & {}^{\approx}r \sqsubseteq {}^{\approx}s \\
\Leftarrow & \\
 & \leqslant \bullet r \bullet \top \supseteq s \bullet \top
\end{array}
$$

Proof:

$$
\begin{array}{ll}
 & {}^{\approx}r \sqsubseteq {}^{\approx}s \\
= & \qquad \{ \text{ Definition 65 } \} \\
 & {}^{\circ}((< \bullet r \bullet \top)_{\prec}) \sqsubseteq {}^{\circ}((< \bullet s \bullet \top)_{\prec}) \\
\Leftarrow & \qquad \{ \text{ Theorem 62 (6) } \} \\
 & (< \bullet r \bullet \top)_{\prec} \supseteq (< \bullet s \bullet \top)_{\prec} \\
\Leftarrow & \qquad \{ \text{ monotonicity of domains } \} \\
 & < \bullet r \bullet \top \supseteq < \bullet s \bullet \top \\
\Leftarrow & \qquad \{ \text{ composition with } <\text{; Theorem 52 } \} \\
 & \leqslant \bullet r \bullet \top \supseteq s \bullet \top
\end{array}
$$

□

In Theorem 70 the expression $\leqslant \bullet t \bullet \top \supseteq t' \bullet \top$ is used. This can be interpreted as $t' \leqslant t$.

Theorem 70

(1) $\qquad {}^{\approx}t \circ {}^{\approx}t' = {}^{\approx}t \sqcup {}^{\approx}t'$

(2) $\qquad {}^{\approx}t \sqsubseteq {}^{\approx}t' \vee {}^{\approx}t' \sqsubseteq {}^{\approx}t$

(3) $\qquad {}^{\approx}(\alpha \bullet t) \sqsubseteq {}^{\approx}t$

Proof:
First and second: to prove these statements, we use properties about points. The details will be omitted.

$$
\begin{array}{ll}
 & {}^{\approx}t \circ {}^{\approx}t' = {}^{\approx}t \sqcup {}^{\approx}t' \\
\Leftarrow & \qquad \{ \text{ Lemma 68 } \} \\
 & {}^{\approx}t \sqsubseteq {}^{\approx}t' \vee {}^{\approx}t' \sqsubseteq {}^{\approx}t \\
\Leftarrow & \qquad \{ \text{ Lemma 69 } \} \\
 & (\leqslant \bullet t \bullet \top \supseteq t' \bullet \top) \ \vee\ (\leqslant \bullet t' \bullet \top \supseteq t \bullet \top) \\
= & \qquad \{ \text{ big step, using properties about points } \} \\
 & \leqslant \cup \geqslant \supseteq t \bullet \top \bullet t' \\
= & \qquad \{ \text{ extensionality closes the quantification over } t \text{ and } t' \ \} \\
 & \leqslant \cup \geqslant \supseteq \mathcal{T} \bullet \top \bullet \mathcal{T} \\
= & \qquad \{ \text{ Definition 51 and Axiom 49 } \} \\
 & \mathit{True}
\end{array}
$$

And third:

$$
\begin{array}{ll}
 & {}^{\approx}(\alpha \bullet t) \sqsubseteq {}^{\approx}t \\
\Leftarrow & \quad \{ \text{ Lemma 69 } \} \\
 & \leqslant \bullet \alpha \bullet t \bullet \top \supseteq t \bullet \top \\
= & \quad \{ \text{ Theorem 54 (1): } \mathcal{T} \bullet t = t\text{; Leibniz } \} \\
 & \leqslant \bullet \alpha \bullet t \bullet \top \supseteq \mathcal{T} \bullet t \bullet \top \\
\Leftarrow & \quad \{ \text{ compose with } t \bullet \top\text{; monotonicity of composition } \} \\
 & \leqslant \bullet \alpha \supseteq \mathcal{T} \\
\Leftarrow & \quad \{ \text{ Definition 51; monotonicity of composition } \} \\
 & < \bullet \alpha \supseteq \mathcal{T} \\
= & \quad \{ \text{ Definition 57 } \} \\
 & \mathit{True}
\end{array}
$$

□

In Theorem 66 the property $I \sqsubseteq {}^{\approx}r$ was derived. This implies for all $n \geqslant 0$ and for all t: $I \sqsubseteq {}^{\approx}(\alpha^n \bullet t)$. Or equivalently: $I \sqsubseteq \sqcap(n : n \geqslant 0 : {}^{\approx}(\alpha^n \bullet t))$ for all t. One may expect that the cap-structure actually *equals* the identity, keeping in mind the unboundedness of α. To prove this a lemma is needed, exploiting the unboundedness:

Lemma 71

$$ {}^{\approx}(\alpha * \bullet t) = I $$

Proof:

$$
\begin{array}{ll}
 & {}^{\approx}(\alpha * \bullet t) \\
= & \quad \{ \text{ Definition 65 } \} \\
 & {}^{\circ}((< \bullet \alpha * \bullet t \bullet \top)_{\prec}) \\
= & \quad \{ \text{ Definition 57 } \} \\
 & {}^{\circ}((\mathcal{T} \bullet \top \bullet \mathcal{T} \bullet t \bullet \top)_{\prec}) \\
= & \quad \{ \mathcal{T} \bullet t = t \neq \bot\text{: Cone Rule 10 } \} \\
 & {}^{\circ}((\mathcal{T} \bullet \top)_{\prec}) \\
= & \quad \{ \text{ Theorem 19; } \mathcal{T} \subseteq \mathcal{I}\text{: Theorem 16 and calculus } \} \\
 & {}^{\circ}\mathcal{T} \\
= & \quad \{ \text{ Theorem 62 (3) } \} \\
 & I
\end{array}
$$

□

Now, the main result follows easily:

Theorem 72

$$ I = \sqcap(n : n \geqslant 0 : {}^{\approx}(\alpha^n \bullet t)) $$

Proof:

$$
\begin{array}{ll}
 & \sqcap(n : n \geqslant 0 : {}^{\approx}(\alpha^n \bullet t)) \\
= & \quad \{ \text{ Theorem 66 (4) } \} \\
 & {}^{\approx}(\cup(n : n \geqslant 0 : \alpha^n \bullet t)) \\
= & \quad \{ \text{ cupjunctivity of composition } \} \\
 & {}^{\approx}(\cup(n : n \geqslant 0 : \alpha^n) \bullet t) \\
= & \quad \{ \text{ Definition 56 } \} \\
 & {}^{\approx}(\alpha * \bullet t) \\
= & \quad \{ \text{ Lemma 71 } \} \\
 & I
\end{array}
$$

□

Now, all the instruments are there to explain and formalise a property, called causality, that solves the preservation problems of feedback. In the next chapter a first attempt is made to formulate the causality property.

4 Causality

In Subsection 2.2.3 a problem was encountered: functionality was not preserved by feedback. In this chapter the notion of *causality* is introduced. This condition on procs should establish the following properties:

Firstly, feedback should preserve functionality and totality:

$$F^{\sigma} \text{ is a function } \Leftarrow \; caus.F$$

$$P^{\sigma} \in A \rightsquigarrow B \;\Leftarrow\; caus.P \;\wedge\; P \in A \rightsquigarrow B \,\|\, A$$

Secondly, causality itself should be preserved by all the composition operations, in particular by the feedback construction:

$$caus.(P^{\sigma}) \;\Leftarrow\; caus.P$$

Finally, the notion of causality should not be unnecessarily restricted. In the following section the notion of causality is defined and explained.

4.1 Causality; a first step

Why does feedback not preserve functionality? Remember the counterexample of the feedback of the function $\gg$ in Subsection 2.2.3: $\gg^{\sigma} = \top\!\!\top$. A problem with the projections is that they react, so to say, instantaneously on the input. This is not a reasonable property of any implementable process: every machine in the real world has some kind of *delay*. Moreover, such machines cannot base their present output on future input.

So, without loss of reasonability, the assumption is made that implementable processes base their present output only on past input. This property is described by *inertia*.

Now, more formally, the following definition of inertia is proposed:

Definition 73 *inertia*

$$\begin{array}{ll} & inert.P \\ \triangleq & \\ & \exists(\,\alpha \;::\; \forall(\,t \;::\; P \circ {\approx}t \circ P^{\cup} \sqsubseteq {\approx}(\alpha \bullet t)\,)\,) \end{array}$$

□

To give insight in this definition the following explanation at the point level is given:

$$\forall(\,f,g,h,i \;:\; h\langle P\rangle f \wedge i\langle P\rangle g \;:\; f\langle {\approx}t\rangle g \;\Rightarrow\; h\langle {\approx}(\alpha \bullet t)\rangle i\,)$$

This means in words: for any two possible input chronicles f and g that are equal up-to moment t, the two possible output chronicles '$P.f$' and '$P.g$' are equal up-to a *later* (because $\mathcal{T} \subseteq \; < \bullet\, \alpha$) moment $\alpha.t$. So, for any moment t, P does not react instantaneously, nor does its output depend on future input.

The property of inertia implies that the process is functional. This is shown as follows:

Theorem 74

$$inert.P \Rightarrow P \text{ is a function}$$

Proof:

$$
\begin{array}{ll}
& \exists(\alpha :: \forall(t :: P \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t))) \\
\Rightarrow & \quad \{ \text{ Theorem 66 (1) } \} \\
& \exists(\alpha :: \forall(t :: P \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t))) \\
= & \quad \{ \alpha \in \mathcal{T} \leftrightarrow \mathcal{T}\text{: forall } t \} \\
& \exists(\alpha :: \forall(n,t :: P \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha^n \bullet t))) \\
= & \quad \{ \text{ plat calculus } \} \\
& \exists(\alpha :: \forall(t :: P \circ P^{\cup} \sqsubseteq \sqcap(n :: {}^{\approx}(\alpha^n \bullet t)))) \\
= & \quad \{ \text{ Theorem 72 } \} \\
& \exists(\alpha :: \forall(t :: P \circ P^{\cup} \sqsubseteq I)) \\
= & \quad \{ \text{ predicate calculus; Axiom 58 } \} \\
& P \circ P^{\cup} \sqsubseteq I \\
= & \quad \{ \text{ Definition 24 } \} \\
& P \text{ is a function}
\end{array}
$$

□

A second problem is the non-preservation of totality, due to the fact that the domain can be 'incomplete'. The following example stresses this:

Example 75

Take $\mathcal{T} = \mathbb{N}$. Construct a sequence of chronicles $[f_0, f_1, \ldots)$ by the following process:

$$\forall(i : i \geqslant 0 : f_i \text{ starts with } i \text{ 0-s, and after that there are all 1-s})$$

Observe that $f_i = f_j \equiv i = j$. Then, define the process P and the interface A by:

$$\forall(g, i : i \geqslant 0 : f_{i+1} \langle P \rangle g \centerdot f_i \wedge f_i \langle A \rangle f_i)$$

Proc P is inert, witnessed by the archimedean function $\alpha.t = t + 1$. Moreover, P is total on $I \| A$. The wish is that P^σ is total on I. But, when P is put in a feedback loop, the resulting proc is $\perp\!\!\!\perp$. This is shown as follows, for all g and h:

$$
\begin{array}{ll}
& g \langle P^\sigma \rangle h \\
= & \quad \{ \text{ interpretation in the model of feedback } \} \\
& g \langle P \rangle h \centerdot g \\
= & \quad \{ f_i \neq f_{i+1}\text{: definition } P \} \\
& false \\
= & \quad \{ \text{ interpretation of } \perp\!\!\!\perp \text{ in the model } \} \\
& g \langle \perp\!\!\!\perp \rangle h
\end{array}
$$

The conclusion is that P^σ is not total on I.

One expects that the limit f_{lim} of the sequence, the chronicle with all 0-s, is the candidate for $f_{lim} \langle P^\sigma \rangle g$. So it follows that f_{lim} has to be in the domain A. A property of P that describes this 'completeness' of interface A is needed.

□

To capture the notion that for all sequences of chronicles in some domain also the limit of that sequence is in the domain, a new property *fullness* is formally defined as:

Definition 76 *fullness*

$$full.P$$

$$\triangleq$$

$$P \neq \perp\!\!\perp \ \wedge \ \sqcap(t :: \top\!\!\top \circ P \circ {}^{\approx}t) \sqsubseteq \top\!\!\top \circ P$$

□

The proc $\perp\!\!\perp$ is excluded from the set of full procs. This is done to avoid the occurrence of the conjunct $P \neq \perp\!\!\perp$ in almost all the important lemmas and theorems that will be proved in the following chapters. Notice that the inclusion in the second conjunct is actually an equality: because of ${}^{\approx}t \sqsupseteq I$, the other inclusion follows.

The definition of fullness is not restricted to interfaces A only: a consequence of relational calculus is $full.P \equiv full.(P>)$, and (right) domains are interfaces. So, the restriction to interfaces only is not necessary.

In the model the fullness of non-empty interface A reads, for all f:

$$\forall(t :: \exists(g : g \in A : g \bullet (< \bullet t)_{\prec} = f \bullet (< \bullet t)_{\prec})) \Rightarrow f \in A$$

Finally we come to the definition of causality. This property specifies a class of processes that will turn out to be very important.

Definition 77 *causality (1)*

$$caus.P$$

$$\triangleq$$

$$inert.P \wedge full.P$$

□

In the following chapter the main objective for introducing causality is shown: functionality and totality are preserved by feedback if the argument proc is causal.

5 Preservation of functionality and totality

The theorems that we want to prove are:

Theorem 78

$$P^{\sigma} \text{ is a function } \Leftarrow caus.P$$

□

For the totality:

Theorem 79

$$P^{\sigma} \in A \sim\!\!\!\!\!\rightharpoondown B \ \Leftarrow \ caus.P \ \wedge \ P \in A \sim\!\!\!\!\!\rightharpoondown B \,\|\, A$$

□

Remember that, according to Theorem 74, causality implies functionality. So, Theorem 78 expresses the desired functionality preservation of feedback.

It will not come as a surprise that some preliminary lemmas are needed before Theorems 78 and 79 can be proved. First, in Section 5.1 the notions of fixpoints and sections of procs are discussed in short. Then, the functionality and the totality preservation of feedback is proved in Sections 5.2 and 5.3.

5.1 Preliminaries

In the literature, the feedback operator is sometimes called the fixpoint operator because it forces (part of) the output to be equal to (part of) the input. This motivates our interest in fixpoints of procs. In the model, a chronicle f is a fixpoint of proc P iff $f\,\langle P\rangle\, f$. In terms of relational algebra the fixpoints are represented by $P \sqcap I$. The following abbreviation is used:

Definition 80 μP

$$\mu P \triangleq P \sqcap I$$

□

which immediately results in:

Theorem 81

(1) $\quad \mu P \sqsubseteq I$

(2) $\quad \mu P \sqsubseteq P$

□

Now, the use of points and extensionality enters the picture. In the lemmas that follow, the formulation $P \circ x \vartriangle I$ pops up. These forms are called *sections* of proc P; in this case it is a *left* section. The structure is well-known in functional programming as currying. For example, the binary function *Add* that adds two numbers has as a curried form (left section) the unary function $Add\, x$ that increments its argument by x. The corresponding relational formulation reads $Add \circ x \vartriangle I$.

The lemma below shows what happens if P^σ is applied to some input x. Here, x is a point and can be thought of as the representation in the algebra $\mathcal{A}$ of a chronicle f in $\mathcal{B}$. The lemma states that the feedback construction gives as result the fixed point of the section $P \circ x \vartriangle I$.

