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ABSTRACT

In this paper we investigate the concept of simple termination. A term rewriting system is called simply terminating if its termination can be proved by means of a simplification order. The basic ingredient of a simplification order is the subterm property, but in the literature two different definitions are given: one based on (strict) partial orders and another one based on preorders (or quasi-orders). In the first part of the paper we argue that there is no reason to choose the second one, while the first one has certain advantages.

Simplification orders are known to be well-founded orders on terms over a finite signature. This important result no longer holds if we consider infinite signatures. Nevertheless, well-known simplification orders like the recursive path order are also well-founded on terms over infinite signatures, provided the underlying precedence is well-founded. We propose a new definition of simplification order, which coincides with the old one (based on partial orders) in case of finite signatures, but which is also well-founded over infinite signatures and covers orders like the recursive path order.

1. Introduction

One of the main problems in the theory of term rewriting is the detection of termination: for a fixed system of rewrite rules, determine whether there exist infinite reduction sequences or not. Huet and Lankford [9] showed that this problem is undecidable in general. However, there are several methods for deciding termination that are successful for many special cases. A well-known method for proving termination is the recursive path order (Dershowitz [2]). The basic idea of such a path order is that, starting from a given order (the so-called *precedence*) on the operation symbols, in a recursive way a well-founded order on terms is defined. If every reduction step in a term rewriting system corresponds to a decrease according this order, one can conclude that the system is terminating. If the order is closed under contexts and substitutions then the decrease only has to be checked for the rewrite rules instead of all reduction steps. The bottleneck of this kind of method is how to prove that a relation defined recursively on terms is indeed a well-founded order. Proving irreflexivity and transitivity often turns out to be feasible, using some induction and case analysis. However, when stating an arbitrary recursive definition of such an order, well-foundedness is very hard to prove directly. Fortunately, the powerful *Tree Theorem* of Kruskal implies that if the order satisfies some simplification property, well-foundedness is obtained for free. An order satisfying this property is called a *simplification order*. This notion of simplification comprises two ingredients:

- a term decreases by removing parts of it, and
- a term decreases by replacing an operation symbol with a smaller (according to the precedence) one.

If the signature is infinite, both of these ingredients are essential for the applicability of Kruskal's Tree Theorem. It is amazing, however, that in the term rewriting literature the notion of simplification order is motivated by the applicability of Kruskal's Tree Theorem but only covers the first ingredient. For infinite signatures one easily defines non-well-founded orders that are simplification orders according to that definition. Therefore, the usual definition of simplification order is only helpful for proving termination of systems over finite signatures. Nevertheless, it is well-known that simplification orders like the recursive path order are also well-founded on terms over infinite signatures (provided the precedence on the signature is well-founded).

In this paper we propose a definition of a simplification order that matches exactly the requirements of Kruskal's Tree Theorem, since that is the basic motivation for the notion of simplification order. According to this new definition all simplification orders are well-founded, both over finite and infinite signatures. For finite signatures the new and the old notion of simplification order coincide. A term rewriting system is called *simply terminating* if there is a simplification order that orients the rewrite rules from left to right. It is immediate from the definition that every recursive path order over a well-founded precedence can be extended to a simplification order, and hence it is well-founded. Even if one is only interested in finite term rewriting systems this is of interest: *semantic labelling* ([18]) often succeeds in proving termination of a finite but "difficult" (non-simply terminating) system by transforming it into an infinite system over an infinite signature to which the recursive path order readily applies.

In the literature simplification orders are sometimes based on preorders (or quasi-orders) instead of (strict) partial orders. A main result of this paper is that there are no compelling reasons for doing so. We prove (constructively) that every term rewriting system which can be shown to be terminating by means of a simplification order based on preorders, can be shown to be terminating by means of a simplification order (based on partial orders). Since basing the notion of simplification order on preorders is more susceptive to mistakes and results in stronger proof obligations, simplification orders should be based on partial orders. (As explained in Section 3 these remarks already apply to finite signatures.) As a consequence, we prefer the partial order variant of *well-quasi-orders*, the so-called *partial well-orders*, in case of infinite signatures. By choosing partial well-orders instead of well-quasi-orders a great part of the theory is not affected, but another part becomes cleaner. For instance, in Section 5 we prove a useful result stating that a term rewriting system is simply terminating if and only if the union of the system and a particular system that captures simplification is terminating. Based on well-quasi-orders a similar result does not hold.

A useful notion of termination for term rewriting systems is *total termination* (see [6, 17]). For finite signature one easily shows that total termination implies simple termination. In Section 6 we show that for infinite signatures this does not hold any more: we construct an infinite term rewriting system whose termination can be proved by a polynomial interpretation, but which is not simply terminating.

2. Termination

In order to fix our notations and terminology, we start with a very brief introduction to term rewriting. Term rewriting is surveyed in Dershowitz and Jouannaud [4] and Klop [11].

A *signature* is a set $\mathcal{F}$ of *function symbols*. Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ be the set of all terms built from $\mathcal{F}$ and a countably infinite set $\mathcal{V}$ of *variables*, disjoint from $\mathcal{F}$. The set of variables occurring in a term t is denoted by $\mathcal{V}ar(t)$. A term t is called *ground* if $\mathcal{V}ar(t) = \varnothing$. The set of all ground terms is denoted by $\mathcal{T}(\mathcal{F})$.

We introduce a fresh constant symbol $\Box$, named *hole*. A *context* C is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\},\mathcal{V})$ containing precisely one hole. The designation *term* is restricted to members of $\mathcal{T}(\mathcal{F},\mathcal{V})$. If C is a context and t a term then $C[t]$ denotes the result of replacing the hole in C by t. A term s is a *subterm* of a term t if there exists a context C such that $t = C[s]$. A subterm s of t is *proper* if $s \neq t$. We assume familiarity with the *position* formalism for describing subterm occurrences. A *substitution* is a map σ from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ with the property that the set $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. If σ is a substitution and t a term then $t\sigma$ denotes the result of applying σ to t. We call $t\sigma$ an *instance* of t. A binary relation R on terms is *closed under contexts* if $C[s]\ R\ C[t]$ whenever $s\ R\ t$, for all contexts C. A binary relation R on terms is *closed under substitutions* if $s\sigma\ R\ t\sigma$ whenever $s\ R\ t$, for all substitutions σ. A *rewrite relation* is a binary relation on terms that is closed under contexts and substitutions.

A *rewrite rule* is a pair (l, r) of terms such that the left-hand side l is not a variable and variables which occur in the right-hand side r occur also in l, i.e., $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. Since we are interested in (simple) termination in this paper, these two restrictions rule out only trivial cases. Rewrite rules (l, r) will henceforth be written as $l \to r$.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F},\mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. We often present a TRS as a set of rewrite rules, without making explicit its signature, assuming that the signature consists of the function symbols occurring in the rewrite rules. The smallest rewrite relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that contains $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$. So $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $l \to r$ in $\mathcal{R}$, a substitution σ, and a context C such that $s = C[l\sigma]$ and $t = C[r\sigma]$. The subterm $l\sigma$ of s is called a *redex* and we say that s rewrites to t by *contracting* redex $l\sigma$. We call $s \to_{\mathcal{R}} t$ a *rewrite* or *reduction step.* The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to^+_{\mathcal{R}}$ and $\to^*_{\mathcal{R}}$ denotes the transitive and reflexive closure of $\to_{\mathcal{R}}$. If $s \to^*_{\mathcal{R}} t$ we say that s *reduces* to t. The converse of $\to^*_{\mathcal{R}}$ is denoted by $\leftarrow^*_{\mathcal{R}}$.

A TRS $(\mathcal{F}, \mathcal{R})$ is called *terminating* if there are no infinite reduction sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. In order to simplify matters, we assume throughout this paper that the signature $\mathcal{F}$ contains a constant symbol. Hence a TRS is terminating if and only if there do not exist infinite reduction sequence involving only ground terms.

A (strict) *partial order* $\succ$ is a transitive and irreflexive relation. The reflexive closure of $\succ$ is denoted by $\succcurlyeq$. The converse of $\succcurlyeq$ is denoted by $\preccurlyeq$. A partial order $\succ$ on a set A is *well-founded* if there are no infinite descending sequences $a_1 \succ a_2 \succ \cdots$ of elements of A. A partial order $\succ$ on A is *total* if for all different elements $a, b \in A$ either $a \succ b$ or $b \succ a$. A *preorder* (or *quasi-order*) $\succsim$ is a transitive and reflexive relation. The converse of $\succsim$ is denoted by $\precsim$. The *strict part* of a preorder $\succsim$ is the partial order $\succ$ defined as $\succsim \backslash \precsim$. Every preorder $\succsim$ induces an equivalence relation $\sim$ defined as $\succsim \cap \precsim$. It is easy to see that $\succ = \succsim \backslash \sim$. A preorder is said to be well-founded if its strict part is a well-founded partial order.

A rewrite relation that is also a partial order is called a *rewrite order.* A well-founded rewrite order is called a *reduction order.* We say that a TRS $(\mathcal{F}, \mathcal{R})$ and a partial order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are *compatible* if $\mathcal{R}$ is contained in $\succ$, i.e., $l \succ r$ for every rewrite rule $l \to r$ of $\mathcal{R}$. It is easy to show that a TRS is terminating if and only if it is compatible with a reduction order.

DEFINITION 2.1. We say that a binary relation R on terms has the *subterm property* if $C[t] \; R \; t$ for all contexts $C \neq \Box$ and terms t.

The task of showing that a given *transitive* relation R has the subterm property amounts to verifying $f(t_1, \ldots, t_n) \; R \; t_i$ for all function symbols f of arity $n \geqslant 1$, terms $t_1, \ldots, t_n$, and $i \in \{1, \ldots, n\}$. This observation will be used freely in the sequel.

DEFINITION 2.2. Let $\mathcal{F}$ be a signature. The TRS $\mathcal{E}mb(\mathcal{F})$ consists of all rewrite rules

$$f(x_1, \ldots, x_n) \quad \to \quad x_i$$

with $f \in \mathcal{F}$ a function symbol of arity $n \geqslant 1$ and $i \in \{1, \ldots, n\}$. Here $x_1, \ldots, x_n$ are pairwise different variables. We abbreviate $\to^+_{\mathcal{E}mb(\mathcal{F})}$ to $\rhd_{emb}$ and $\leftarrow^*_{\mathcal{E}mb(\mathcal{F})}$ to $\unlhd_{emb}$. The latter relation is called *embedding.*

The following easy result relates the subterm property to embedding.

LEMMA 2.3. *A rewrite order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has the subterm property if and only if it is compatible with the TRS $\mathcal{E}mb(\mathcal{F})$.* □

PROOF.

- ⇒ The subterm property yields $f(x_1, \ldots, x_n) \succ x_i$ and hence $l \succ r$ for every rewrite rule $l \to r \in \mathcal{E}mb(\mathcal{F})$.
- ⇐ We have to show $f(t_1, \ldots, t_n) \succ t_i$ for all function symbols f of arity $n \geqslant 1$, terms $t_1, \ldots, t_n$, and $i \in \{1, \ldots, n\}$. By assumption $f(x_1, \ldots, x_n) \succ x_i$. Closure under substitutions yields $f(t_1, \ldots, t_n) \succ t_i$.

□

3. Simple Termination — Finite Signatures

Throughout this section we are dealing with *finite* signatures only.

DEFINITION 3.1. A *simplification order* is a rewrite order with the subterm property. A TRS $(\mathcal{F}, \mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Since we are only interested in signatures consisting of function symbols with fixed arity, we have no need for the *deletion property* (cf. [2]). It should also be noted that many authors (e.g. [1, 2, 3, 7, 10, 16]) do not require that simplification orders are closed under substitutions. Since we don't really want to check whether a simplification order orients *all instances* of rewrite rules from left to right in order to conclude termination, and concrete simplification orders like the recursive path order are closed under substitutions, closure under substitutions should be part of the definition. Moreover, it is easy to show that the class of simply terminating TRSs is not affected by imposing closure under substitutions. Dershowitz [1, 2] showed that every simply terminating TRS is terminating. The proof is based on the beautiful Tree Theorem of Kruskal [12].

DEFINITION 3.2. An infinite sequence $t_1, t_2, t_3, \ldots$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is *self-embedding* if there exist $1 \leqslant i < j$ such that $t_i \trianglelefteq_{emb} t_j$.

THEOREM 3.3 (KRUSKAL'S TREE THEOREM—FINITE VERSION). *Every infinite sequence of ground terms is self-embedding.* □

We refrain from proving Theorem 3.3 since it is a special case of the general version of Kruskal's Tree Theorem, which is presented and proved in Section 4.

THEOREM 3.4. *Every simply terminating TRS is terminating.*

PROOF. Suppose there exists a simply terminating TRS $(\mathcal{F}, \mathcal{R})$ that is not terminating. So $(\mathcal{F}, \mathcal{R})$ is compatible with a simplification order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ and there exists an infinite reduction sequence $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ involving only ground terms. From Kruskal's Tree Theorem we learn the existence of $1 \leqslant i < j$ such that $t_i \trianglelefteq_{emb} t_j$. From Lemma 2.3 we easily obtain $t_j \succcurlyeq t_i$. However, since $(\mathcal{F}, \mathcal{R})$ is compatible with $\succ$, $t_i \to_{\mathcal{R}}^{+} t_j$ implies $t_i \succ t_j$. Hence we have a contradiction with the fact that $\succ$ is a partial order. We conclude that $(\mathcal{F}, \mathcal{R})$ is terminating. □

The following well-known result is especially useful for showing that a given TRS is *not* simply terminating, see [17].