Lemma 82

$$\forall (x :: P^\sigma \circ x \;=\; \mu(P \circ x \vartriangle I) \circ \top\!\!\top)$$

Proof:

In the proof, the property is used that $x \vartriangle I$ is a function. This follows from functionality of x (Definition 42) and I (Theorem 38 (1)) and the functionality preservation of $\vartriangle$ (Theorem 29 (2)).

$$
\begin{array}{cl}
 & P^\sigma \circ x \\
= & \quad \{ \text{ Definition 39 } \} \\
 & (P \sqcap \gg) \circ I \vartriangle \top\!\!\top \circ x \\
= & \quad \{ \text{ Corollary 14 (2) } \} \\
 & (P \sqcap \gg) \circ x \vartriangle \top\!\!\top \\
= & \quad \{ \text{ Theorem 43 (2) } \} \\
 & (P \sqcap \gg) \circ x \vartriangle I \circ \top\!\!\top \\
= & \quad \{ \; x \vartriangle I \text{ is a function: Theorem 28 (1) } \} \\
 & (P \circ x \vartriangle I \;\sqcap\; \gg \circ\, x \vartriangle I) \;\circ\; \top\!\!\top \\
= & \quad \{ \text{ split computation 9 (4); Theorem 43 (4): } \top\!\!\top \circ x = \top\!\!\top \text{: } \} \\
 & (P \circ x \vartriangle I \;\sqcap\; I) \;\circ\; \top\!\!\top \\
= & \quad \{ \text{ Definition 80 } \} \\
 & \mu(P \circ x \vartriangle I) \;\circ\; \top\!\!\top
\end{array}
$$

□

The lemma above points out that the left section $P \circ x \vartriangle I$ is an interesting one. The next lemma shows that (left) sectioning preserves inertia of an inert proc.

Lemma 83 *inertia preservation of sectioning*

$$\forall(\, x \;::\; inert.(P \circ x \vartriangle I)\,) \;\Leftarrow\; inert.P$$

Proof:

It is shown that the archimedean function α which witnesses the inertia of P also witnesses the inertia of $P \circ x \vartriangle I$:

$$
\begin{array}{ll}
 & P \circ x \vartriangle I \circ {}^{\approx}t \circ (P \circ x \vartriangle I)^{\cup} \\
= & \quad \{ \text{ Theorem 43 (2); reverse through composition 2 } \} \\
 & P \circ I \| {}^{\approx}t \circ x \vartriangle I \circ (x \vartriangle I)^{\cup} \circ P^{\cup} \\
= & \quad \{ \text{ Theorem 43 (3) and (1) } \} \\
 & P \circ I \| {}^{\approx}t \circ x{<} \| I \circ P^{\cup} \\
= & \quad \{ \text{ parallel-parallel fusion Theorem 8 (2) } \} \\
 & P \circ x{<} \| {}^{\approx}t \circ P^{\cup} \\
\sqsubseteq & \quad \{ \text{ Theorems 20 (1) and 66 (1): } x{<} \sqsubseteq I \sqsubseteq {}^{\approx}t\text{; monotonicity of parallel } \} \\
 & P \circ {}^{\approx}t \| {}^{\approx}t \circ P^{\cup} \\
\sqsubseteq & \quad \{ \text{ Theorem 66 (6) } \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \\
\sqsubseteq & \quad \{ \; inert.P \text{ is assumed; Definition 73 } \} \\
 & {}^{\approx}(\alpha \bullet t)
\end{array}
$$

□

The lemma that follows shows that (left) sectioning preserves totality of the argument proc.

Lemma 84 *typing of sectioning*

$$\forall(\, x \;:\; B \circ x = x \;:\; P \circ x \vartriangle I \in A \sim A\,) \;\Leftarrow\; P \in A \sim B \| A$$

Proof:

The assumption on P translates to $A \circ P = P$ and $\top\!\top \circ B \| A \;=\; \top\!\top \circ P$, Theorem 35 (1).

$$
\begin{array}{ll}
 & P \circ x \vartriangle I \in A \sim A \\
= & \quad \{ \text{ Theorem 35 (1) } \} \\
 & A \circ P \circ x \vartriangle I = P \circ x \vartriangle I \;\wedge\; \top\!\top \circ P \circ x \vartriangle I = \top\!\top \circ A \\
= & \quad \{ \text{ assumption on } P\text{; predicate calculus } \} \\
 & \top\!\top \circ B \| A \circ x \vartriangle I = \top\!\top \circ A
\end{array}
$$

This is linearly proved:

$$
\begin{array}{ll}
 & \top\!\top \circ B \| A \circ x \vartriangle I \\
= & \quad \{ \text{ parallel-split fusion Theorem 8 (1) } \} \\
 & \top\!\top \circ (B \circ x) \vartriangle A \\
= & \quad \{ \text{ assumption } B \circ x = x \; \} \\
 & \top\!\top \circ x \vartriangle A \\
= & \quad \{ \text{ Theorem 36 } \} \\
 & \top\!\top \circ x \sqcap \top\!\top \circ A \\
= & \quad \{ \text{ Theorem 43 (4) } \} \\
 & \top\!\top \sqcap \top\!\top \circ A \\
= & \quad \{ \text{ plat calculus } \} \\
 & \top\!\top \circ A
\end{array}
$$

□

To prove that from the fullness of $B \| A$ the fullness of interface B follows, we need the following useful lemma.

Lemma 85

$$full.(P \| Q) \;\Rightarrow\; full.P \wedge full.Q$$

Proof:

The non-emptyness of P (and Q) follows straightforwardly from the non-emptyness of $P \| Q$ by the following rule from relational calculus:

$$P \| Q \neq \perp\!\!\!\perp \;\equiv\; P \neq \perp\!\!\!\perp \wedge Q \neq \perp\!\!\!\perp$$

For the second conjunct of the definition of fullness, the result from relational calculus

$$\ll \circ R \| S \circ I \vartriangle \top\!\!\!\top \;=\; R \;\Leftarrow\; S \neq \perp\!\!\!\perp$$

will be used. This result gives a way to introduce or remove $R \| S$. Then:

$$\begin{array}{ll}
 & \top\!\!\!\top \circ P \\
= & \quad \{\ Q \neq \perp\!\!\!\perp\text{: above result }\} \\
 & \top\!\!\!\top \circ \ll \circ P \| Q \circ I \vartriangle \top\!\!\!\top \\
= & \quad \{\ \text{Theorem 38 (6): } \top\!\!\!\top \circ \ll \;=\; \top\!\!\!\top \circ I \| I\ \} \\
 & \top\!\!\!\top \circ P \| Q \circ I \vartriangle \top\!\!\!\top \\
\sqsupseteq & \quad \{\ P \| Q \text{ is full; monotonicity of composition }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ P \| Q \circ {}^{\approx}t) \circ I \vartriangle \top\!\!\!\top \\
= & \quad \{\ \text{Theorem 38 (6): } \top\!\!\!\top \circ I \| I \;=\; \top\!\!\!\top \circ \ll\text{; parallel-split fusion 8 (1) }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ \ll \circ P \| Q \circ {}^{\approx}t) \circ I \| I \circ I \vartriangle \top\!\!\!\top \\
= & \quad \{\ I \| I \text{ is a function and } \mathcal{T} \text{ is non-empty: Theorem 28 (1); Theorem 66 (5) }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ \ll \circ P \| Q \circ {}^{\approx}t \| {}^{\approx}t) \circ I \vartriangle \top\!\!\!\top \\
= & \quad \{\ \text{parallel computation 23 (3) and parallel-parallel fusion 8 (2) }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ P \circ {}^{\approx}t \circ \ll \circ I \| (Q \circ {}^{\approx}t)) \circ I \vartriangle \top\!\!\!\top \\
\sqsupseteq & \quad \{\ \text{Theorem 66 (1); monotonicity }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ P \circ {}^{\approx}t \circ \ll \circ I \| Q) \circ I \vartriangle \top\!\!\!\top \\
\sqsupseteq & \quad \{\ \text{monotonicity of composition }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ P \circ {}^{\approx}t) \circ \ll \circ I \| Q \circ I \vartriangle \top\!\!\!\top \\
= & \quad \{\ Q \neq \perp\!\!\!\perp\text{: above result }\} \\
 & \sqcap(t :: \top\!\!\!\top \circ P \circ {}^{\approx}t)
\end{array}$$

□

The summary of all the previous results is a lemma that states the causality and typing of the left section $P \circ x \vartriangle I$:

Lemma 86

$$\begin{array}{ll}
 & \forall(x : B \circ x = x : caus.(P \circ x \vartriangle I) \;\wedge\; P \circ x \vartriangle I \in A \rightsquigarrow A) \\
\Leftarrow & \\
 & caus.P \;\wedge\; P \in A \rightsquigarrow B \| A
\end{array}$$

Proof:

For all x such that $B \circ x = x$:

$$\begin{array}{ll}
 & caus.(P \circ x \vartriangle I) \;\wedge\; P \circ x \vartriangle I \in A \rightsquigarrow A \\
= & \quad \{\ \text{typing of } P\text{: Lemma 84; Definition 77 }\}
\end{array}$$

$$
\begin{array}{ll}
& inert.(P \circ x \vartriangle I) \;\wedge\; full.(P \circ x \vartriangle I) \\
= & \quad \{ \text{ causality of } P \text{ implies } inert.P\text{: Lemma 83 } \} \\
& full.(P \circ x \vartriangle I) \\
= & \quad \{ \text{ typing of } P \text{ and Lemma 84 } \} \\
& full.A \\
\Leftarrow & \quad \{ \text{ Lemma 85 } \} \\
& full.(B \parallel A) \\
= & \quad \{ \text{ typing and fullness of } P \text{ } \} \\
& true
\end{array}
$$

□

This concludes the set of preliminary results.

5.2 Preservation of functionality

One wishes to show that, given a causal proc P, for any two chronicles f and g such that $f \langle P \rangle f$ and $g \langle P \rangle g$ it follows that $f = g$. In relational algebra this translates to $\mu P \circ \top\!\!\top \circ \mu P \sqsubseteq I$. This is stated and proved in the next lemma:

Lemma 87 *At most one fixpoint for inert P*

$$\mu P \circ \top\!\!\top \circ \mu P \sqsubseteq I \;\Leftarrow\; inert.P$$

Proof:

From the definition of *inert.P*, 73, one obtains that there exists an archimedean function α such that for all t:

$$
\begin{array}{ll}
& P \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
\Rightarrow & \quad \{ \; \mu P \sqsubseteq P\text{; monotonicity of reverse and composition } \} \\
& \mu P \circ {}^{\approx}t \circ (\mu P)^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
= & \quad \{ \text{ Theorem 81 (1): Theorem 16 (2) } \} \\
& \mu P \circ {}^{\approx}t \circ \mu P \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
\Rightarrow & \quad \{ \text{ compose with } \mu P\text{; Corollary 17 (1) } \} \\
& \mu P \circ {}^{\approx}t \circ \mu P \sqsubseteq \mu P \circ {}^{\approx}(\alpha \bullet t) \circ \mu P \\
= & \quad \{ \text{ induction } \} \\
& \forall(n \;:\; n \geqslant 0 \;:\; \mu P \circ {}^{\approx}t \circ \mu P \sqsubseteq \mu P \circ {}^{\approx}(\alpha^{n} \bullet t) \circ \mu P) \\
\Rightarrow & \quad \{ \text{ Theorem 81 (1); monotonicity of composition } \} \\
& \forall(n \;:\; n \geqslant 0 \;:\; \mu P \circ {}^{\approx}t \circ \mu P \sqsubseteq {}^{\approx}(\alpha^{n} \bullet t)) \\
= & \quad \{ \text{ plat calculus } \} \\
& \mu P \circ {}^{\approx}t \circ \mu P \sqsubseteq \sqcap(n \;:\; n \geqslant 0 \;:\; {}^{\approx}(\alpha^{n} \bullet t)) \\
= & \quad \{ \text{ Theorem 72 } \} \\
& \mu P \circ {}^{\approx}t \circ \mu P \sqsubseteq I \\
\Rightarrow & \quad \{ \text{ instantiate } t_0 \text{ for } t\text{; Theorem 67 } \} \\
& \mu P \circ \top\!\!\top \circ \mu P \sqsubseteq I
\end{array}
$$

□

With this result it is straightforward to prove the functionality of the feedback when the argument proc is a causal proc.

Proof of Theorem 78:

According to Definition 24 the proof obligation is $P^{\sigma} \circ (P^{\sigma})^{\cup} \sqsubseteq I$ which, by an extensional argument, is equivalent to

$$\forall(x \;::\; P^{\sigma} \circ x \circ (P^{\sigma} \circ x)^{\cup} \sqsubseteq I)$$

Notice that causality implies, by definition, inertia. So, for all points x:

$$
\begin{array}{ll}
& P^{\sigma} \circ x \circ (P^{\sigma} \circ x)^{\cup} \\
= & \quad \{ \text{ Lemma 82 } \} \\
& \mu(P \circ x \vartriangle I) \circ \top\!\!\top \circ (\mu(P \circ x \vartriangle I) \circ \top\!\!\top)^{\cup} \\
= & \quad \{ \text{ reverse through composition } \} \\
& \mu(P \circ x \vartriangle I) \circ \top\!\!\top \circ \top\!\!\top^{\cup} \circ (\mu(P \circ x \vartriangle I))^{\cup} \\
= & \quad \{ \ \top\!\!\top^{\cup} = \top\!\!\top; \text{ Theorem 3 } \} \\
& \mu(P \circ x \vartriangle I) \circ \top\!\!\top \circ (\mu(P \circ x \vartriangle I))^{\cup} \\
= & \quad \{ \text{ Theorem 81 (1): Theorem 16 (2) } \} \\
& \mu(P \circ x \vartriangle I) \circ \top\!\!\top \circ \mu(P \circ x \vartriangle I) \\
\sqsubseteq & \quad \{ \ \mathit{caus}.P\text{: Theorem 83 and Lemma 87 } \} \\
& I
\end{array}
$$

□

This concludes the discussion on preservation of functionality.

5.3 Preservation of totality

In this section the totality preservation is proved, Theorem 79. To prove totality of a causal proc P, it is first shown as an intermediate result that there exists a fixpoint for P, that is, $\mu P \neq \perp\!\!\perp$. This is the counterpart of Lemma 87.

Two general results about the typing of the feedback construction are derived. Then, the claim that causal procs have at least one fixpoint is verified. These results pave the way to the proof of Theorem 79.

First, it is shown that it is allowed to restrict the typing of the argument proc of the feedback construction.