LEMMA 3.5. *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}))$ is terminating.*

PROOF.

⇒ Let $(\mathcal{F}, \mathcal{R})$ be compatible with the simplification order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. From Lemma 2.3 we learn that $\succ$ is compatible with the TRS $\mathcal{E}mb(\mathcal{F})$. Hence the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}))$ is simply terminating. Theorem 3.4 yields the termination of $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}))$.

⇐ Let $\succ$ be the rewrite order associated with $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}))$ (i.e., the transitive closure of its rewrite relation). Clearly $\succ$ is compatible with $\mathcal{E}mb(\mathcal{F})$. Lemma 2.3 shows that it is a simplification order. Since also the TRS $(\mathcal{F}, \mathcal{R})$ is compatible with $\succ$, it is simply terminating.

□

In the term rewriting literature the notion of simplification order is sometimes based on preorders instead of partial orders. Dershowitz [2] obtained the following result.

THEOREM 3.6. *Let $(\mathcal{F},\mathcal{R})$ be a TRS. Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F},\mathcal{V})$ which is closed under contexts and has the subterm property. If $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ then $(\mathcal{F},\mathcal{R})$ is terminating.* □

A preorder that is closed under contexts and has the subterm property is sometimes called a *quasi-simplification order.* Observe that we require *$l\sigma \succ r\sigma$ for all substitutions σ* in Theorem 3.6. It should be stressed that this requirement cannot be weakened to the compatibility of $(\mathcal{F},\mathcal{R})$ and $\succ$ (i.e., $l \succ r$ for all rules $l \to r \in \mathcal{R}$) if we additionally require that $\succsim$ is closed under substitutions, as is incorrectly done in Dershowitz and Jouannaud [4]. For instance, the relation $\to_{\mathcal{R}}^*$ associated with the TRS

$$\mathcal{R} = \begin{cases} f(g(x)) & \to & f(f(x)) \\ f(g(x)) & \to & g(g(x)) \\ f(x) & \to & x \\ g(x) & \to & x \end{cases}$$

is a rewrite relation with the subterm property (because $\mathcal{R}$ contains $\mathcal{E}mb(\{f,g\})$). Moreover, $l \to_{\mathcal{R}}^* r$ but not $r \to_{\mathcal{R}}^* l$, for every rewrite rule $l \to r \in \mathcal{R}$. So $\mathcal{R}$ is included in the strict part of $\to_{\mathcal{R}}^*$. Nevertheless, $\mathcal{R}$ is not terminating:

$$f(g(g(x))) \to_{\mathcal{R}} f(f(g(x)) \to_{\mathcal{R}} f(g(g(x))) \to_{\mathcal{R}} \cdots.$$

The point is that the strict part of $\to_{\mathcal{R}}^*$ is not closed under substitutions. Hence to conclude termination from compatibility with $\succsim$ it is essential that $\succ$ is closed under substitutions. A simpler TRS illustrating the same point, due to Enno Ohlebusch (personal communication), is $\{f(x) \to f(a),\ f(x) \to x\}$.

Dershowitz [2] writes that Theorem 3.6 generalizes Theorem 3.4. We have the following result.

THEOREM 3.7. *A TRS $(\mathcal{F},\mathcal{R})$ is simply terminating if and only if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that is closed under contexts, has the subterm property, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ.* □

The proof is given in Section 5, where the above theorem is generalized to TRSs over arbitrary, not necessarily finite, signatures.

So every TRS whose termination can be shown by means of Theorem 3.6 is simply terminating, i.e., its termination can be shown by a simplification order. Since it is easier to check $l \succ r$ for finitely many rewrite rules $l \to r$ than $l\sigma \succsim r\sigma$ but not $r\sigma \succsim l\sigma$ for finitely many rewrite rules $l \to r$ and infinitely many substitutions σ, there is no reason to base the definition of simplification order on preorders.

4. Partial Well-Orders

Theorem 3.4 does not hold if we allow infinite signatures. Consider for instance the TRS $(\mathcal{F},\mathcal{R})$ consisting of infinitely many constants a_i and rewrite rules $a_i \to a_{i+1}$ for all $i \geqslant 1$. The rewrite order $\to_{\mathcal{R}}^+$ vacuously satisfies the subterm property, but $(\mathcal{F},\mathcal{R})$ is not terminating:

$$a_1 \to_{\mathcal{R}} a_2 \to_{\mathcal{R}} a_3 \to_{\mathcal{R}} \cdots$$

So in case $\mathcal{F}$ is infinite, compatibility with $\mathcal{E}mb(\mathcal{F})$ does not ensure termination. In the next section we will see that the results of the previous section can be recovered by suitably extending the TRS $\mathcal{E}mb(\mathcal{F})$.

DEFINITION 4.1. Let $\succ$ be a partial order on a signature $\mathcal{F}$. The TRS $\mathcal{E}mb(\mathcal{F},\succ)$ consists of all rewrite rules of $\mathcal{E}mb(\mathcal{F})$ together with all rewrite rules

$$f(x_1,\ldots,x_n) \rightarrow g(x_{i_1},\ldots,x_{i_m})$$

with f an n-ary function symbol in $\mathcal{F}$, g an m-ary function symbol in $\mathcal{F}$, $n \geqslant m \geqslant 0$, $f \succ g$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. Here $x_1,\ldots,x_n$ are pairwise different variables. We abbreviate $\rightarrow^+_{\mathcal{E}mb(\mathcal{F},\succ)}$ to $\succ_{emb}$ and $\leftarrow^*_{\mathcal{E}mb(\mathcal{F},\succ)}$ to $\preccurlyeq_{emb}$. The latter relation is called *homeomorphic embedding*.

Since $\mathcal{E}mb(\mathcal{F},\varnothing) = \mathcal{E}mb(\mathcal{F})$, homeomorphic embedding generalizes embedding. Consider for instance the signature $\mathcal{F}$ consisting of constants a and b, a unary function symbol g, and binary functions symbols f and h. Define the partial order $\succ$ on $\mathcal{F}$ by $a \succ b \succ f \succ g \succ h$. In the TRS

$$\mathcal{E}mb(\mathcal{F},\succ) = \mathcal{E}mb(\mathcal{F}) \cup \left\{ \begin{array}{rcl} a & \rightarrow & b \\ f(x,y) & \rightarrow & g(x) \\ f(x,y) & \rightarrow & g(y) \\ f(x,y) & \rightarrow & h(x,y) \end{array} \right\}$$

we have the reduction sequence $f(h(a,b),g(a)) \rightarrow f(a,g(a)) \rightarrow f(a,a) \rightarrow f(a,b)$, hence the term $f(a,b)$ is homeomorphically embedded in $f(h(a,b),g(a))$. Since there is no reduction sequence in the TRS $\mathcal{E}mb(\mathcal{F})$ from $f(h(a,b),g(a))$ to $f(a,b)$, the term $f(a,b)$ is not embedded in $f(h(a,b),g(a))$.