Lemma 88 *type restriction*

$$P^{\sigma} = (P \circ I \,\|\, P{<})^{\sigma}$$

Proof:

$$
\begin{array}{ll}
& P^{\sigma} \\
= & \quad \{ \text{ Definition 39 } \} \\
& (P \sqcap \gg) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ Theorem 20 (3) } \} \\
& (P{<} \circ P \sqcap \gg) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \ P{<} \text{ is an interface: Theorem 16 (3) } \} \\
& (P \sqcap P{<} \circ \gg) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ parallel computation Theorem 23 (4) } \} \\
& (P \sqcap \gg \circ I \,\|\, P{<}) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \ I \,\|\, P{<} \text{ is an interface: Theorem 16 (3) } \} \\
& (P \circ I \,\|\, P{<} \sqcap \gg) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ Definition 39 } \} \\
& (P \circ I \,\|\, P{<})^{\sigma}
\end{array}
$$

□

This means that without loss of generality one may restrict proc P to type $A \sim B \,\|\, A$ for some interfaces A and B when proc P is the argument of feedback. Assuming this type restriction on P, the following can be said about the type of the feedback, P^{σ}:

Lemma 89 *typing*

$$P^{\sigma} \in A \sim B \;\Leftarrow\; P \in A \sim B \,\|\, A$$

Proof:

According to Definition 21 and Theorem 22 (3) the assumption on P translates to $P = A \circ P \circ B \| A$. The proof obligation then reads $P^{\sigma} = A \circ P^{\sigma} \circ B$.

$$
\begin{array}{ll}
 & A \circ P^{\sigma} \circ B \\
= & \quad \{ \text{ Definition 39 } \} \\
 & A \circ (P \sqcap \gg) \circ I \vartriangle \top\!\!\top \circ B \\
= & \quad \{ \text{ Corollary 17 (3); Corollary 14 (2) } \} \\
 & (A \circ P \sqcap A \circ \gg) \circ B \vartriangle \top\!\!\top \\
= & \quad \{ \text{ parallel computation Theorem 23 (4); parallel-split fusion Theorem 8 (1) } \} \\
 & (A \circ P \sqcap \gg \circ I \| A) \circ B \| I \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ Corollary 17 (3); parallel-parallel fusion Theorem 8 (2) } \} \\
 & (A \circ P \sqcap \gg \circ B \| A) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ Corollary 17 (3) } \} \\
 & (A \circ P \circ B \| A \sqcap \gg) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ assumption on } P \} \\
 & (P \sqcap \gg) \circ I \vartriangle \top\!\!\top \\
= & \quad \{ \text{ Definition 39 } \} \\
 & P^{\sigma}
\end{array}
$$

□

Then, the counterpart of Lemma 87 is proved: a causal proc, which satisfies an additional typing property, has at least one fixpoint. The candidate in the model for the fixpoint is suggested by Example 75.

Lemma 90 *At least one fixpoint for causal P*

$$\mu P \neq \perp\!\!\perp \;\Leftarrow\; caus.P \wedge P \in A \sim A$$

Proof:

This lemma is proved in the model. We will use the functionality of causal proc P. Let α witness the inertia of P. First, a sequence $[f_0, f_1, \ldots)$ in A is constructed using the function α: take $f_0 \in A$ (fullness of A guarantees the existence of chronicle f_0) and $f_{n+1} \langle P \rangle f_n$ for all n (functionality and typing of P guarantees the well-definedness of f_n). Then, the limit f_{lim} of this sequence is characterised by:

$$\forall (n \;:\; n \geqslant 0 \;:\; f_{lim} \langle {}^{\approx}(\alpha^n \bullet t_0) \rangle f_n)$$

The chronicle f_{lim} is well-defined. This follows from Theorem 72. f_{lim} is the candidate for μP: $f_{lim} \langle P \rangle f_{lim}$ or $f_{lim} = P.f_{lim}$. This is proved by showing that $P.f_{lim}$ satisfies the defining property of f_{lim}:

$$\forall (n \;:\; n \geqslant 0 \;:\; P.f_{lim} \langle {}^{\approx}(\alpha^n \bullet t_0) \rangle f_n)$$

First, it has to be assumed that f_{lim} is in the domain of P, but this is exactly described by the fullness of P. Then, for all n by case analysis:

$n = 0$:

$$
\begin{array}{ll}
 & P.f_{lim} \langle {}^{\approx}(\alpha^0 \bullet t_0) \rangle f_0 \\
= & \quad \{ \text{ Definition 56; identity } \} \\
 & P.f_{lim} \langle {}^{\approx} t_0 \rangle f_0 \\
= & \quad \{ \text{ Theorem 67 } \} \\
 & P.f_{lim} \langle \top\!\!\top \rangle f_0 \\
= & \quad \{ \text{ definition of } \top\!\!\top \text{ in the model } \} \\
 & \mathit{True}
\end{array}
$$

$n > 0$:

$$
\begin{array}{ll}
 & P.f_{lim}\,\langle{}^{\approx}(\alpha^n \bullet t_0)\rangle\, f_n \\
= & \quad \{\ n > 0\text{; Definition 56 }\} \\
 & P.f_{lim}\,\langle{}^{\approx}(\alpha \bullet \alpha^{n-1} \bullet t_0)\rangle\, f_n \\
\Leftarrow & \quad \{\ \text{causality of } P\text{: Definition 73 }\} \\
 & P.f_{lim}\,\langle P \circ {}^{\approx}(\alpha^{n-1} \bullet t_0) \circ P^{\cup}\rangle\, f_n \\
= & \quad \{\ \text{definition of } \circ \text{ in the model }\} \\
 & \exists(g,h\ ::\ P.f_{lim}\,\langle P\rangle\, g\ \wedge\ g\,\langle{}^{\approx}(\alpha^{n-1} \bullet t_0)\rangle\, h\ \wedge\ h\,\langle P^{\cup}\rangle\, f_n\,) \\
= & \quad \{\ \text{definition of } {}^{\cup} \text{ in the model }\} \\
 & \exists(g,h\ ::\ P.f_{lim}\,\langle P\rangle\, g\ \wedge\ g\,\langle{}^{\approx}(\alpha^{n-1} \bullet t_0)\rangle\, h\ \wedge\ f_n\,\langle P\rangle\, h\,) \\
\Leftarrow & \quad \{\ \text{predicate calculus }\} \\
 & P.f_{lim}\,\langle P\rangle\, f_{lim}\ \wedge\ f_{lim}\,\langle{}^{\approx}(\alpha^{n-1} \bullet t_0)\rangle\, f_{n-1}\ \wedge\ f_n\,\langle P\rangle\, f_{n-1} \\
= & \quad \{\ f_{lim} \in A\text{; definition limit } f_{lim}\text{; definition } f_n\ \} \\
 & \mathit{True}
\end{array}
$$

□

After all this preliminary work on typing and fixpoints, the proof of Theorem 79 can be presented.

Proof of Theorem 79:

The assumptions on P are: $caus.P$ and $P \in A \rightsquigarrow B \| A$. Then, the proof proceeds as follows:

$$
\begin{array}{ll}
 & P^{\sigma} \in A \rightsquigarrow B \\
= & \quad \{\ \text{typing; definition totality: 30 }\} \\
 & P^{\sigma} \in A \rightsquigarrow B\ \wedge\ \top\!\!\top \circ B \sqsubseteq \top\!\!\top \circ P^{\sigma} \\
= & \quad \{\ P \in A \rightsquigarrow B \| A\text{: Lemma 89 }\} \\
 & \top\!\!\top \circ B \sqsubseteq \top\!\!\top \circ P^{\sigma}
\end{array}
$$

This last expression is, by an extensional argument and making use of the Cone Rule 10, equivalent to

$$\forall(x\ :\ B \circ x = x\ :\ P^{\sigma} \circ x \neq \bot\!\!\bot\,)$$

Now, the proof continues for all x such that $B \circ x = x$:

$$
\begin{array}{ll}
 & P^{\sigma} \circ x \neq \bot\!\!\bot \\
= & \quad \{\ \text{Lemma 82 }\} \\
 & \mu(P \circ x \vartriangle I)\ \circ\ \top\!\!\top\ \neq\ \bot\!\!\bot \\
= & \quad \{\ \text{Cone Rule 10 and Theorem 3 }\} \\
 & \mu(P \circ x \vartriangle I)\ \neq\ \bot\!\!\bot \\
\Leftarrow & \quad \{\ \text{Theorem 90 }\} \\
 & caus.(P \circ x \vartriangle I)\ \wedge\ (P \circ x \vartriangle I) \in A \rightsquigarrow A \\
\Leftarrow & \quad \{\ B \circ x = x\text{: Lemma 86 }\} \\
 & caus.P\ \wedge\ P \in A \rightsquigarrow B \| A
\end{array}
$$

□

6 Preservation of causality

In this chapter an important aspect is handled: causality of several procs and preservation of causality by several constructions. First, the plat structure is tackled. Next, the emphasis will be on the monoid of composition. The third layer of the relational algebra, the reverse layer, will not cause much trouble. After that, the layer of parallel composition is considered. Finally, the preservation properties of feedback are investigated.

6.1 Lattice

In this section, the following theorem is proved:

Theorem 91

(1) $caus.(P \sqcup Q) \;\Leftarrow\; caus.P \;\wedge\; caus.Q \;\wedge\; \forall(t :: P{>} \circ {}^{\approx}t \circ Q{>} \sqsubseteq {}^{\approx}t \circ P^{\cup} \circ Q \circ {}^{\approx}t)$

(2) $\neg(caus.\perp\!\!\!\perp)$

(3) $caus.(P \sqcap Q) \;\Leftarrow\; caus.P \;\wedge\; caus.Q \;\wedge\; \perp\!\!\!\perp \neq P \circ Q^{\cup} \sqsubseteq I$

(4) $caus.\top\!\!\!\top \;\equiv\; \top\!\!\!\top \sqsubseteq I$

□

The large conjunct in the condition of the first statement (very) informally says: P and Q are not distinguishable on their common domain.

First the property of inertia is proved; after that, the proof for fullness is given. For the inertia it is assumed that *inert.P* is witnessed by α; it is also assumed that β witnesses the inertia of Q. The proof shows that $\downarrow \bullet\, \alpha_{\blacktriangle}\beta$ witnesses the inertia of $P \sqcup Q$. For all t:

$$
\begin{array}{ll}
 & (P \sqcup Q) \circ {}^{\approx}t \circ (P \sqcup Q)^{\cup} \sqsubseteq {}^{\approx}(\downarrow \bullet\, \alpha_{\blacktriangle}\beta \bullet t) \\
= & \quad \{ \text{ cupjunctivity of composition; reverse over cup } \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \sqcup P \circ {}^{\approx}t \circ Q^{\cup} \sqcup Q \circ {}^{\approx}t \circ P^{\cup} \sqcup Q \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}(\downarrow \bullet\, \alpha_{\blacktriangle}\beta \bullet t) \\
= & \quad \{ \text{ plat calculus } \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\downarrow \bullet\, \alpha_{\blacktriangle}\beta \bullet t) \;\wedge\; P \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}(\downarrow \bullet\, \alpha_{\blacktriangle}\beta \bullet t) \\
\wedge & Q \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\downarrow \bullet\, \alpha_{\blacktriangle}\beta \bullet t) \;\wedge\; Q \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}(\downarrow \bullet\, \alpha_{\blacktriangle}\beta \bullet t) \\
\Leftarrow & \quad \{ \text{ Lemma 130 and Theorem 70 (1); Corollary 131 } \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \;\wedge\; P \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \circ {}^{\approx}(\beta \bullet t) \\
\wedge & Q \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \circ {}^{\approx}(\beta \bullet t) \;\wedge\; Q \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}(\beta \bullet t) \\
= & \quad \{ \text{ inertia of } P \text{ and } Q\text{; reverse } \} \\
 & P \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \circ {}^{\approx}(\beta \bullet t) \\
\Leftarrow & \quad \{ \text{ Theorem 20 (3); inertia of } P \text{ and } Q \text{ } \} \\
 & P \circ P{>} \circ {}^{\approx}t \circ Q{>} \circ Q^{\cup} \sqsubseteq P \circ {}^{\approx}t \circ P^{\cup} \circ Q \circ {}^{\approx}t \circ Q^{\cup} \\
\Leftarrow & \quad \{ \text{ assumption and monotonicity of composition } \} \\
 & \mathit{true}
\end{array}
$$

For the non-emptyness we observe that:

$$
\begin{array}{ll}
 & P \sqcup Q \neq \perp\!\!\!\perp \\
= & \quad \{ \text{ relational calculus } \} \\
 & P \neq \perp\!\!\!\perp \vee Q \neq \perp\!\!\!\perp \\
= & \quad \{ \text{ fullness of } P \text{ or } Q \text{ } \} \\
 & \mathit{true}
\end{array}
$$

Now, the following calculation is sufficient to prove fullness of $P \sqcup Q$:

$$
\begin{array}{ll}
 & \top\!\!\!\top \circ (P \sqcup Q) \\
= & \quad \{ \text{ distribution of composition over cup } \} \\
 & \top\!\!\!\top \circ P \sqcup \top\!\!\!\top \circ Q \\
\sqsupseteq & \quad \{ \text{ fullness of } P \text{ and } Q \text{ } \} \\
 & \sqcap(t :: \top\!\!\!\top \circ P \circ {}^{\approx}t) \sqcup \sqcap(t' :: \top\!\!\!\top \circ Q \circ {}^{\approx}t') \\
= & \quad \{ \text{ distribution of cup over cap } \} \\
 & \sqcap(t, t' :: \top\!\!\!\top \circ P \circ {}^{\approx}t \sqcup \top\!\!\!\top \circ Q \circ {}^{\approx}t') \\
= & \quad \{ \text{ Theorem 70 (2): } {}^{\approx}t \sqsupseteq {}^{\approx}t' \vee {}^{\approx}t' \sqsupseteq {}^{\approx}t\text{; range disjunction } \}
\end{array}
$$

$$
\begin{array}{ll}
 & \sqcap(t,t' : {}^{\approx}t \sqsupseteq {}^{\approx}t' : \top\!\top \circ P \circ {}^{\approx}t \sqcup \top\!\top \circ Q \circ {}^{\approx}t') \sqcap \sqcap(t,t' : {}^{\approx}t' \sqsupseteq {}^{\approx}t : \top\!\top \circ P \circ {}^{\approx}t \sqcup \top\!\top \circ Q \circ {}^{\approx}t') \\
\sqsupseteq & \quad \{ \text{ monotonicity } \} \\
 & \sqcap(t' :: \top\!\top \circ P \circ {}^{\approx}t' \sqcup \top\!\top \circ Q \circ {}^{\approx}t') \sqcap \sqcap(t :: \top\!\top \circ P \circ {}^{\approx}t \sqcup \top\!\top \circ Q \circ {}^{\approx}t) \\
= & \quad \{ \text{ idempotency of cap } \} \\
 & \sqcap(t :: \top\!\top \circ P \circ {}^{\approx}t \sqcup \top\!\top \circ Q \circ {}^{\approx}t) \\
= & \quad \{ \text{ cupjunctivity } \} \\
 & \sqcap(t :: \top\!\top \circ (P \sqcup Q) \circ {}^{\approx}t)
\end{array}
$$

□

The fact that $\perp\!\!\perp$ is not causal follows straightforwardly from the observation that it is not full.