In the next section we show that all results of the previous section carry over to infinite signatures if we require compatibility with $\mathcal{E}mb(\mathcal{F},\succ)$, provided the partial order $\succ$ satisfies a stronger property than well-foundedness. This property is explained below.

DEFINITION 4.2. Let $\succ$ be a partial order on a set A.

- An infinite sequence $(a_i)_{i\geqslant 1}$ over A is called *good* if there exist indices $1 \leqslant i < j$ with $a_i \preccurlyeq a_j$, otherwise it is called *bad*.
- An infinite sequence $(a_i)_{i\geqslant 1}$ over A is called a *chain* if $a_i \preccurlyeq a_{i+1}$ for all $i \geqslant 1$. We say that $(a_i)_{i\geqslant 1}$ contains a chain if it has a subsequence that is a chain.
- An infinite sequence $(a_i)_{i\geqslant 1}$ over A is called an *antichain* if neither $a_i \preccurlyeq a_j$ nor $a_j \preccurlyeq a_i$, for all $1 \leqslant i < j$.

LEMMA 4.3. *Let $\succ$ be a partial order on a set A. The following statements are equivalent.*

(1) *Every partial order that extends $\succ$ (including $\succ$ itself) is well-founded.*
(2) *Every infinite sequence over A is good.*
(3) *Every infinite sequence over A contains a chain.*
(4) *The partial order $\succ$ is well-founded and does not admit antichains.*

PROOF.

(1) $\Rightarrow$ (2) Suppose $(a_i)_{i\geqslant 1}$ is a bad sequence. Define $\succ' = (\succ \cup \{(a_i, a_{i+1}) \mid i \geqslant 1\})^+$. Assume $a \succ' a$ for some $a \in A$. Since $\succ$ is irreflexive there is a non-empty sequence of numbers $i_1, \ldots, i_n$ such that

$$a \succcurlyeq a_{i_1},\ a_{i_1+1} \succcurlyeq a_{i_2},\ a_{i_2+1} \succcurlyeq a_{i_3},\ \ldots,\ a_{i_{n-1}+1} \succcurlyeq a_{i_n},\ a_{i_n+1} \succcurlyeq a.$$

Since $(a_i)_{i\geqslant 1}$ is bad $a_i \succcurlyeq a_j$ is only possible for $i \leqslant j$. Hence we obtain the impossible

$$i_1 < i_1 + 1 \leqslant i_2 < i_2 + 1 \leqslant i_3 < \cdots < i_{n-1} + 1 \leqslant i_n < i_n + 1 \leqslant i_1.$$

We conclude that $\succ'$ is irreflexive. By definition it is transitive, hence it is a partial order extending $\succ$. However, since $a_1 \succ' a_2 \succ' a_3 \succ' \cdots$, it is not well-founded.

(2) $\Rightarrow$ (3) Let $(a_i)_{i\geqslant 1}$ be any infinite sequence over A. Consider the subsequence consisting of all elements a_i with the property that $a_i \preccurlyeq a_j$ holds for no $j > i$. If this subsequence is infinite then it is a bad sequence, contradicting (2). Hence it is finite, and thus there exists an index $N \geqslant 1$ such that for every $i \geqslant N$ there exists a $j > i$ with $a_i \preccurlyeq a_j$. Define inductively

$$\phi(i) = \begin{cases} N & \text{if } i = 1, \\ \min\{j \mid j > \phi(i-1) \text{ and } a_{\phi(i-1)} \preccurlyeq a_j\} & \text{if } i > 1. \end{cases}$$

Now $a_{\phi(1)}, a_{\phi(2)}, a_{\phi(3)}, \ldots$ is a chain.

(3) $\Rightarrow$ (4) If $\succ$ is not well-founded then there exists an infinite sequence $a_1 \succ a_2 \succ \cdots$. Clearly $a_i \preccurlyeq a_j$ doesn't hold for any $1 \leqslant i < j$. Hence this sequence doesn't contain a chain. If $\succ$ admits an antichain then this antichain is an infinite sequence not containing a chain.

(4) $\Rightarrow$ (1) For a proof by contradiction, let $\succ$ be a well-founded partial order not satisfying (1). Then there is an extension $\succ'$ of $\succ$ that is not well-founded. So there exists an infinite sequence $a_1 \succ' a_2 \succ' \cdots$. Since $\succ$ is well-founded, the sequence $(a_i)_{i\geqslant 1}$ contains an element a_i with the property that for no $j > i$ $a_i \succ a_j$ holds. Actually, $(a_i)_{i\geqslant 1}$ contains infinitely many such elements. We claim that the infinite subsequence $(a_{\phi(i)})_{i\geqslant 1}$ consisting of those elements is an antichain (with respect to $\succ$). Let $1 \leqslant i < j$. By construction $a_{\phi(i)} \succ a_{\phi(j)}$ is impossible. If $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$ then also $a_{\phi(i)} \preccurlyeq' a_{\phi(j)}$, contradicting $a_{\phi(i)} \succ' a_{\phi(j)}$. Hence $\succ$ admits a anti-chain.

□

DEFINITION 4.4. A partial order $\succ$ on a set A is called a *partial well-order* (PWO for short) if it satisfies one of the four equivalent assertions of Lemma 4.3.

Using the terminology of PWOs, Theorem 3.3 can now be read as follows: if $\mathcal{F}$ is a finite signature then $\rhd_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$.

By definition every PWO is a well-founded order, but the reverse does not hold. For instance, the empty relation on an infinite set is a well-founded order but not a PWO. Clearly every total well-founded order (or well-order) is a PWO. Any partial order extending a PWO is a PWO. The following lemma states how new PWOs can be obtained by restricting existing PWOs.

LEMMA 4.5. *Let $\succ$ be a PWO on a set A and let $\sqsupset$ be a PWO on a set B. Let $\varphi\colon A \to B$ be any function. The partial order $\succ'$ on A defined by $a \succ' b$ if and only if $a \succ b$ and $\varphi(a) \sqsupset \varphi(b)$ is a PWO.*

PROOF. Let $(a_i)_{i\geqslant 1}$ be any infinite sequence over A. Since $\succ$ is a PWO this sequence admits a chain

$$a_{\phi(1)} \preccurlyeq a_{\phi(2)} \preccurlyeq a_{\phi(3)} \preccurlyeq \cdots.$$

Since $\sqsupset$ is a PWO on B there exist $1 \leqslant i < j$ with $\varphi(a_{\phi(i)}) \sqsubseteq \varphi(a_{\phi(j)})$. Transitivity of $\preccurlyeq$ yields $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$. Hence $a_{\phi(i)} \preccurlyeq' a_{\phi(j)}$, while $\phi(i) < \phi(j)$. We conclude that $(a_i)_{i\geqslant 1}$ is a good sequence with respect to $\succ'$, so $\succ'$ is a PWO. □

COROLLARY 4.6. *The intersection of two PWOs on a set A is a PWO on A.*

PROOF. Choose the function φ in Lemma 4.5 to be the identity on A. □

THEOREM 4.7 (KRUSKAL'S TREE THEOREM—GENERAL VERSION). *If $\succ$ is a PWO on a signature $\mathcal{F}$ then $\succ_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$.* □

For the sake of completeness, below we present a proof of this beautiful theorem, even though it is very similar to the proof of the Kruskal's Tree Theorem formulated in terms of *well-quasi-orders* (see e.g. Gallier [7]). First we show a related result for strings, known as *Higman's Lemma* (Higman [8]).

DEFINITION 4.8. Let $\succ$ be a partial order on a set A. We define a relation $\succ^*$ on A^* as follows: if $w_1 = a_1a_2\cdots a_n$ and $w_2 = b_1b_2\cdots b_m$ are elements of A^* then $w_1 \succ^* w_2$ if and only if $w_1 \neq w_2$ and either

- $m = 0$, or
- $n \geqslant m > 0$ and there exist indices $i_1, \ldots, i_m$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and $a_{i_j} \succcurlyeq b_j$ for all $j \in \{1, \ldots, m\}$.

The next result can be viewed as an alternative definition of $\succ^*$.

LEMMA 4.9. *Let $\succ$ be a partial order on a set A. The relation $\succ^*$ is the least partial order $\sqsupset$ on A^* satisfying the following two properties:*

(1) *$w_1aw_2 \sqsupset w_1w_2$ for all $w_1, w_2 \in A^*$ and $a \in A$,*

(2) *$w_1aw_2 \sqsupset w_1bw_2$ for all $w_1, w_2 \in A^*$ and $a, b \in A$ with $a \succ b$.*

PROOF. First we show that $\succ^*$ is a partial order. Irreflexivity is obvious. Let $w_1 = a_1\cdots a_n$, $w_2 = b_1\cdots b_m$, and $w_3 = c_1\cdots c_l$ be elements of A^* such that $w_1 \succ^* w_2 \succ^* w_3$. If $l = 0$ then $m > 0$ (because $w_2 \neq w_3$) and $n \geqslant m > 0$. Hence $w_1 \succ^* w_3$. Suppose $l > 0$. We have $n \geqslant m \geqslant l$. There exist indices $i_1, \ldots, i_l$ and $j_1, \ldots, j_m$ such that $1 \leqslant i_1 < \cdots < i_l \leqslant m$, $b_{i_k} \succcurlyeq c_k$ for all $k \in \{1, \ldots, l\}$, $1 \leqslant j_1 < \cdots < j_m \leqslant n$, and $a_{j_k} \succcurlyeq b_k$ for all $k \in \{1, \ldots, m\}$. Since $1 \leqslant j_{i_1} < \cdots < j_{i_l} \leqslant n$ and $a_{j_{i_k}} \succcurlyeq b_{i_k} \succcurlyeq c_k$ for all $k \in \{1, \ldots, l\}$, we have $w_1 \succ^* w_3$. This concludes the proof of the transitivity of $\succ^*$. It is very easy to see that $\succ^*$ satisfies properties (1) and (2). Conversely, let $\sqsupset$ be any partial order on A^* that satisfies properties (1) and (2). We will show that $\succ^* \subseteq \sqsupset$. Suppose $w_1 = a_1\cdots a_n \succ^* b_1\cdots b_m = w_2$. If $m = 0$ then $n > 0$ and hence the sequence $w_1 = a_1\cdots a_n \sqsupset a_2\cdots a_n \sqsupset \cdots \sqsupset a_n \sqsupset \varepsilon = w_2$ is non-empty, showing that $w_1 \sqsupset w_2$. If $n \geqslant m > 0$ then there exist indices $i_1, \ldots, i_m$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and $a_{i_j} \succcurlyeq b_j$ for all $j \in \{1, \ldots, m\}$. Let $w_3 = a_{i_1}\cdots a_{i_m}$. We have $w_1 \sqsupseteq w_3$ by successively removing elements a_i from w_1 whose index i does not belong to the set $\{i_1, \ldots, i_m\}$. (Clearly $w_1 = w_3$ if and only if $n = m$.) We have $w_3 \sqsupseteq w_2$ by replacing a_{i_j} with b_j whenever $a_{i_j} \succ b_j$. Therefore $w_1 \sqsupseteq w_2$ and since $w_1 \neq w_2$ we obtain $w_1 \sqsupset w_2$. □

LEMMA 4.10 (HIGMAN'S LEMMA). *If* $\succ$ *is a PWO on a set* A *then* $\succ^*$ *is a PWO on* A^*.

PROOF. The following proof is essentially due to Nash-Williams [14]. We have to show that there are no bad sequences over A^*. Suppose to the contrary that there exist bad sequences over A^*. We construct a *minimal bad sequence* $(w_i)_{i\geqslant 1}$ as follows:

> Suppose we already chose the first $n-1$ strings $w_1, \ldots, w_{n-1}$. Define w_n to be a shortest string such that there are bad sequences that start with $w_1, \ldots, w_n$.

Because $\varepsilon \preccurlyeq^* w$ for all $w \in A^*$, we have $w_i \neq \varepsilon$ for all $i \geqslant 1$. Hence we may write $w_i = a_i v_i$ ($i \geqslant 1$). Since $\succ$ is a PWO on A, the infinite sequence $(a_i)_{i\geqslant 1}$ contains a chain, say $(a_{\phi(i)})_{i\geqslant 1}$. Because $v_{\phi(1)}$ is shorter than $w_{\phi(1)}$, the sequence

$$w_1,\ \ldots,\ w_{\phi(1)-1},\ v_{\phi(1)},\ v_{\phi(2)},\ \ldots$$

must be good. Clearly $w_i \preccurlyeq^* w_j$ ($1 \leqslant i < j \leqslant \phi(1)-1$) is impossible as $(w_i)_{i\geqslant 1}$ is bad. Likewise, $w_i \preccurlyeq^* v_{\phi(j)}$ ($1 \leqslant i \leqslant \phi(1)-1$ and $1 \leqslant j$) contradicts the badness of $(w_i)_{i\geqslant 1}$ since $v_{\phi(j)} \preccurlyeq^* w_{\phi(j)}$ and therefore $w_i \preccurlyeq^* w_{\phi(j)}$. Hence we must have $v_{\phi(i)} \preccurlyeq^* v_{\phi(j)}$ for some $1 \leqslant i < j$. Combining this with $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$ easily yields $w_{\phi(i)} = a_{\phi(i)}v_{\phi(i)} \preccurlyeq^* a_{\phi(j)}v_{\phi(j)} = w_{\phi(j)}$, contradicting the badness of $(w_i)_{i\geqslant 1}$. We conclude that there are no bad sequences over A^*. □

PROOF OF KRUSKAL'S TREE THEOREM—GENERAL VERSION. The proof, essentially due to Nash-Williams [14], has the same structure as the proof of Higman's Lemma. We have to show that there are no bad sequences of terms in $\mathcal{T}(\mathcal{F})$. Suppose to the contrary that there exist bad sequences of ground terms. We construct a minimal bad sequence $(t_i)_{i\geqslant 1}$ as follows:

> Suppose we already chose the first $n-1$ terms $t_1, \ldots, t_{n-1}$. Define t_n to be a smallest (with respect to size) term such that there are bad sequences that start with $t_1, \ldots, t_n$.