We continue with the causality preservation of cap. First, inertia is considered:

$$
\begin{array}{ll}
 & (P \sqcap Q) \circ {}^{\approx}t \circ (P \sqcap Q)^{\cup} \\
\sqsubseteq & \quad \{ \text{ monotonicity of composition } \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \\
\sqsubseteq & \quad \{ \ P \text{ is inert, so there exists a archimedean } \alpha \ \} \\
 & {}^{\approx}(\alpha \bullet t)
\end{array}
$$

That was easy. Now for the fullness:

$$
\begin{array}{ll}
 & P \sqcap Q = \perp\!\!\perp \\
= & \quad \{ \text{ Big step: Dedekind 12 } \} \\
 & P \circ Q^{\cup} \sqcap I = \perp\!\!\perp \\
= & \quad \{ \text{ assumption } P \circ Q^{\cup} \sqsubseteq I \ \} \\
 & P \circ Q^{\cup} = \perp\!\!\perp
\end{array}
$$

And this is a false statement according to the assumptions. The second conjunct of fullness is verified as follows:

$$
\begin{array}{ll}
 & \sqcap(t :: \top\!\top \circ (P \sqcap Q) \circ {}^{\approx}t) \\
\sqsubseteq & \quad \{ \text{ monotonicity of cap } \} \\
 & \sqcap(t :: \top\!\top \circ P \circ {}^{\approx}t) \sqcap \sqcap(t :: \top\!\top \circ Q \circ {}^{\approx}t) \\
\sqsubseteq & \quad \{ \text{ fullness of } P \text{ and } Q \text{ assumed } \} \\
 & \top\!\top \circ P \sqcap \top\!\top \circ Q \\
\sqsubseteq & \quad \{ \text{ below } \} \\
 & \top\!\top \circ (P \sqcap Q)
\end{array}
$$

And the postponed proof obligation is discharged as follows:

$$
\begin{array}{ll}
 & \top\!\top \circ (P \sqcap Q) \sqsupseteq \top\!\top \circ P \sqcap \top\!\top \circ Q \\
\Leftarrow & \quad \{ \text{ Dedekind 12 } \} \\
 & P \sqcap Q \sqsupseteq P \sqcap \top\!\top^{\cup} \circ \top\!\top \circ Q \\
= & \quad \{ \ \top\!\top^{\cup} \circ \top\!\top = \top\!\top; \text{ plat calculus } \} \\
 & Q \sqsupseteq P \sqcap \top\!\top \circ Q \\
\Leftarrow & \quad \{ \text{ Dedekind 12 } \} \\
 & I \sqsupseteq \top\!\top \sqcap P \circ Q^{\cup} \\
= & \quad \{ \text{ plat calculus } \} \\
 & I \sqsupseteq P \circ Q^{\cup} \\
= & \quad \{ \text{ assumption } \} \\
 & \mathit{true}
\end{array}
$$

□

The last statement of Theorem 91 expresses the causality of $\top\!\!\top$:

$$
\begin{array}{ll}
 & caus.\top\!\!\top \\
= & \quad \{\ \text{Definitions 77 and 76}\ \} \\
 & inert.\top\!\!\top \ \wedge\ \top\!\!\top \neq \bot\!\!\bot \ \wedge\ \sqcap(t :: \top\!\!\top \circ \top\!\!\top \circ {}^{\approx}t) \sqsubseteq \top\!\!\top \circ \top\!\!\top \\
= & \quad \{\ \text{Theorems 11 and 3; } \top\!\!\top \text{ is top element; Definition 73}\ \} \\
 & \exists(\alpha :: \forall(t :: \top\!\!\top \circ {}^{\approx}t \circ \top\!\!\top^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t))) \\
= & \quad \{\ \top\!\!\top^{\cup} = \top\!\!\top;\ I \sqsubseteq {}^{\approx}t \Rightarrow \top\!\!\top = \top\!\!\top \circ {}^{\approx}t\ \} \\
 & \exists(\alpha :: \forall(t :: \top\!\!\top \sqsubseteq {}^{\approx}(\alpha \bullet t))) \\
= & \quad \{\ \text{for all } t\ \} \\
 & \exists(\alpha :: \forall(t :: \forall(n : n \geqslant 0 : \top\!\!\top \sqsubseteq {}^{\approx}(\alpha^{n} \bullet t)))) \\
= & \quad \{\ \text{plat calculus}\ \} \\
 & \exists(\alpha :: \forall(t :: \top\!\!\top \sqsubseteq \sqcap(n : n \geqslant 0 : {}^{\approx}(\alpha^{n} \bullet t)))) \\
= & \quad \{\ \text{Theorem 72}\ \} \\
 & \exists(\alpha :: \forall(t :: \top\!\!\top \sqsubseteq I)) \\
= & \quad \{\ \text{predicate calculus; Axiom 58}\ \} \\
 & \top\!\!\top \sqsubseteq I
\end{array}
$$

□

Because the inclusion $\top\!\!\top \sqsubseteq I$ is highly undesirable, and is also independent of the axioms listed in Section 2.1, one could conclude that $\top\!\!\top$ is not causal. In Chapter 7 the causality of $\top\!\!\top$ will be the subject of investigation again.

6.2 Sequential composition

Theorem 92

(1) $\quad caus.(P \circ Q) \ \Leftarrow\ caus.P \ \wedge\ caus.Q \ \wedge\ P \in I \sim A \ \wedge\ Q \in A \sim I$

(2) $\quad caus.I \ \equiv\ \top\!\!\top \sqsubseteq I$

□

The preservation of causality by sequential composition is delightfully simple. The parallel composition will cause some more troubles.

First the property of inertia is proved; after that the proof for fullness is given. The assumption on P and Q is that they are inert, witnessed by some α and β, respectively. The calculation below shows that α witnesses the inertia of $P \circ Q$. For all t:

$$
\begin{array}{ll}
 & P \circ Q \circ {}^{\approx}t \circ (P \circ Q)^{\cup} \\
= & \quad \{\ \text{reverse through composition}\ \} \\
 & P \circ Q \circ {}^{\approx}t \circ Q^{\cup} \circ P^{\cup} \\
\sqsubseteq & \quad \{\ \text{inertia of } Q\text{; monotonicity}\ \} \\
 & P \circ {}^{\approx}(\beta \bullet t) \circ P^{\cup} \\
\sqsubseteq & \quad \{\ \text{Theorem 70 (3)}\ \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \\
\sqsubseteq & \quad \{\ \text{inertia of } P\ \} \\
 & {}^{\approx}(\alpha \bullet t)
\end{array}
$$

And for the fullness, first observe that $\top\!\!\top \circ P \circ Q = \top\!\!\top \circ Q$ by the typing assumptions on P and Q, Theorem 35 (1), and Definition 21. Then:

$$
\begin{array}{ll}
 & P \circ Q \neq \bot\!\!\bot \\
= & \quad \{\ \text{Cone Rule 10}\ \} \\
 & \top\!\!\top \circ P \circ Q \circ \top\!\!\top = \top\!\!\top \\
= & \quad \{\ \text{above observation}\ \}
\end{array}
$$

$$
\begin{array}{ll}
 & \top\!\!\top \circ Q \circ \top\!\!\top = \top\!\!\top \\
= & \quad \{ \text{ Cone Rule 10 } \} \\
 & Q \neq \perp\!\!\perp \\
\Leftarrow & \quad \{ \text{ fullness of Q } \} \\
 & \mathit{true}
\end{array}
$$

And

$$
\begin{array}{ll}
 & \sqcap(t :: \top\!\!\top \circ P \circ Q \circ {}^{\approx}t) \\
= & \quad \{ \text{ above observation } \} \\
 & \sqcap(t :: \top\!\!\top \circ Q \circ {}^{\approx}t) \\
\sqsubseteq & \quad \{ \text{ fullness of Q } \} \\
 & \top\!\!\top \circ Q \\
= & \quad \{ \text{ above observation } \} \\
 & \top\!\!\top \circ P \circ Q
\end{array}
$$

□

The typing requirements in the antecedent of Theorem 92 are needed to ensure that proc Q does not produce output for which P is not defined. Observe that in this case, the angelic sequential composition of P and Q coincides with the demonic sequential composition which was one of the objectives. This concludes the treatment of sequential composition.

For the (non-)causality of I the calculation proceeds as follows:

$$
\begin{array}{ll}
 & \mathit{caus}.I \\
= & \quad \{ \text{ Definition 77 and 76 } \} \\
 & \mathit{inert}.I \wedge I \neq \perp\!\!\perp \wedge \sqcap(t :: \top\!\!\top \circ I \circ {}^{\approx}t) \sqsubseteq \top\!\!\top \circ I \\
= & \quad \{ \text{ Theorem 11 (1); } I \text{ is identity of composition; } \top\!\!\top \text{ is top element; Definition 73 } \} \\
 & \exists(\alpha :: \forall(t :: {}^{\approx}t \sqsubseteq {}^{\approx}(\alpha \bullet t))) \\
= & \quad \{ \text{ induction } \} \\
 & \exists(\alpha :: \forall(t :: \forall(n : n \geqslant 0 : {}^{\approx}t \sqsubseteq {}^{\approx}(\alpha^{n} \bullet t)))) \\
= & \quad \{ \text{ plat calculus } \} \\
 & \exists(\alpha :: \forall(t :: {}^{\approx}t \sqsubseteq \sqcap(n : n \geqslant 0 : {}^{\approx}(\alpha^{n} \bullet t)))) \\
= & \quad \{ \text{ Theorem 72 } \} \\
 & \exists(\alpha :: \forall(t :: {}^{\approx}t \sqsubseteq I)) \\
= & \quad \{ \text{ predicate calculus; Axiom 58 } \} \\
 & \forall(t :: {}^{\approx}t \sqsubseteq I) \\
= & \quad \{ \Rightarrow\text{: instantiate } t \text{ to } t_0\text{: Theorem 67; } \Leftarrow\text{: } {}^{\approx}t \sqsubseteq \top\!\!\top\text{, transitivity } \} \\
 & \top\!\!\top \sqsubseteq I
\end{array}
$$

□

This concludes the proof of Theorem 92.

6.3 Reverse

The handling of the preservation of causality by reverse is very short: reverse completely frustrates the property of causality. Informally, this is motivated as follows: consider a causal proc P that delays its input by d time units. Then, the reversed proc $P^{\cup}$ can make use of input in a range of d time units in the future. This behaviour rules out the possibility that $P^{\cup}$ is inert.

6.4 Split and parallel

In this section the construction of parallel composition will be considered. The preservation of causality will get complicated.

Theorem 93

(1) $caus.(P \parallel Q) \Leftarrow caus.P \wedge caus.Q$

(2) $caus.(P \triangle Q) \Leftarrow caus.P \wedge caus.Q \wedge P \circ Q^{\cup} \neq \perp\!\!\perp$

(3) $caus.\ll \; \equiv \; \top\!\!\top \sqsubseteq I$

(4) $caus.\gg \; \equiv \; \top\!\!\top \sqsubseteq I$

□

In Appendix A, the function $\downarrow$ is investigated. It is the minimum operator on moments in time. Its definition is:

$$\downarrow \; \triangleq \; (\mathcal{T} \blacktriangle \geqslant \; \cup \; \geqslant \blacktriangle \mathcal{T})^{-1}$$

In the model, the interpretation reads:

$$\forall (t, t', t'' \; :: \; t \langle \downarrow \rangle (t', t'') \; \equiv \; (t = t' \wedge t \leqslant t'') \vee (t \leqslant t' \wedge t = t''))$$

With this operator, the causality preservation of parallel can be proved.

First the property of inertia is proved; after that the proof for fullness is given. The assumption on P and Q is that they are inert, witnessed by some α and β, respectively. For all moments in time t:

$P \parallel Q \circ {}^{\approx}t \circ (P \parallel Q)^{\cup}$

$=$ { reverse distributes over parallel }

$P \parallel Q \circ {}^{\approx}t \circ P^{\cup} \parallel Q^{\cup}$

$=$ { Theorem 66 (5) }

$P \parallel Q \circ {}^{\approx}t \parallel {}^{\approx}t \circ P^{\cup} \parallel Q^{\cup}$

$=$ { parallel-parallel fusion, Theorem 8 (2) }

$(P \circ {}^{\approx}t \circ P^{\cup}) \parallel (Q \circ {}^{\approx}t \circ Q^{\cup})$

$\sqsubseteq$ { inertia of P and Q; monotonicity of parallel }

${}^{\approx}(\alpha \bullet t) \parallel {}^{\approx}(\beta \bullet t)$

$\sqsubseteq$ { Corollary 131; monotonicity of parallel }

${}^{\approx}(\downarrow \bullet \alpha \blacktriangle \beta \bullet t) \parallel {}^{\approx}(\downarrow \bullet \alpha \blacktriangle \beta \bullet t)$

$\sqsubseteq$ { Theorem 66 (6) }

${}^{\approx}(\downarrow \bullet \alpha \blacktriangle \beta \bullet t)$

So, inertia of $P \parallel Q$ is witnessed by the archimedean function $\downarrow \bullet \alpha \blacktriangle \beta$.

For the fullness part, two lemmas are needed:

Lemma 94

$$full.(I \parallel I)$$

Proof:

The non-emptyness of $I \parallel I$ follows from the non-emptyness of I and the equivalence

$$P \parallel Q \neq \perp\!\!\perp \; \equiv \; P \neq \perp\!\!\perp \wedge Q \neq \perp\!\!\perp$$

For the second conjunct of fullness the following auxiliary result is proved with extensionality:

$$
\begin{array}{ll}
 & \sqcup(t :: (<\bullet t)_{\prec}) \\
= & \quad \{ \text{ cupjunctivity of domains and composition } \} \\
 & (< \bullet \sqcup(t :: t))_{\prec} \\
= & \quad \{ \text{ extensionality } \} \\
 & (<\bullet \mathcal{T})_{\prec} \\
= & \quad \{ \text{ Theorem 59; domains } \} \\
 & \mathcal{T}
\end{array}
$$

Then:

$$
\begin{array}{ll}
 & f\langle \sqcap(t :: \top\!\!\top \circ I \| I \circ {}^{\approx}t)\rangle g \\
= & \quad \{ \text{ interpretations in the model; Definition 65 } \} \\
 & \forall(t :: \exists(h,i :: h_{\blacktriangle}i \supseteq g \bullet (<\bullet t)_{\prec})) \\
\Rightarrow & \quad \{ \text{ monotonicity } \} \\
 & \forall(t :: \top_{\blacktriangle}\top \supseteq g \bullet (<\bullet t)_{\prec}) \\
= & \quad \{ \text{ plat calculus } \} \\
 & \top_{\blacktriangle}\top \supseteq \sqcup(t :: g \bullet (<\bullet t)_{\prec}) \\
= & \quad \{ \text{ cupjunctivity of composition; above result } \} \\
 & \top_{\blacktriangle}\top \supseteq g \bullet \mathcal{T} \\
= & \quad \{ \ g \in \mathcal{D}\text{: Definition 21 } \} \\
 & \top_{\blacktriangle}\top \supseteq g \\
= & \quad \{ \text{ relational calculus } \} \\
 & \exists(h,i :: h_{\blacktriangle}i = g) \\
= & \quad \{ \text{ interpretations in the model } \} \\
 & f\langle \top\!\!\top \circ I \| I\rangle g
\end{array}
$$

□

A corollary of this lemma is:

Lemma 95

$$\sqcap(t :: \top\!\!\top \circ P \| Q \circ {}^{\approx}t) \in I \rightsquigarrow I \| I$$

Proof:

$$
\begin{array}{ll}
 & \sqcap(t :: \top\!\!\top \circ P \| Q \circ {}^{\approx}t) \in I \rightsquigarrow I \| I \\
= & \quad \{ \text{ Definition 21: Theorem 22 (1) and (2) } \} \\
 & \sqcap(t :: \top\!\!\top \circ P \| Q \circ {}^{\approx}t) \sqsubseteq \top\!\!\top \circ I \| I \\
\Leftarrow & \quad \{ \ \top\!\!\top \circ P \| Q \sqsubseteq \top\!\!\top \circ I \| I \ \} \\
 & \sqcap(t :: \top\!\!\top \circ I \| I \circ {}^{\approx}t) \sqsubseteq \top\!\!\top \circ I \| I \\
= & \quad \{ \text{ Definition 76 and Lemma 94 } \} \\
 & \mathit{true}
\end{array}
$$

□

Now we are ready to prove the fullness of $P \| Q$, given the fullness of P and Q. The non-emptyness of $P \| Q$ follows from the non-emptyness of P and Q. For the other part the equality $\top\!\!\top \circ I \| I = \top\!\!\top \circ \top\!\!\top \| \top\!\!\top$ is used. Also, distributivity properties of composition over cap are used:

$$
\begin{array}{ll}
 & \sqcap(t :: \top\!\!\top \circ P \| Q \circ {}^{\approx}t) \\
= & \quad \{ \text{ Lemma 95 and Definition 21 } \} \\
 & \sqcap(t :: \top\!\!\top \circ P \| Q \circ {}^{\approx}t) \circ I \| I \\
= & \quad \{ \ I \| I \text{ is a function: Theorem 28 (1); Theorem 66 (5) } \} \\
 & \sqcap(t :: \top\!\!\top \circ P \| Q \circ {}^{\approx}t \| {}^{\approx}t) \\
= & \quad \{ \ \top\!\!\top \circ I \| I = \top\!\!\top \circ \top\!\!\top \| \top\!\!\top\text{; parallel-parallel fusion } \}
\end{array}
$$

$$
\begin{array}{ll}
 & \sqcap(t :: \top\!\!\top \circ (\top\!\!\top \circ P \circ {\approx}t) \parallel (\top\!\!\top \circ Q \circ {\approx}t)) \\
= & \quad \{ \text{ non-trivial distribution } \} \\
 & \top\!\!\top \circ \sqcap(t :: (\top\!\!\top \circ P \circ {\approx}t) \parallel (\top\!\!\top \circ Q \circ {\approx}t)) \\
= & \quad \{ \text{ capjunctivity of parallel composition } \} \\
 & \top\!\!\top \circ \sqcap(t :: (\top\!\!\top \circ P \circ {\approx}t)) \parallel \sqcap(t :: (\top\!\!\top \circ Q \circ {\approx}t)) \\
\sqsubseteq & \quad \{ \text{ fullness of } P \text{ and } Q\text{; monotonicity of parallel composition } \} \\
 & \top\!\!\top \circ (\top\!\!\top \circ P) \parallel (\top\!\!\top \circ Q) \\
= & \quad \{ \ \top\!\!\top \circ \top\!\!\top \parallel \top\!\!\top = \top\!\!\top \circ I \parallel I \ \} \\
 & \top\!\!\top \circ P \parallel Q
\end{array}
$$

□

We turn to the split operator. The extra conjunct $P \circ Q^{\cup} \neq \perp\!\!\perp$ in the second statement of Theorem 93 can be interpreted as the property that P and Q have some domain elements in common.

For the inertia part, we first show the lemma:

Lemma 96

$$
I \vartriangle I \circ {\approx}t \circ (I \vartriangle I)^{\cup} \sqsubseteq {\approx}t
$$

Proof:

$$
\begin{array}{ll}
 & I \vartriangle I \circ {\approx}t \circ (I \vartriangle I)^{\cup} \\
\sqsubseteq & \quad \{ \text{ distribution over split } \} \\
 & {\approx}t \vartriangle {\approx}t \circ (I \vartriangle I)^{\cup} \\
= & \quad \{ \text{ parallel-split fusion } \} \\
 & {\approx}t \parallel {\approx}t \circ I \vartriangle I \circ (I \vartriangle I)^{\cup} \\
\sqsubseteq & \quad \{ \ I \vartriangle I \text{ is a function: Definition 24 } \} \\
 & {\approx}t \parallel {\approx}t \\
\sqsubseteq & \quad \{ \text{ Theorem 66 (6) } \} \\
 & {\approx}t
\end{array}
$$

□

Then the inertia of $P \vartriangle Q$ follows from the inertia of $P \parallel Q$:

$$
\begin{array}{ll}
 & P \vartriangle Q \circ {\approx}t \circ (P \vartriangle Q)^{\cup} \\
= & \quad \{ \text{ parallel-split fusion; reverse over compositions } \} \\
 & P \parallel Q \circ I \vartriangle I \circ {\approx}t \circ (I \vartriangle I)^{\cup} \circ P^{\cup} \parallel Q^{\cup} \\
\sqsubseteq & \quad \{ \text{ Lemma 96 } \} \\
 & P \parallel Q \circ {\approx}t \circ P^{\cup} \parallel Q^{\cup} \\
\sqsubseteq & \quad \{ \text{ causality of } P \text{ and } Q\text{; Theorem 93 (1) } \} \\
 & {\approx}(\downarrow \bullet \alpha \centerdot \beta \bullet t)
\end{array}
$$

The fullness of $P \vartriangle Q$ follows from a calculation which is analogous to the proof of fullness of $P \parallel Q$. For the non-emptyness: $P \vartriangle Q \neq \perp\!\!\perp \equiv P \circ Q^{\cup} \neq \perp\!\!\perp$. And for the second conjunct of fullness:

$$
\begin{array}{ll}
 & \sqcap(t :: \top\!\!\top \circ P \vartriangle Q \circ {\approx}t) \\
\sqsubseteq & \quad \{ \ \top\!\!\top \circ I \parallel I = \top\!\!\top \circ \top\!\!\top \parallel \top\!\!\top\text{; distribution over split } \} \\
 & \sqcap(t :: \top\!\!\top \circ (\top\!\!\top \circ P \circ {\approx}t) \vartriangle (\top\!\!\top \circ Q \circ {\approx}t)) \\
= & \quad \{ \text{ non-trivial distribution } \} \\
 & \top\!\!\top \circ \sqcap(t :: (\top\!\!\top \circ P \circ {\approx}t) \vartriangle (\top\!\!\top \circ Q \circ {\approx}t)) \\
= & \quad \{ \text{ capjunctivity of split } \} \\
 & \top\!\!\top \circ \sqcap(t :: (\top\!\!\top \circ P \circ {\approx}t)) \vartriangle \sqcap(t :: (\top\!\!\top \circ Q \circ {\approx}t)) \\
\sqsubseteq & \quad \{ \text{ fullness of } P \text{ and } Q\text{; monotonicity of split } \} \\
 & \top\!\!\top \circ (\top\!\!\top \circ P) \vartriangle (\top\!\!\top \circ Q) \\
= & \quad \{ \ \top\!\!\top \circ \top\!\!\top \parallel \top\!\!\top = \top\!\!\top \circ I \parallel I \ \} \\
 & \top\!\!\top \circ P \vartriangle Q
\end{array}
$$

□

The proof of the (non-)causality of the two projections is very similar to the proof of the (non-)causality of $\top\!\!\top$ and I, so it is omitted.

6.5 Feedback

This section concerns the feedback operator. During the calculations, properties of points and the principles of extensionality and induction are used. However, one should keep in mind that only the final results are going to be axiomatised, in that way abstracting from the use of extensionality and induction.

Theorem 97

$$caus.(P^{\sigma}) \;\Leftarrow\; caus.P \;\wedge\; P \in A \sim B \| A$$

□

The proof of Theorem 97 requires a preliminary lemma, but, it is 'expensive' in the sense that the proof is not straightforward and the applicability of the result itself is very restricted. The lemma expresses for inert P and for proc Q:

$$f\langle Q\rangle g \Rightarrow \exists(h \,::\, f\langle {}^{\approx}t'\rangle h \wedge h\langle P\rangle g)$$
$$\Rightarrow$$
$$fix\,\langle P\rangle\, fix \;\wedge\; gix\,\langle Q\rangle\, gix \;\Rightarrow\; fix\,\langle {}^{\approx}t'\rangle\, gix$$

Or informally: if P and Q are indistinguishable up-to moment t' then the fixpoints of P and Q (if they exist) are indistinguishable up-to moment t'.

Lemma 98

$$\mu P \circ \top\!\!\top \circ \mu Q \sqsubseteq {}^{\approx}t'$$
$$\Leftarrow$$
$$inert.P \;\wedge\; Q \sqsubseteq {}^{\approx}t' \circ P$$

Proof:
First, the inclusion $\mu P \circ {}^{\approx}t \circ \mu Q \sqsubseteq {}^{\approx}(\alpha \bullet t) \sqcup {}^{\approx}t'$ is proved. Let the archimedean function α be a witness for the inertia of P. For all t:

$\mu P \circ {}^{\approx}t \circ \mu Q$
$\sqsubseteq$ { Theorem 81 (1,2) }
$P \circ {}^{\approx}t \circ Q^{\cup}$
$\sqsubseteq$ { $Q \sqsubseteq {}^{\approx}t' \circ P$ so $Q^{\cup} \sqsubseteq P^{\cup} \circ {}^{\approx}t'$ by reverse and Theorem 66 (3) }
$P \circ {}^{\approx}t \circ P^{\cup} \circ {}^{\approx}t'$
$\sqsubseteq$ { inertia of P }
${}^{\approx}(\alpha \bullet t) \circ {}^{\approx}t'$
$=$ { Theorem 70 (1) }
${}^{\approx}(\alpha \bullet t) \sqcup {}^{\approx}t'$

Then, the principle of induction is applied. The step is taken without further explanation to avoid a cumbersome calculation that would not contribute to the main calculation.

$\forall(n \,:\, n \geqslant 0 \,:\, \mu P \circ {}^{\approx}t \circ \mu Q \sqsubseteq {}^{\approx}(\alpha^{n} \bullet t) \sqcup {}^{\approx}t')$
$=$ { plat calculus }
$\mu P \circ {}^{\approx}t \circ \mu Q \sqsubseteq \sqcap(n \,:\, n \geqslant 0 \,:\, {}^{\approx}(\alpha^{n} \bullet t) \sqcup {}^{\approx}t')$
$=$ { capjunctivity of $\sqcup {}^{\approx}t'$ }
$\mu P \circ {}^{\approx}t \circ \mu Q \sqsubseteq \sqcap(n \,:\, n \geqslant 0 \,:\, {}^{\approx}(\alpha^{n} \bullet t)) \sqcup {}^{\approx}t'$
$=$ { Theorem 72 }

$$
\begin{array}{ll}
 & \mu P \circ {}^{\approx}t \circ \mu Q \sqsubseteq I \sqcup {}^{\approx}t' \\
= & \quad \{ \ I \sqsubseteq {}^{\approx}t'\text{: plat calculus } \} \\
 & \mu P \circ {}^{\approx}t \circ \mu Q \sqsubseteq {}^{\approx}t' \\
\Rightarrow & \quad \{ \text{ instantiate } t_0 \text{ for } t\text{; Theorem 67 } \} \\
 & \mu P \circ \top\!\!\top \circ \mu Q \sqsubseteq {}^{\approx}t'
\end{array}
$$

□

The seeming asymmetry in P and Q can be removed by applying reverse to the consequence of Lemma 98. The lemma is a vital step in the

Proof of Theorem 97:

First we will tackle the inertia part. The assumptions on P that can be used are $P \in A \rightsquigarrow B \,\|\, A$ and the inertia of P. The proof obligation is that there exists an archimedean function α such that for all moments t:

$$
\begin{array}{ll}
 & P^{\sigma} \circ {}^{\approx}t \circ (P^{\sigma})^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
= & \quad \{ \text{ Lemma 89; Definition 21 } \} \\
 & P^{\sigma} \circ B \circ {}^{\approx}t \circ (P^{\sigma} \circ B)^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t)
\end{array}
$$

which is, by an extensional argument, equivalent to: For all x, y such that $B \circ x = x$, $B \circ y = y$ and $x \circ y^{\cup} \sqsubseteq {}^{\approx}t$:

$$
\begin{array}{ll}
 & P^{\sigma} \circ x \circ y^{\cup} \circ (P^{\sigma})^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
= & \quad \{ \text{ reverse through composition; Lemma 82 } \} \\
 & \mu(P \circ x \vartriangle I) \circ \top\!\!\top \circ \mu(P \circ y \vartriangle I) \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
\Leftarrow & \quad \{ \text{ Lemma 98 } \} \\
 & \mathit{inert}.(P \circ y \vartriangle I) \ \wedge \ P \circ x \vartriangle I \sqsubseteq {}^{\approx}(\alpha \bullet t) \circ P \circ y \vartriangle I \\
= & \quad \{ \text{ assumption } \mathit{inert}.P\text{: Theorem 83 } \} \\
 & P \circ x \vartriangle I \sqsubseteq {}^{\approx}(\alpha \bullet t) \circ P \circ y \vartriangle I \\
= & \quad \{ \text{ Lemma 84: } \top\!\!\top \circ P \circ x \vartriangle I = \top\!\!\top \circ P \circ y \vartriangle I \ \} \\
 & P \circ x \vartriangle I \sqcap \top\!\!\top \circ P \circ y \vartriangle I \sqsubseteq {}^{\approx}(\alpha \bullet t) \circ P \circ y \vartriangle I \\
\Leftarrow & \quad \{ \text{ Dedekind 12 } \} \\
 & P \circ x \vartriangle I \circ (P \circ y \vartriangle I)^{\cup} \sqcap \top\!\!\top \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
= & \quad \{ \ \top\!\!\top \text{ is unit of cap; reverse through composition } \} \\
 & P \circ x \vartriangle I \circ (y \vartriangle I)^{\cup} \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
\Leftarrow & \quad \{ \text{ Theorem 43 (3) } \} \\
 & P \circ (x \circ y^{\cup}) \,\|\, I \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
\Leftarrow & \quad \{ \ x \circ y^{\cup} \sqsubseteq B \circ {}^{\approx}t \circ B \ \} \\
 & P \circ (B \circ {}^{\approx}t \circ B) \,\|\, I \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t) \\
\Leftarrow & \quad \{ \text{ Definition 15 and Theorem 66 (1,6): } (B \circ {}^{\approx}t \circ B) \,\|\, I \sqsubseteq {}^{\approx}t \ \} \\
 & P \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t)
\end{array}
$$

which follows from the inertia of P. The fullness part of causality of P^{σ} is delightfully simple:

$$
\begin{array}{ll}
 & \mathit{full}.P^{\sigma} \\
= & \quad \{ \text{ assumptions on } P\text{: Theorem 79; Theorem 35 (1) } \} \\
 & \mathit{full}.B \\
\Leftarrow & \quad \{ \text{ Lemma 85 } \} \\
 & \mathit{full}.(B \,\|\, A) \\
= & \quad \{ \text{ assumption on } P\text{: Theorem 35 (1) } \} \\
 & \mathit{full}.P
\end{array}
$$

□

This concludes the discussion on preservation properties of the feedback

7 Causality again

Theorem 74 shows that the Definition 77 of causality is only satisfied for functional procs. This chapter extends the notion of causality to arbitrary procs. Section 7.1 explains and formulates an extended definition of causality. The last two sections cover several preservation properties such as functionality and totality preservation.

7.1 Causality; the final solution

To define causality for procs, advantage is taken of the view that a relation is the union of all the deterministic relations that are included in that relation and that are defined on the same domain [2]. The notion of 'included' and 'same domain' is sometimes dubbed *refinement*:

Definition 99 *refinement*

$$P \leq Q$$

$$\triangleq$$

$$P \sqsubseteq Q \ \wedge \ \top\!\!\top \circ P = \top\!\!\top \circ Q$$

□

$P \leq Q$ is pronounced as "P refines Q". In [1], [6] and [9] the direction of the refinement symbol is reversed. Moreover, the notion mentioned in those papers is also a little weaker. The choice for the direction in the definition above originates from the ordering on procs, which is due to set inclusion in the model.