For every $i \geqslant 1$, let f_i be the root symbol of t_i and let A_i be the set of arguments of t_i (if t_i is a constant then $A_i = \varnothing$). Moreover, let w_i be the string of arguments (from left to right) of t_i. Finally, let $A = \bigcup_{i\geqslant 1} A_i$.

We claim that $\succ_{emb}$ is a PWO on the subset A of $\mathcal{T}(\mathcal{F})$. For a proof by contradiction, suppose $(a_i)_{i\geqslant 1}$ is a bad sequence over A. Let $a_1 \in A_k$. Since $A' = \bigcup_{i=1}^{k-1} A_i$ is a finite set and all elements of $(a_i)_{i\geqslant 1}$ are different, only finitely many elements of $(a_i)_{i\geqslant 1}$ belong to A'. Thus there exists an index $l > 1$ such that $a_i \in A \backslash A'$ for all $i \geqslant l$. Because a_1 is a proper subterm of t_k, the sequence

$$t_1,\ \ldots,\ t_{k-1},\ a_1,\ a_l,\ a_{l+1},\ \ldots$$

must be good. Clearly $t_i \preccurlyeq_{emb} t_j$ ($1 \leqslant i < j \leqslant k-1$) is impossible as $(t_i)_{i\geqslant 1}$ is bad. Likewise, $t_i \preccurlyeq_{emb} a_j$ ($1 \leqslant i \leqslant k-1$ and $j = 1$ or $l \leqslant j$) contradicts the badness of $(t_i)_{i\geqslant 1}$ since $a_j \preccurlyeq_{emb} t_m$ for some $m \geqslant k$—recall that a_1 is a proper subterm of t_k and if $j \geqslant l$ then $a_j \in A\backslash A'$—and thus $t_i \preccurlyeq_{emb} t_j$. Hence we must have $a_i \preccurlyeq_{emb} a_j$ for some $1 \leqslant i < j$ (and $i, j \notin \{2, \ldots, l-1\}$), contradicting the badness of $(a_i)_{i\geqslant 1}$. Hence $\succ_{emb}$ is a PWO on A. From Higman's Lemma we infer that $\succ_{emb}^*$ is a PWO on A^*.

Since $\succ$ is a PWO on $\mathcal{F}$, the infinite sequence $(f_i)_{i\geqslant 1}$ contains a chain, say $(f_{\phi(i)})_{i\geqslant 1}$. Consider the infinite sequence $(w_{\phi(i)})_{i\geqslant 1}$ over A^*. Since $\succ_{emb}^*$ is a PWO on A^*, we have $w_{\phi(i)} \preccurlyeq_{emb}^* w_{\phi(j)}$ for some $1 \leqslant i < j$. A straightforward case analysis reveals that $f_{\phi(i)} \preccurlyeq f_{\phi(j)}$ and $w_{\phi(i)} \preccurlyeq_{emb}^* w_{\phi(j)}$ imply $t_{\phi(i)} \preccurlyeq_{emb} t_{\phi(j)}$. Hence we obtained a contradiction with the badness of $(t_i)_{i\geqslant 1}$. We conclude that there are no bad sequences over $\mathcal{T}(\mathcal{F})$. □

PWOs are closely related to the more familiar concept of well-quasi-order.

DEFINITION 4.11. A *well-quasi-order* (WQO for short) is a preorder that contains a PWO.

The above definition is equivalent to all other definitions of WQO found in the literature. Kruskal's Tree Theorem is usually presented in terms of WQOs. This is not more powerful than the PWO version: notwithstanding the fact that the strict part of a WQO is not necessarily a PWO, it is very easy to show that the WQO version of Kruskal's Tree Theorem is a corollary of Theorem 4.7, and vice-versa.

Let $\succ$ be a PWO on a signature $\mathcal{F}$. A natural question is whether we can restrict $\succ_{emb}$ while retaining the property of being a PWO on $\mathcal{T}(\mathcal{F})$. In particular, do we really need all rewrite rules in $\mathcal{E}mb\,(\mathcal{F},\succ)$? In case there is a uniform bound on the arities of the function symbols in $\mathcal{F}$, we can greatly reduce the set $\mathcal{E}mb\,(\mathcal{F},\succ)$. That is, suppose there exists an $N \geqslant 0$ such that all function symbols in $\mathcal{F}$ have arity less than or equal to N. Now we can apply Lemma 4.5: choose φ to be the function that assigns to every function symbol its arity and take $\sqsupset$ to be the empty relation on $\{1,\ldots,N\}$. Hence the partial order $\succ'$ on $\mathcal{F}$ defined by $f \succ' g$ if and only if f and g have the same arity and $f \succ g$ is a PWO. The corresponding set $\mathcal{E}mb\,(\mathcal{F},\succ')$ consists, besides all rewrite rules of the form $f(x_1,\ldots,x_n) \to x_i$, of all rewrite rules $f(x_1,\ldots,x_n) \to g(x_1,\ldots,x_n)$ with f and g n-ary function symbols such that $f \succ g$. This construction does not work if the arities of function symbols in $\mathcal{F}$ are not uniformly bounded. Consider for instance a signature $\mathcal{F}$ consisting of a constant a and n-ary function symbols f_n for every $n \geqslant 1$ (and let $\succ$ be any PWO on $\mathcal{F}$). The sequence

$$f_1(a),\ f_2(a,a),\ f_3(a,a,a),\ \ldots$$

is bad with respect to $\succ'_{emb}$. Finally, one may wonder whether the restriction to all rewrite rules $f(x_1,\ldots,x_n) \to g(x_{i+1},\ldots,x_{i+m})$ with f an n-ary function symbol, g an m-ary function symbol, $n \geqslant m \geqslant 0$, $n-m \geqslant i \geqslant 0$, and $f \succ g$ is sufficient. This is also not the case, as can be seen by extending the previous signature with a constant b and considering the sequence

$$f_2(b,b),\ f_3(b,a,b),\ f_4(b,a,a,b),\ \ldots\ .$$

Of course, if the signature $\mathcal{F}$ is finite then the rules of $\mathcal{E}mb\,(\mathcal{F})$ are sufficient since the empty relation is a PWO on any finite set.