Theorem 100

(1) $\leq$ is a partial order

(2) $F \leq G \ \equiv \ F = G$

(3) $P \leq \top\!\!\top \ \equiv \ \top\!\!\top \circ P = \top\!\!\top$

□

In words $\top\!\!\top \circ P = \top\!\!\top$ means, according to Definition 30, P is total on I. Several operators from the relational algebra, such as $\sqcup$ and $\vartriangle$, preserve refinement. Other operators such as $\sqcap$, $\circ$, $^{\cup}$ and $^{\sigma}$ do not preserve refinement, due to the domain requirement in Definition 99. Additional typing information (or even causality information) of the arguments avoids this problem. Causality for arbitrary procs P is defined as follows:

Definition 101 *causality (2)*

$$causal.P$$

$$\triangleq$$

$$P = \sqcup(\, F \ : \ F \leq P \wedge caus.F \ : \ F\,) \ \wedge \ P \neq \perp\!\!\perp$$

□

A healthiness condition is that this definition, when applied to functions, reduces to Definition 77. This is proved as follows:

Theorem 102 *coincide*

$$causal.F \ \equiv \ caus.F$$

[2] This is an extensional argument.

Proof:

$$
\begin{array}{ll}
 & causal.F \\
= & \quad \{ \text{ Definition 101 } \} \\
 & F = \sqcup(G \ : \ G \leq F \wedge caus.G \ : \ G) \ \wedge \ F \neq \perp\!\!\perp \\
= & \quad \{ \text{ Theorem 100 (2); predicate calculus } \} \\
 & F = \sqcup(G \ : \ G = F \wedge caus.G \ : \ G) \ \wedge \ F \neq \perp\!\!\perp \ \wedge \ (caus.F \vee \neg(caus.F)) \\
= & \quad \{ \text{ Leibniz; } \wedge \text{ distributes over } \vee \ \} \\
 & \quad (F = \sqcup(G \ : \ G = F \wedge caus.F \ : \ F) \ \wedge \ F \neq \perp\!\!\perp \ \wedge \ caus.F) \\
 & \vee \ (F = \sqcup(G \ : \ G = F \wedge caus.F \ : \ F) \ \wedge \ F \neq \perp\!\!\perp \ \wedge \ \neg(caus.F)) \\
= & \quad \{ \text{ plat calculus } \} \\
 & (F = F \wedge F \neq \perp\!\!\perp \wedge caus.F) \ \vee \ (F = \perp\!\!\perp \wedge F \neq \perp\!\!\perp \wedge \neg(caus.F)) \\
= & \quad \{ \ caus.F \Rightarrow F \neq \perp\!\!\perp\text{; predicate calculus } \} \\
 & caus.F
\end{array}
$$

□

This discussion on causality continues by proving the functionality and totality preservation when a causal proc is put into a feedback loop.

7.2 Preservation of functionality and totality

The theorem proved in this section is:

Theorem 103

(1) $\quad F^{\sigma}$ is a function $\Leftarrow$ $causal.F$

(2) $\quad P^{\sigma} \in A \rightsquigarrow B \ \Leftarrow \ causal.P \ \wedge \ P \in A \rightsquigarrow B \| A$

□

Because all the work has been done in the previous sections, the proof of Theorem 103 can be given without further preparations. Firstly, functionality preservation:

$$
\begin{array}{ll}
 & F^{\sigma} \text{ is a function} \\
\Leftarrow & \quad \{ \text{ Theorem 78 } \} \\
 & caus.F \\
= & \quad \{ \text{ Theorem 102 } \} \\
 & causal.F
\end{array}
$$

□

And secondly, we handle the totality preservation. The fact is used that $F \leq P$ and $P \in A \rightsquigarrow B \| A$ imply $F \in A \rightsquigarrow B \| A$. Then:

$$
\begin{array}{ll}
 & P^{\sigma} \in A \rightsquigarrow B \\
= & \quad \{ \text{ Definitions 31 and 30 } \} \\
 & P^{\sigma} \in A \rightsquigarrow B \ \wedge \ \top\!\!\top \circ B \sqsubseteq \top\!\!\top \circ P^{\sigma} \\
= & \quad \{ \text{ typing of } P\text{: Lemma 89; } P \text{ is causal: Definition 101 } \} \\
 & \top\!\!\top \circ B \sqsubseteq \top\!\!\top \circ (\sqcup(F \ : \ F \leq P \wedge caus.F \ : \ F))^{\sigma} \\
= & \quad \{ \text{ cupjunctivity of feedback (Theorem 40) and } \top\!\!\top \circ \ \} \\
 & \top\!\!\top \circ B \sqsubseteq \sqcup(F \ : \ F \leq P \wedge caus.F \ : \ \top\!\!\top \circ F^{\sigma}) \\
= & \quad \{ \ F \in A \rightsquigarrow B \| A \text{ and } caus.F\text{: Theorem 79 and Theorem 35 (1) } \} \\
 & \top\!\!\top \circ B \sqsubseteq \sqcup(F \ : \ F \leq P \wedge caus.F \ : \ \top\!\!\top \circ B) \\
\Leftarrow & \quad \{ \text{ plat and predicate calculus } \} \\
 & \exists(F \ :: \ F \leq P \wedge caus.F) \\
= & \quad \{ \ causal.P \Rightarrow P \neq \perp\!\!\perp \ \} \\
 & true
\end{array}
$$

□

7.3 Preservation of causality

This section covers two things. First, the causality of $\top\!\!\!\top$ is proved (see Theorem 91 (4)). Secondly, the important composition constructions are considered with respect to their causality preservation properties.

7.3.1 Causality of $\top\!\!\!\top$

For proving the causality of $\top\!\!\!\top$, we need to divide this proc in functions that are causal. In Section 2.3 it is stated that extensionality equals the fact that $\top\!\!\!\top$ is the union of all points. So, first a lemma is proved, showing the causality of points:

Lemma 104

$$\forall(x :: caus.x)$$

Proof:

According to Definition 77, the inertia and fullness of points has to be shown:

$$
\begin{array}{ll}
 & inert.x \\
= & \quad \{ \text{ Definition 73 } \} \\
 & \exists(\alpha :: \forall(t :: x \circ {}^{\approx}t \circ x^{\cup} \sqsubseteq {}^{\approx}(\alpha \bullet t))) \\
\Leftarrow & \quad \{ \text{ Theorem 66 (1) } \} \\
 & \exists(\alpha :: \forall(t :: x \circ {}^{\approx}t \circ x^{\cup} \sqsubseteq I)) \\
= & \quad \{ \text{ predicate calculus; Axiom 58 } \} \\
 & \forall(t :: x \circ {}^{\approx}t \circ x^{\cup} \sqsubseteq I) \\
\Leftarrow & \quad \{ {}^{\approx}t \sqsubseteq \top\!\!\!\top\text{; predicate calculus } \} \\
 & x \circ \top\!\!\!\top \circ x^{\cup} \sqsubseteq I \\
= & \quad \{ \text{ Definition 42: } x \circ \top\!\!\!\top = x \text{ } \} \\
 & x \circ x^{\cup} \sqsubseteq I \\
= & \quad \{ \text{ Definition 42: a point is a function; Definition 24 } \} \\
 & True
\end{array}
$$

And for the fullness part:

$$
\begin{array}{ll}
 & full.x \\
= & \quad \{ \text{ Definition 76 } \} \\
 & x \neq \bot\!\!\!\bot \wedge \sqcap(t :: \top\!\!\!\top \circ x \circ {}^{\approx}t) \sqsubseteq \top\!\!\!\top \circ x \\
= & \quad \{ \text{ Cone Rule 10 } \} \\
 & \top\!\!\!\top \circ x \circ \top\!\!\!\top = \top\!\!\!\top \wedge \sqcap(t :: \top\!\!\!\top \circ x \circ {}^{\approx}t) \sqsubseteq \top\!\!\!\top \circ x \\
= & \quad \{ \text{ Theorem 43 (4) } \} \\
 & \top\!\!\!\top \circ \top\!\!\!\top = \top\!\!\!\top \wedge \sqcap(t :: \top\!\!\!\top \circ {}^{\approx}t) \sqsubseteq \top\!\!\!\top \\
= & \quad \{ \text{ Theorem 3; } \top\!\!\!\top \text{ is top element } \} \\
 & True
\end{array}
$$

□

This lemma paves the way to causality of $\top\!\!\!\top$. The principle of extensionality is used in proving this theorem.

Theorem 105

$$causal.\top\!\!\!\top$$

Proof:

$$
\begin{array}{ll}
 & causal.\top\!\!\top \\
= & \quad \{\ \text{Definition 101; } \top\!\!\top \text{ is top element }\ \} \\
 & \top\!\!\top \sqsubseteq \sqcup(F\ :\ F \leq \top\!\!\top \wedge caus.F\ :\ F) \wedge \top\!\!\top \neq \bot\!\!\bot \\
= & \quad \{\ \text{Theorem 100 (3); Theorem 11 (2)}\ \} \\
 & \top\!\!\top \sqsubseteq \sqcup(F\ :\ \top\!\!\top \circ F = \top\!\!\top\ \wedge\ caus.F\ :\ F) \\
\Leftarrow & \quad \{\ \text{Definition 42: a point is a function}\ \} \\
 & \top\!\!\top \sqsubseteq \sqcup(x\ :\ \top\!\!\top \circ x = \top\!\!\top\ \wedge\ caus.x\ :\ x) \\
= & \quad \{\ \text{Theorem 43 (4); Lemma 104}\ \} \\
 & \top\!\!\top \sqsubseteq \sqcup(x\ ::\ x) \\
= & \quad \{\ \text{equivalent form of Extensionality 44}\ \} \\
 & true
\end{array}
$$

□

Now, causality of all the basic procs in the relational algebra is treated.

7.3.2 Preservation of causality by several operators

The operators $\circ$, $\|$, $\vartriangle$ and $^\sigma$ are investigated with respect to their preservation properties of the general definition of causality.

Theorem 106

(1) $causal.(P \circ Q) \Leftarrow causal.P \wedge causal.Q \wedge P \in I \rightsquigarrow A \wedge Q \in A \rightsquigarrow I$

(2) $causal.(P \| Q) \Leftarrow causal.P \wedge causal.Q$

(3) $causal.(P \vartriangle Q) \Leftarrow causal.P \wedge causal.Q \wedge P \circ Q^{\cup} \neq \bot\!\!\bot$

(4) $causal.P^{\sigma} \Leftarrow causal.P \wedge P \in A \rightsquigarrow B \| A$

□

This theorem is a true generalisation of all the corresponding theorems in Chapter 6. Not surprisingly, these previous results are used in the proof of Theorem 106. First, sequential composition is tackled. The following fact, expressing preservation of refinement by sequential composition, is used:

$$
\begin{array}{ll}
 & F \circ G \leq P \circ Q \ \wedge\ F \in I \rightsquigarrow A \ \wedge\ G \in A \rightsquigarrow I \\
\Leftarrow & \\
 & F \leq P \ \wedge\ G \leq Q \ \wedge\ P \in I \rightsquigarrow A \ \wedge\ Q \in A \rightsquigarrow I
\end{array}
$$

Then:

$$
\begin{array}{ll}
 & P \circ Q \\
= & \quad \{\ causal.P \text{ and } causal.Q\ \} \\
 & \sqcup(F\ :\ F \leq P \wedge caus.F\ :\ F) \circ \sqcup(G\ :\ G \leq Q \wedge caus.G\ :\ G) \\
= & \quad \{\ \text{cupjunctivity of composition}\ \} \\
 & \sqcup(F, G\ :\ F \leq P \wedge G \leq Q \wedge caus.F \wedge caus.G\ :\ F \circ G) \\
\sqsubseteq & \quad \{\ \text{above fact; Theorem 92; monotonicity of cup}\ \} \\
 & \sqcup(F, G\ :\ F \circ G \leq P \circ Q\ \wedge\ caus.(F \circ G)\ :\ F \circ G) \\
\sqsubseteq & \quad \{\ \text{functionality preservation of composition 29 (1); monotonicity of cup}\ \} \\
 & \sqcup(F'\ :\ F' \leq P \circ Q\ \wedge\ caus.F'\ :\ F') \\
\sqsubseteq & \quad \{\ \text{monotonicity}\ \} \\
 & P \circ Q
\end{array}
$$

And for the non-emptyness:

$$
\begin{array}{ll}
& P \circ Q \neq \perp\!\!\!\perp \\
= & \quad \{ \text{ Cone Rule 10 } \} \\
& \top\!\!\!\top \circ P \circ Q \circ \top\!\!\!\top = \top\!\!\!\top \\
= & \quad \{ \text{ assumption on } P\text{: Theorem 35 (1) } \} \\
& \top\!\!\!\top \circ A \circ Q \circ \top\!\!\!\top = \top\!\!\!\top \\
= & \quad \{ \text{ assumption on } Q\text{: Theorem 35 (1) } \} \\
& \top\!\!\!\top \circ Q \circ \top\!\!\!\top = \top\!\!\!\top \\
= & \quad \{ \text{ Cone Rule 10 } \} \\
& Q \neq \perp\!\!\!\perp \\
= & \quad \{ \text{ causality of } Q \ \} \\
& \mathit{true}
\end{array}
$$

□

This concludes the treatment of sequential composition. Next, the construction of parallel composition will be considered. In the proof the property

$$F \parallel G \leq P \parallel Q \ \Leftarrow \ F \leq P \wedge G \leq Q$$

is used. Then:

$$
\begin{array}{ll}
& P \parallel Q \\
= & \quad \{ \ \mathit{causal}.P \text{ and } \mathit{causal}.Q \ \} \\
& \sqcup(F \ : \ F \leq P \wedge \mathit{caus}.F \ : \ F) \parallel \sqcup(G \ : \ G \leq Q \wedge \mathit{caus}.G \ : \ G) \\
= & \quad \{ \text{ cupjunctivity of parallel } \} \\
& \sqcup(F, G \ : \ F \leq P \wedge G \leq Q \wedge \mathit{caus}.F \wedge \mathit{caus}.G \ : \ F \parallel G) \\
\sqsubseteq & \quad \{ \text{ above fact; Theorem 93 (1); monotonicity of cup } \} \\
& \sqcup(F, G \ : \ F \parallel G \leq P \parallel Q \ \wedge \ \mathit{caus}.(F \parallel G) \ : \ F \parallel G) \\
\sqsubseteq & \quad \{ \text{ functionality preservation of parallel 29 (3); monotonicity of cup } \} \\
& \sqcup(F' \ : \ F' \leq P \parallel Q \ \wedge \ \mathit{caus}.F' \ : \ F') \\
\sqsubseteq & \quad \{ \text{ monotonicity } \} \\
& P \parallel Q
\end{array}
$$

For the non-emptyness the fact $P \parallel Q \neq \perp\!\!\!\perp \ \equiv \ P \neq \perp\!\!\!\perp \wedge Q \neq \perp\!\!\!\perp$ suffices.