5. Simple Termination — Infinite Signatures

Kurihara and Ohuchi [13] were the first to use the terminology simple termination. They call a TRS $(\mathcal{F},\mathcal{R})$ simply terminating if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$. Since compatibility with a simplification order doesn't ensure the termination of TRSs over infinite signatures, see the example at the beginning of the previous section, this definition of simple termination is clearly not the right one. Ohlebusch [15] and others call a TRS $(\mathcal{F},\mathcal{R})$ simply terminating if it is compatible with a *well-founded* simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$. This is a very artificial way to ensure that every simply terminating is terminating, more precisely, termination of simply terminating TRSs has nothing to do with Kruskal's Tree Theorem; simply terminating TRSs are terminating by definition. We propose instead to bring the definition of simple termination in accordance with (the general version of) Kruskal's Tree Theorem.

DEFINITION 5.1. A *simplification order* is a rewrite order on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that contains $\succ_{emb}$ for some PWO $\succ$ on $\mathcal{F}$. A TRS $(\mathcal{F},\mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$.

This definition coincides with the one in Section 3 in case of finite signatures:

LEMMA 5.2. *A rewrite order $\sqsupset$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ with $\mathcal{F}$ finite is a simplification order if and only if it has the subterm property, i.e.,* $\rhd \subseteq \sqsupset$.

PROOF.

$\Rightarrow$ By definition there exists a PWO $\succ$ on $\mathcal{F}$ such that $\succ_{emb} \subseteq \sqsupset$. Since $\rhd \subseteq \succ_{emb}$, $\sqsupset$ has the subterm property.

$\Leftarrow$ The empty relation $\varnothing$ is a PWO on any finite set. The subterm property yields $\varnothing_{emb} = \rhd \subseteq \sqsupset$. Hence $\sqsupset$ is a simplification order.

□

THEOREM 5.3. *Every simply terminating TRS is terminating.*

PROOF. Let $(\mathcal{F},\mathcal{R})$ be compatible with a simplification order $\sqsupset$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$. Let $\succ$ be any PWO such that $\succ_{emb}$ is included in $\sqsupset$. Theorem 4.7 shows that the restriction of $\succ_{emb}$ to ground terms is a PWO. Hence the extension $\sqsupset$ of $\succ_{emb}$ is well-founded on ground terms. Therefore $(\mathcal{F},\mathcal{R})$ is terminating. □

The following result extends the very useful Lemma 3.5 to arbitrary TRSs. In the proof of Theorem 5.9 below and in the final example of Section 6 we make use of this result.

LEMMA 5.4. *A TRS $(\mathcal{F},\mathcal{R})$ is simply terminating if and only if the TRS $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is terminating for some PWO $\succ$ on $\mathcal{F}$.*

PROOF.

$\Rightarrow$ Let $(\mathcal{F},\mathcal{R})$ be compatible with the simplification order $\sqsupset$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$. By definition there exists a PWO $\succ$ on $\mathcal{F}$ such that $\succ_{emb} \subseteq \sqsupset$. If $l \to r \in \mathcal{E}mb(\mathcal{F},\succ)$ then $l \succ_{emb} r$ and therefore $l \sqsupset r$. Hence $\mathcal{E}mb(\mathcal{F},\succ)$ is also compatible with $\sqsupset$. So $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is simply terminating. Theorem 5.3 shows that $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is terminating.

$\Leftarrow$ Suppose $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is terminating for some PWO $\succ$ on $\mathcal{F}$. Let $\sqsupset$ be the rewrite order associated with the TRS $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$. Clearly $\succ_{emb} \subseteq \sqsupset$. Hence $\sqsupset$ is a simplification order. Since $(\mathcal{F},\mathcal{R})$ is compatible with $\sqsupset$, we conclude that it is simply terminating.

□

It should be stressed that there is no equivalent to the above lemma if we base the definition of simplification order on WQOs. This is one of the reasons why we favor PWOs.

In the remainder of this section we generalize Theorem 3.7 (and hence Theorem 3.6) to arbitrary TRSs. Our proof is based on the elegant proof sketch of Theorem 3.6 given by Plaisted [16]. The proof employs *multiset extensions* of preorders. A *multiset* is a collection in which elements are allowed to occur more than once. If A is a set then the set of all finite multisets over A is denoted by $\mathcal{M}(A)$. The *multiset extension* of a partial order $\succ$ on A is the partial order $\succ_{mul}$ defined on $\mathcal{M}(A)$ defined as follows: $M_1 \succ_{mul} M_2$ if $M_2 = (M_1 - X) \uplus Y$ for some multisets $X, Y \in \mathcal{M}(A)$ that satisfy $\varnothing \neq X \subseteq M_1$ and for all $y \in Y$ there exists an $x \in X$ such that $x \succ y$. Using Higman's Lemma, it is quite easy to show that multiset extension preserves PWO. From this we infer that the multiset extension of a well-founded partial order is well-founded, using the well-known facts that (1) every well-founded partial order can be extended to a total well-founded order (in particular a PWO) and (2) multiset extension is monotonic (i.e., if $\succ \subseteq \sqsupset$ then $\succ_{mul} \subseteq \sqsupset_{mul}$). Using König's Lemma, Dershowitz and Manna [5] gave a direct proof that multiset extension preserves well-founded partial orders.

DEFINITION 5.5. Let $\succsim$ be a preorder on a set A. For every $a \in A$, let $[a]$ denote the equivalence class with respect to the equivalence relation $\sim$ containing a. Let $A\backslash\!\sim\; = \{[a] \mid a \in A\}$ be the set of all equivalence classes of A. The preorder $\succsim$ on A induces a partial order $\succ$ on $A\backslash\!\sim$ as follows: $[a] \succ [b]$ if and only if $a \succ b$. (The latter $\succ$ denotes the strict part of the preorder $\succsim$.) For every multiset $M \in \mathcal{M}(A)$, let $[M] \in \mathcal{M}(A\backslash\!\sim)$ denote the multiset obtained from M by replacing every element a by $[a]$. We now define the *multiset extension* $\succsim_{mul}$ of the preorder $\succsim$ as follows: $M_1 \succsim_{mul} M_2$ if and only if $[M_1] \succ^{=}_{mul} [M_2]$ where $\succ^{=}_{mul}$ denotes the reflexive closure of the multiset extension of the partial order $\succ$ on $A\backslash\!\sim$.