□

We go on with split. In the proof the property

$$
\begin{array}{ll}
& F \vartriangle G \leq P \vartriangle Q \\
\Leftarrow & \\
& F \leq P \ \wedge \ G \leq Q
\end{array}
$$

is used.

$$
\begin{array}{ll}
& P \vartriangle Q \\
= & \quad \{ \ \mathit{causal}.P \text{ and } \mathit{causal}.Q \ \} \\
& \sqcup(F \ : \ F \leq P \wedge \mathit{caus}.F \ : \ F) \vartriangle \sqcup(G \ : \ G \leq Q \wedge \mathit{caus}.G \ : \ G) \\
= & \quad \{ \text{ cupjunctivity of split } \} \\
& \sqcup(F, G \ : \ F \leq P \wedge G \leq Q \wedge \mathit{caus}.F \wedge \mathit{caus}.G \ : \ F \vartriangle G) \\
\sqsubseteq & \quad \{ \text{ above fact; Theorem 93 (2); monotonicity of cup } \} \\
& \sqcup(F, G \ : \ F \vartriangle G \leq P \vartriangle Q \ \wedge \ \mathit{caus}.(F \vartriangle G) \ : \ F \vartriangle G) \\
\sqsubseteq & \quad \{ \text{ functionality preservation of split 29 (2); monotonicity of cup } \} \\
& \sqcup(F' \ : \ F' \leq P \vartriangle Q \ \wedge \ \mathit{caus}.F' \ : \ F') \\
\sqsubseteq & \quad \{ \text{ monotonicity } \} \\
& P \vartriangle Q
\end{array}
$$

For the non-emptyness the fact $P \triangle Q \neq \perp\!\!\perp \;\equiv\; P \circ Q^{\cup} \neq \perp\!\!\perp$ suffices.

□

Finally, there is just one theorem left to be proved.

$$
\begin{array}{ll}
 & P^{\sigma} \\
= & \quad \{ \ \textit{causal}.P\text{: Definition 101} \ \} \\
 & (\sqcup(F \ : \ F \leq P \wedge \textit{caus}.F \ : \ F))^{\sigma} \\
= & \quad \{ \ \text{cupjunctivity of feedback} \ \} \\
 & \sqcup(F \ : \ F \leq P \wedge \textit{caus}.F \ : \ F^{\sigma})
\end{array}
$$

The conjunction of the assumptions of the theorem, *causal*.P and $P \in A \rightsquigarrow B \| A$, together with the refinement $F \leq P$ and *caus*.F gives:

$$
\begin{array}{ll}
 & F \leq P \ \wedge \ \textit{causal}.P \ \wedge \ P \in A \rightsquigarrow B \| A \ \wedge \ \textit{caus}.F \\
= & \quad \{ \ F \leq P \ \wedge \ P \in A \rightsquigarrow B \| A \ \Rightarrow \ F \in A \rightsquigarrow B \| A \ \} \\
 & F \leq P \ \wedge \ \textit{causal}.P \ \wedge \ P \in A \rightsquigarrow B \| A \ \wedge \ \textit{caus}.F \ \wedge \ F \in A \rightsquigarrow B \| A \\
\Rightarrow & \quad \{ \ \text{Theorem 78; Definition 99; Theorems 103 (2), 79 and 97} \ \} \\
 & F^{\sigma} \text{ is functional} \ \wedge \ F \sqsubseteq P \ \wedge \ P^{\sigma} \in A \rightsquigarrow B \ \wedge \ F^{\sigma} \in A \rightsquigarrow B \ \wedge \ \textit{caus}.(F^{\sigma}) \\
\Rightarrow & \quad \{ \ \text{monotonicity of feedback; Theorem 35 (1)} \ \} \\
 & F^{\sigma} \text{ is functional} \ \wedge \ F^{\sigma} \sqsubseteq P^{\sigma} \ \wedge \ \top\!\!\top \circ F^{\sigma} = \top\!\!\top \circ P^{\sigma} \ \wedge \ \textit{caus}.(F^{\sigma}) \\
= & \quad \{ \ \text{Definition 99} \ \} \\
 & F^{\sigma} \text{ is functional} \ \wedge \ F^{\sigma} \leq P^{\sigma} \ \wedge \ \textit{caus}.(F^{\sigma})
\end{array}
$$

So, the proof is continued:

$$
\begin{array}{ll}
 & \sqcup(F \ : \ F \leq P \wedge \textit{caus}.F \ : \ F^{\sigma}) \\
\sqsubseteq & \quad \{ \ \text{above calculation; monotonicity of cup in the range} \ \} \\
 & \sqcup(F \ : \ F^{\sigma} \text{ is functional} \wedge F^{\sigma} \leq P^{\sigma} \wedge \textit{caus}.(F^{\sigma}) \ : \ F^{\sigma}) \\
\sqsubseteq & \quad \{ \ \text{plat calculus} \ \} \\
 & \sqcup(F' \ : \ F' \leq P^{\sigma} \wedge \textit{caus}.F' \ : \ F') \\
\sqsubseteq & \quad \{ \ \text{monotonicity} \ \} \\
 & P^{\sigma}
\end{array}
$$

The non-emptyness of P^{σ} is proved by contraposition:

$$
\begin{array}{ll}
 & P^{\sigma} = \perp\!\!\perp \\
= & \quad \{ \ \text{assumptions on } P\text{: Theorem 103 (2) and Theorem 35 (1)} \ \} \\
 & B = \perp\!\!\perp \\
\Rightarrow & \quad \{ \ \text{parallel composition is bottom strict} \ \} \\
 & B \| A = \perp\!\!\perp \\
= & \quad \{ \ P \in A \rightsquigarrow B \| A\text{: Theorem 35 (1)} \ \} \\
 & P = \perp\!\!\perp \\
= & \quad \{ \ \textit{causal}.P \ \} \\
 & \textit{false}
\end{array}
$$

□

This concludes the discussion on preservation properties.

8 Weak causality

In Theorem 92 (1) the inertia preservation of sequential composition was proved. The witness for inertia of $P \circ Q$ for inert P and Q used in the proof did not depend on the witness for inertia of Q. Of course, one can give the "symmetric" proof of the theorem and show that inertia of $P \circ Q$ is established by the witness of Q *only*.

8.1 Weak causality defined

The above discussion suggests that plain inertia of both arguments of sequential composition is really overkill. More specificly: the (new) assumption on inertia of Q in Theorem 92 (1)

$$\forall(t :: Q \circ {}^{\approx}t \circ Q^{\cup} \sqsubseteq {}^{\approx}t)$$

is sufficient to get the same result by an even shorter proof! This assumption on Q is dubbed *weak inertia* and formally defined as follows:

Definition 107 *weak inertia*

$$winert.P$$
$$\triangleq$$
$$\forall(t :: P \circ {}^{\approx}t \circ P^{\cup} \sqsubseteq {}^{\approx}t)$$

□

Remember that, informally, inertia expresses the fact that the proc does not react instantaneously, nor does is look into the future. The aspect of "instantaneously reaction" is dropped in the definition of weak inertia. This weaker formulation of inertia induces a weaker formulation of causality:

Definition 108 *weak causality*

(1) $wcaus.P \triangleq winert.P \wedge full.P$

(2) $wcausal.P \triangleq P = \sqcup(F : F \leq P \wedge wcaus.F : F) \wedge P \neq \perp\!\!\perp$

□

The following section summarises a number of properties concerning the new notion of weak causality.

8.2 Properties

The property of weak causality is still sufficient to guarantee functionality. Moreover, inertia implies weak inertia. Consequently, causality implies weak causality.

Lemma 109

(1) P is a function $\Leftarrow winert.P$

(2) $winert.P \Leftarrow inert.P$

(3) $wcaus.P \Leftarrow caus.P$

(4) $wcausal.P \Leftarrow causal.P$

□

The proof, being very straightforward, is omitted.

In Chapter 6 it was shown that several basic procs like I and the projection are not causal (assuming the reasonable axiom $\top\!\!\top \neq I$). It will not come as a surprise, though, that these functions are weakly causal.

The weak causality of the identity and the projections is stated and proved in the following lemma:

Lemma 110

(1) $wcaus.I$

(2) $wcaus.\ll$

(3) $wcaus.\gg$

(4) $wcaus.(I \vartriangle I)$

Proof:

Only the inertia part of the second statement is proved. Notice that the fullness part for the projections follows from Theorems 38 (5,6) and 35 (2), and Lemma 94. Also observe that Lemma 96 already proves that $I \vartriangle I$ is weakly inert. Now, weak inertia of the left projection is proved. For all t:

$$
\begin{array}{ll}
 & \ll \circ {}^{\approx}t \circ \ll^{\cup} \\
= & \quad \{ \text{ Theorem 38 (5): } \ll \; = \; \ll \circ I \| I \text{; reverse } \} \\
 & \ll \circ I \| I \circ {}^{\approx}t \circ I \| I \circ \ll^{\cup} \\
= & \quad \{ \text{ Theorem 66 (5) } \} \\
 & \ll \circ {}^{\approx}t \| {}^{\approx}t \circ \ll^{\cup} \\
= & \quad \{ \text{ Theorem 23 (3) } \} \\
 & {}^{\approx}t \circ \ll \circ I \| ({}^{\approx}t)\!> \circ \ll^{\cup} \\
\sqsubseteq & \quad \{ \; I \| ({}^{\approx}t)\!> \; \sqsubseteq I \| I \sqsubseteq I \; \} \\
 & {}^{\approx}t \circ \ll \circ \ll^{\cup} \\
\sqsubseteq & \quad \{ \text{ Theorem 38 (5): } \ll \text{ is a function, or } \ll \circ \ll^{\cup} \sqsubseteq I \; \} \\
 & {}^{\approx}t
\end{array}
$$

□

The procs I, $\ll$, $\gg$ and $I \vartriangle I$ are sometimes referred to by *plumbing*, because their only action is redirecting information.

With the notion of weak causality, Theorem 106 (1) can be strengthened by weakening the assumptions on the arguments.

Theorem 111

$$
\begin{array}{lll}
causal.(P \circ Q) & \Leftarrow & ((causal.P \wedge wcausal.Q) \vee (wcausal.P \wedge causal.Q)) \\
 & & \wedge \; P \in I \rightsquigarrow A \; \wedge \; Q \in A \sim I
\end{array}
$$

□

The proof of this theorem is almost an exact copy of the proofs of Theorems 92 (1) and 106 (1). An easy exercise is the preservation of weak causality by several compositions.

Theorem 112

(1) $wcausal.(P \circ Q) \;\Leftarrow\; wcausal.P \;\wedge\; wcausal.Q \;\wedge\; P \in I \rightsquigarrow A \;\wedge\; Q \in A \sim I$

(2) $wcausal.(P \| Q) \;\Leftarrow\; wcausal.P \;\wedge\; wcausal.Q$

(3) $wcausal.(P \vartriangle Q) \;\Leftarrow\; wcausal.P \;\wedge\; wcausal.Q \;\wedge\; P \circ Q^{\cup} \neq \perp\!\!\!\perp$

□

Weak causality of P^{σ} for weakly causal P is highly unlikely, because the fullness part depends on (plain) inertia of P. It is exactly this part that is dropped in the definition of weak causality. This concludes the theoretical discussion on causality, fullness and a lot of preservation properties.

9 In conclusion

This paper introduces a relational algebra together with several useful notions such as functionality, injectivity, totality and surjectivity. It has been shown that the class of causal processes is a very important one: a feedback loop preserves functionality and totality of its argument proc if it is causal. Furthermore, (weak) causality itself is preserved (under certain reasonable conditions) by the composition constructions of the relational algebra.

The result of all the calculations could be the axiomatisation of (weak) causality of the constants in the relational algebra:

Axiom 113 *Causality*

(1) $causal.\top\!\!\top$

(2) $wcausal.I$

(3) $wcausal.\ll$

(4) $wcausal.\gg$

□

the connection between causality and weak causality:

Axiom 114 *Causality implies weak causality*

$$wcausal.P \Leftarrow causal.P$$

□

the derivation rules of causality for several composition constructions:

Axiom 115 *Causality preservation*

(1) $causal.(P \circ Q) \Leftarrow ((causal.P \wedge wcausal.Q) \vee (wcausal.P \wedge causal.Q)) \wedge P \in I \rightsquigarrow A \wedge Q \in A \sim I$

(2) $causal.(P \,\|\, Q) \Leftarrow causal.P \wedge causal.Q$

(3) $causal.(P \vartriangle Q) \Leftarrow causal.P \wedge causal.Q \wedge P \circ Q^{\cup} \neq \perp\!\!\perp$

(4) $causal.P^{\sigma} \Leftarrow causal.P \wedge P \in A \rightsquigarrow B \,\|\, A$

□

the derivation rules of weak causality for several composition constructions:

Axiom 116 *Weak causality preservation*

(1) $wcausal.(P \circ Q) \Leftarrow wcausal.P \wedge wcausal.Q \wedge P \in I \rightsquigarrow A \wedge Q \in A \sim I$

(2) $wcausal.(P \,\|\, Q) \Leftarrow wcausal.P \wedge wcausal.Q$

(3) $wcausal.(P \vartriangle Q) \Leftarrow wcausal.P \wedge wcausal.Q \wedge P \circ Q^{\cup} \neq \perp\!\!\perp$

□

and finally, the properties it was all about:

Axiom 117 *Functionality and totality*

(1) F^σ is a function $\Leftarrow$ $causal.F$

(2) $P^\sigma \in A \rightsquigarrow B \Leftarrow causal.P \wedge P \in A \rightsquigarrow B \| A$
□

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A

In Chapter 6 the function $\downarrow$ was introduced to prove the inertia preservation of $\sqcup$ and $\|$. This appendix shows that the function $\downarrow \bullet\, \alpha_{\blacktriangle}\beta$ is archimedean for archimedeans α and β. The leading prose will be very limited.

Definition 118 *min*

$$\downarrow \triangleq (\mathcal{T}_{\blacktriangle}\geqslant \cup \geqslant_{\blacktriangle}\mathcal{T})^{-1}$$

□

The typing of $\downarrow$ obeys:

Lemma 119

$$\downarrow \in \mathcal{T} \sim \mathcal{T} \times \mathcal{T}$$

Proof:

In this proof, several typing rules known from relational calculus are used.

$$\begin{array}{ll}
 & \downarrow \in \mathcal{T} \sim \mathcal{T}\times\mathcal{T} \\
= & \qquad \{ \text{ Definition 118 } \} \\
 & (\mathcal{T}_{\blacktriangle}\geqslant \cup \geqslant_{\blacktriangle}\mathcal{T})^{-1} \in \mathcal{T} \sim \mathcal{T}\times\mathcal{T} \\
= & \qquad \{ \text{ reverse } \} \\
 & \mathcal{T}_{\blacktriangle}\geqslant \cup \geqslant_{\blacktriangle}\mathcal{T} \in \mathcal{T}\times\mathcal{T} \sim \mathcal{T} \\
\Leftarrow & \qquad \{ \text{ typing } \} \\
 & \mathcal{T}_{\blacktriangle}\geqslant \in \mathcal{T}\times\mathcal{T} \sim \mathcal{T} \ \wedge\ \geqslant_{\blacktriangle}\mathcal{T} \in \mathcal{T}\times\mathcal{T} \sim \mathcal{T} \\
\Leftarrow & \qquad \{ \text{ typing } \} \\
 & \mathcal{T} \in \mathcal{T}\sim\mathcal{T} \ \wedge\ \geqslant \in \mathcal{T}\sim\mathcal{T} \\
= & \qquad \{ \text{ Definition 51; typing } \} \\
 & \mathcal{T} \in \mathcal{T}\sim\mathcal{T} \ \wedge\ > \in \mathcal{T}\sim\mathcal{T} \\
= & \qquad \{ \ \mathcal{T}\subseteq\mathcal{I} \text{ and Definition 21; Theorem 50 (2) } \} \\
 & \mathit{true}
\end{array}$$

□

The fact that $\downarrow$ is a function follows from the fact that $<$ is antisymmetric, a property of all transitive and irreflexive orders.

Lemma 120

$\downarrow$ is a function

Proof:

The following result from relational calculus is used:

$$\begin{array}{ll}
 & (r \cup s) \text{ is a function} \\
\equiv & \\
 & r \bullet s^{-1} \subseteq \mathcal{I} \ \wedge\ r \text{ and } s \text{ are functions}
\end{array}$$

Then:

$$\begin{array}{ll}
 & \downarrow \text{ is a function} \\
= & \qquad \{ \text{ definition } \downarrow\text{; above; } (\mathcal{T}_{\blacktriangle}\geqslant)^{-1} \text{ and } (\geqslant_{\blacktriangle}\mathcal{T})^{-1} \text{ are functions } \} \\
 & (\mathcal{T}_{\blacktriangle}\geqslant)^{-1} \bullet \geqslant_{\blacktriangle}\mathcal{T} \subseteq \mathcal{I} \\
= & \qquad \{ \text{ Axiom 6; } \leqslant, \geqslant \in \mathcal{T}\sim\mathcal{T} \ \}
\end{array}$$

$$
\begin{array}{ll}
 & \leqslant \cap \geqslant \subseteq \mathcal{I} \\
= & \qquad \{ \text{ Definition 51 } \} \\
 & (< \cup \mathcal{T}) \cap (\mathcal{T} \cup >) \subseteq \mathcal{I} \\
= & \qquad \{ \ \mathcal{T} \subseteq \mathcal{I}\text{: monotonicity } \} \\
 & < \cap > \subseteq \mathcal{I} \\
= & \qquad \{ \ < \text{ is antisymmetric } \} \\
 & \mathit{true}
\end{array}
$$

□

For the totality part of $\downarrow$:

Lemma 121

$$\top \bullet \downarrow \; = \; \top \bullet \mathcal{T} \times \mathcal{T}$$

Proof:

$$
\begin{array}{ll}
 & \top \bullet \downarrow \; = \; \top \bullet \mathcal{T} \times \mathcal{T} \\
= & \qquad \{ \text{ reverse.} \} \\
 & \downarrow^{-1} \bullet \top \; = \; \mathcal{T} \times \mathcal{T} \bullet \top \\
= & \qquad \{ \text{ Definition 118; } \mathcal{I} \times \mathcal{I} \bullet \top \; = \; \top \blacktriangle \mathcal{I} \bullet \top \ \} \\
 & (\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet \top \; = \; \mathcal{T} \times \mathcal{T} \bullet \top \blacktriangle \mathcal{I} \bullet \top \\
= & \qquad \{ \text{ cupjunctivity of composition; product-split fusion 8 (1) } \} \\
 & \mathcal{T} \blacktriangle \geqslant \bullet \top \cup \geqslant \blacktriangle \mathcal{T} \bullet \top \; = \; (\mathcal{T} \bullet \top) \blacktriangle \mathcal{T} \bullet \top \\
= & \qquad \{ \ r \blacktriangle s \bullet \top \; = \; (r \bullet s^{-1}) \blacktriangle \mathcal{I} \bullet \top\text{; Definition 51 (2) } \} \\
 & \leqslant \blacktriangle \mathcal{T} \bullet \top \cup \geqslant \blacktriangle \mathcal{T} \bullet \top \; = \; (\mathcal{T} \bullet \top \bullet \mathcal{T}) \blacktriangle \mathcal{T} \bullet \top \\
= & \qquad \{ \text{ cupjunctivity properties of composition and split } \} \\
 & (\leqslant \cup \geqslant) \blacktriangle \mathcal{T} \bullet \top \; = \; (\mathcal{T} \bullet \top \bullet \mathcal{T}) \blacktriangle \mathcal{T} \bullet \top \\
= & \qquad \{ \text{ Definition 51 and Axiom 49 } \} \\
 & \mathit{true}
\end{array}
$$

□

The consequence of Lemmas 120 and 121 is the following:

Lemma 122 *type*

$$\downarrow \; \in \; \mathcal{T} \leftarrow\!\!\dashv \mathcal{T} \times \mathcal{T}$$

□

Moreover, by the functionality and totality preservation of composition:

Theorem 123

$$\forall (\alpha, \beta \ :: \ \downarrow \bullet \, \alpha \blacktriangle \beta \; \in \; \mathcal{T} \leftarrow\!\!\dashv \mathcal{T})$$

□

We go on with the augmenting property of $\downarrow \bullet \alpha \blacktriangle \beta$.

Lemma 124

$$\forall(r,s :: \downarrow \bullet r \blacktriangle s \;=\; (r \cap \leqslant \bullet s) \cup (\leqslant \bullet r \cap s))$$

Proof:

$$\begin{array}{ll}
 & \downarrow \bullet r \blacktriangle s \\
= & \quad \{ \text{ Definition 118 } \} \\
 & (\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T})^{-1} \bullet r \blacktriangle s \\
= & \quad \{ \text{ cupjunctivity of reverse and composition } \} \\
 & (\mathcal{T} \blacktriangle \geqslant)^{-1} \bullet r \blacktriangle s \cup (\geqslant \blacktriangle \mathcal{T})^{-1} \bullet r \blacktriangle s \\
= & \quad \{ \text{ Axiom 6 } \} \\
 & (\mathcal{T} \bullet r \cap \leqslant \bullet s) \cup (\leqslant \bullet r \cap \mathcal{T} \bullet s) \\
= & \quad \{ \text{ Theorem 16 (3) } \} \\
 & (r \cap \mathcal{T} \bullet \leqslant \bullet s) \cup (\mathcal{T} \bullet \leqslant \bullet r \cap s) \\
= & \quad \{ \leqslant \in \mathcal{T} \sim \mathcal{T} \} \\
 & (r \cap \leqslant \bullet s) \cup (\leqslant \bullet r \cap s)
\end{array}$$

□

This lemma paves the way to the next theorem:

Theorem 125 *augmenting*

$$\forall(\alpha,\beta :: \downarrow \bullet \alpha \blacktriangle \beta \subseteq >)$$

Proof:

$$\begin{array}{ll}
 & \downarrow \bullet \alpha \blacktriangle \beta \\
= & \quad \{ \text{ Lemma 124 } \} \\
 & (\alpha \cap \leqslant \bullet \beta) \cup (\leqslant \bullet \alpha \cap \beta) \\
\subseteq & \quad \{ \text{ plat calculus } \} \\
 & \alpha \cup \beta \\
\subseteq & \quad \{ \alpha \text{ and } \beta \text{ are augmenting } \} \\
 & > \cup > \\
= & \quad \{ \text{ idempotency } \} \\
 & >
\end{array}$$

□

To prove the fact that $\downarrow \bullet \alpha \blacktriangle \beta$ is increasing we need one lemma, expressing a kind of monotonicity of $\downarrow$ with respect to $<$:

Lemma 126

$$\downarrow \bullet < \times < \;=\; < \bullet \downarrow$$

Proof:

$$\begin{array}{ll}
 & \downarrow \bullet < \times < \\
= & \quad \{ \text{ definition product 7 } \} \\
 & \downarrow \bullet (< \bullet \ll) \blacktriangle (< \bullet \gg) \\
= & \quad \{ \text{ Lemma 124 } \} \\
 & (< \bullet \ll \cap \leqslant \bullet < \bullet \gg) \cup (\leqslant \bullet < \bullet \ll \cap < \bullet \gg) \\
= & \quad \{ \text{ Theorem 52 } \} \\
 & (< \bullet \ll \cap < \bullet \gg) \cup (< \bullet \ll \cap < \bullet \gg)
\end{array}$$

$=$ { idempotency cup }
$< \bullet \ll \cap < \bullet \gg$
$=$ { reverse; definition split 7 }
$(> \blacktriangle >)^{-1}$

After noticing that $< \bullet \downarrow = ((\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet >)^{-1}$ first:

$(\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet >$
$=$ { cupjunctivity }
$\mathcal{T} \blacktriangle \geqslant \bullet > \cup \geqslant \blacktriangle \mathcal{T} \bullet >$
$\subseteq$ { distribution over split; $> \in \mathcal{T} \sim \mathcal{T}$ }
$> \blacktriangle (\geqslant \bullet >) \cup (\geqslant \bullet >) \blacktriangle >$
$=$ { Theorem 52 }
$> \blacktriangle > \cup > \blacktriangle >$
$=$ { idempotency cup }
$> \blacktriangle >$

And second:

$> \blacktriangle > \subseteq (\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet >$
$=$ { $> \in \mathcal{T} \sim \mathcal{T}$ }
$(\mathcal{T} \bullet >) \blacktriangle (\mathcal{T} \bullet >) \subseteq (\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet >$
$=$ { parallel-split fusion 8 (1); Lemma 121: ${\downarrow}_{\succ} = \mathcal{T} \times \mathcal{T}$ }
${\downarrow}_{\succ} \bullet > \blacktriangle > \subseteq (\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet >$
$=$ { Definition 118; reverse }
$(\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T})_{\prec} \bullet > \blacktriangle > \subseteq (\mathcal{T} \blacktriangle \geqslant \cup \geqslant \blacktriangle \mathcal{T}) \bullet >$
$=$ { cupjunctivity of domains and composition }
$(\mathcal{T} \blacktriangle \geqslant)_{\prec} \bullet > \blacktriangle > \cup (\geqslant \blacktriangle \mathcal{T})_{\prec} \bullet > \blacktriangle > \subseteq \mathcal{T} \blacktriangle \geqslant \bullet > \cup \geqslant \blacktriangle \mathcal{T} \bullet >$

Then, continuing with the first disjunct:

$(\mathcal{T} \blacktriangle \geqslant)_{\prec} \bullet > \blacktriangle >$
$\subseteq$ { $r_{\prec} \subseteq r \bullet r^{-1}$ }
$\mathcal{T} \blacktriangle \geqslant \bullet (\mathcal{T} \blacktriangle \geqslant)^{-1} \bullet > \blacktriangle >$
$=$ { Axiom 6 }
$\mathcal{T} \blacktriangle \geqslant \bullet (\mathcal{T} \bullet > \cap \leqslant \bullet >)$
$\subseteq$ { $\mathcal{T} \subseteq \mathcal{I}$; plat calculus }
$\mathcal{T} \blacktriangle \geqslant \bullet >$

□

Then the fact that $\downarrow \bullet \alpha \blacktriangle \beta$ is increasing is easily proved:

Theorem 127 *increasing*

$$\forall(\alpha, \beta :: \downarrow \bullet \alpha \blacktriangle \beta \bullet < \subseteq < \bullet \downarrow \bullet \alpha \blacktriangle \beta)$$

Proof:

$\downarrow \bullet \alpha \blacktriangle \beta \bullet <$
$\subseteq$ { distribution over split }
$\downarrow \bullet (\alpha \bullet <) \blacktriangle (\beta \bullet <)$
$\subseteq$ { α and β are increasing }
$\downarrow \bullet (< \bullet \alpha) \blacktriangle (< \bullet \beta)$
$=$ { product-split fusion }
$\downarrow \bullet < \times < \bullet \alpha \blacktriangle \beta$
$=$ { Lemma 126 }
$< \bullet \downarrow \bullet \alpha \blacktriangle \beta$

□

The last proof obligation for $\downarrow \bullet \alpha\blacktriangle\beta$ being archimedean is its unboundedness:

Theorem 128 *unbounded*

$$\forall(\alpha,\beta \;::\; < \bullet (\downarrow \bullet \alpha\blacktriangle\beta)* \;=\; \mathcal{T}\bullet\top\bullet\mathcal{T}\,)$$

□

The rather long proof of Theorem 128 is omitted[3]. The corollary of Theorems 123, 125, 127 and 128 is the important theorem:

Theorem 129 *archimedean*

$$\forall(\alpha,\beta \;::\; arch.(\downarrow \bullet \alpha\blacktriangle\beta)\,)$$

□

So much for this archimedean property of $\downarrow$. The discussion is completed by the following lemma and corollary:

Lemma 130

$$\forall(r,s,t \;::\; {}^{\approx}(\downarrow \bullet r\blacktriangle s \bullet t) = {}^{\approx}(r\bullet t)\sqcup{}^{\approx}(s\bullet t)\,)$$

Proof:

Remember the conventions that r and s are arbitrary relations, and that t is a moment in time.

$$\begin{array}{ll}
 & {}^{\approx}(r\bullet t)\sqcup{}^{\approx}(s\bullet t) \\
= & \quad \{\ \text{Theorem 70 (1)}\ \} \\
 & {}^{\approx}(r\bullet t)\circ{}^{\approx}(s\bullet t) \\
= & \quad \{\ \text{Definition 65}\ \} \\
 & {}^{\circ}((<\bullet r\bullet t\bullet\top)_{\prec})\circ{}^{\circ}((<\bullet s\bullet t\bullet\top)_{\prec}) \\
= & \quad \{\ \text{Theorem 62 (4)}\ \} \\
 & {}^{\circ}((<\bullet r\bullet t\bullet\top)_{\prec}\bullet(<\bullet s\bullet t\bullet\top)_{\prec}) \\
= & \quad \{\ \text{Corollary 17 (2)}\ \} \\
 & {}^{\circ}((<\bullet r\bullet t\bullet\top)_{\prec}\cap(<\bullet s\bullet t\bullet\top)_{\prec}) \\
= & \quad \{\ \text{distribution property of domains from defining Galois connection}\ \} \\
 & {}^{\circ}((<\bullet r\bullet t\bullet\top\cap<\bullet s\bullet t\bullet\top)_{\prec}) \\
= & \quad \{\ t\bullet\top\ \text{is a function: Theorem 28 (1)}\ \} \\
 & {}^{\circ}(((<\bullet r\cap<\bullet s)\bullet t\bullet\top)_{\prec}) \\
= & \quad \{\ \text{below}\ \} \\
 & {}^{\circ}((<\bullet\downarrow\bullet r\blacktriangle s\bullet t\bullet\top)_{\prec}) \\
= & \quad \{\ \text{Definition 65}\ \} \\
 & {}^{\approx}(\downarrow\bullet r\blacktriangle s\bullet t)
\end{array}$$

Below:

$$\begin{array}{ll}
 & <\bullet\downarrow\bullet r\blacktriangle s \\
= & \quad \{\ \text{Lemma 126}\ \} \\
 & \downarrow\bullet<\times<\bullet r\blacktriangle s \\
= & \quad \{\ \text{parallel-split fusion 8 (1)}\ \} \\
 & \downarrow\bullet(<\bullet r)\blacktriangle(<\bullet s) \\
= & \quad \{\ \text{Lemma 124}\ \} \\
 & (<\bullet r\cap\leqslant\bullet<\bullet s)\cup(\leqslant\bullet<\bullet r\cap<\bullet s) \\
= & \quad \{\ \text{Theorem 52}\ \} \\
 & (<\bullet r\cap<\bullet s)\cup(<\bullet r\cap<\bullet s) \\
= & \quad \{\ \text{idempotency of cup}\ \} \\
 & <\bullet r\cap<\bullet s
\end{array}$$

□

[3]The proof was given by L. Meertens.

With the straightforward corollary:

Corollary 131

$$\forall(\alpha,\beta,t\ ::\ {}^{\approx}(\downarrow\bullet\alpha\blacktriangle\beta\bullet t)\sqsupseteq{}^{\approx}(\alpha\bullet t)\ \wedge\ {}^{\approx}(\downarrow\bullet\alpha\blacktriangle\beta\bullet t)\sqsupseteq{}^{\approx}(\beta\bullet t))$$

□

This concludes this appendix on $\downarrow$.