It is easy to show that $\succsim_{mul}$ is a preorder on $\mathcal{M}(A)$. The associated equivalence relation $\sim_{mul} = \succsim_{mul} \cap \precsim_{mul}$ can be characterized in the following simple way: $M_1 \sim_{mul} M_2$ if and only if $[M_1] = [M_2]$. Likewise, its strict part $\succnsim_{mul} = \succsim_{mul} \backslash \precsim_{mul} = \succsim_{mul} \backslash \sim_{mul}$ has the following simple characterization: $M_1 \succnsim_{mul} M_2$ if and only if $[M_1] \succ_{mul} [M_2]$. Observe that we denote the strict part of $\succsim_{mul}$ by $\succnsim_{mul}$ in order to avoid confusion with the multiset extension $\succ_{mul}$ of the strict part $\succ$ of $\succsim$, which is a smaller relation.

The above definition of multiset extension of a preorder can be shown to be equivalent to the more operational ones in Dershowitz [3] and Gallier [7], but since we define the multiset extension of a preorder in terms of the well-known multiset extension of a partial order, we get all desired properties basically for free. In particular, using the fact that multiset extension preserves well-founded partial orders, it is very easy to show that the multiset extension of a well-founded preorder is well-founded.

DEFINITION 5.6. If $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ then $S(t) \in \mathcal{M}(\mathcal{T}(\mathcal{F},\mathcal{V}))$ denotes the finite multiset of all subterm occurrences in t and $F(t) \in \mathcal{M}(\mathcal{F})$ denotes the finite multiset of all function symbol occurrences in t. Formally,

$$M(t) = \begin{cases} \{t\} & \text{if } t \text{ is a variable,} \\ \{t\} \uplus \biguplus_{i=1}^{n} M(t_i) & \text{if } t = f(t_1,\ldots,t_n). \end{cases}$$

$$F(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \{f\} \uplus \biguplus_{i=1}^{n} F(t_i) & \text{if } t = f(t_1,\ldots,t_n). \end{cases}$$

LEMMA 5.7. *Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F},\mathcal{V})$ with the subterm property. If $s \succ t$ then $S(s) \succnsim_{mul} S(t)$.*

PROOF. We show that $s \succ t'$ for all $t' \in S(t)$. This implies $\{s\} \succnsim_{mul} S(t)$ and hence also $S(s) \succnsim_{mul} S(t)$. If $t' = t$ then $s \succ t'$ by assumption. Otherwise t' is a proper subterm of t and hence $t \succsim t'$ by the subterm property. Combining this with $s \succ t$ yields $s \succ t$. □

LEMMA 5.8. *Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F},\mathcal{V})$ which is closed under contexts. Suppose $s \succsim t$ and let C be an arbitrary context.*

- *If $S(s) \succnsim_{mul} S(t)$ then $S(C[s]) \succnsim_{mul} S(C[t])$.*
- *If $S(s) \succsim_{mul} S(t)$ then $S(C[s]) \succsim_{mul} S(C[t])$.*

PROOF. Let $S_1 = S(C[s]) - S(s)$ and $S_2 = S(C[t]) - S(t)$. For both statements it suffices to prove that $S_1 \succsim_{mul} S_2$. Let $p \in \mathcal{P}os(C[s])$ be the position of the displayed s in $C[s]$. There is a one-to-one correspondence between terms in S_1 (S_2) and positions in $\mathcal{P}os(C) - \{p\}$. Hence it suffices to show that $s' \succsim t'$ where $s' = C[s]_{|q}$ and $t' = C[t]_{|q}$ are the to position q corresponding terms in S_1 and S_2, for all $q \in \mathcal{P}os(C) - \{p\}$. If p and q are disjoint positions then $s' = t'$. Otherwise $q < p$ and there exists a context C' such that $s' = C'[s]$ and $t' = C'[t]$. By assumption $s \succsim t$. Closure under contexts yields $s' \succsim t'$. We conclude that $S_1 \succsim_{mul} S_2$. □

After these two preliminary results we are ready for the generalization of Theorem 3.7 to arbitrary TRSs.

THEOREM 5.9. *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that is closed under contexts, contains the relation $\sqsupset_{emb}$ for some PWO $\sqsupset$ on $\mathcal{F}$, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ.*

PROOF. The "only if" direction is obvious since the reflexive closure $\succcurlyeq$ of the simplification order $\succ$ used to prove simple termination is a preorder with the desired properties. For the "if" direction it suffices to show that $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset))$ is a terminating TRS, according to Lemma 5.4. First we show that either $S(s) \succnsim_{mul} S(t)$ or $S(s) \sim_{mul} S(t)$ and $F(s) \sqsupset_{mul} F(t)$ whenever $s \to t$ is a reduction step in the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset))$. So let $s = C[l\sigma]$ and $t = C[r\sigma]$ with $l \to r \in \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset)$. We distinguish three cases.

- If $l \to r \in \mathcal{R}$ then $l\sigma \succ r\sigma$ by assumption and $S(l\sigma) \succnsim_{mul} S(r\sigma)$ according to Lemma 5.7. The first part of Lemma 5.8 yields $S(s) \succnsim_{mul} S(t)$.
- If $l \to r \in \mathcal{E}mb(\mathcal{F})$ then $l\sigma = f(t_1, \ldots, t_n)$ and $r\sigma = t_i$ for some $i \in \{1, \ldots, n\}$. Therefore $S(l\sigma) \succnsim_{mul} S(r\sigma)$ since $S(t_i)$ is properly contained in $S(f(t_1, \ldots, t_n))$. Clearly $l\sigma \sqsupset_{emb} r\sigma$ and thus also $l\sigma \succsim r\sigma$. An application of the first part of Lemma 5.8 yields $S(s) \succnsim_{mul} S(t)$.
- If $l \to r \in \mathcal{E}mb(\mathcal{F}, \sqsupset) - Emb(\mathcal{F})$ then $l\sigma = f(t_1, \ldots, t_n)$ and $r\sigma = g(t_{i_1}, \ldots, t_{i_m})$ with $f \sqsupset g$, $n \geqslant m \geqslant 0$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. We have of course $l\sigma \sqsupset_{emb} r\sigma$ and thus also $l\sigma \succsim r\sigma$. Since the multiset $\{t_{i_1}, \ldots, t_{i_m}\}$ is contained in the multiset $\{t_1, \ldots, t_n\}$, we obtain $S(l\sigma) \succsim_{mul} S(r\sigma)$ and $F(l\sigma) \sqsupset_{mul} F(r\sigma)$. The second part of Lemma 5.8 yields $S(s) \succsim_{mul} S(t)$. We obtain $F(s) \sqsupset_{mul} F(t)$ from $F(l\sigma) \sqsupset_{mul} F(r\sigma)$.

Kruskal's Tree Theorem shows that $\sqsupset_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$. Hence $\succsim$ is a well-founded preorder on $\mathcal{T}(\mathcal{F})$. Since multiset extension preserves well-founded preorders, $\succsim_{mul}$ is a well-founded preorder on $\mathcal{M}(\mathcal{T}(\mathcal{F}))$. Because $\sqsupset$ is a PWO on the signature $\mathcal{F}$ it is a well-founded partial order. Hence its multiset extension $\sqsupset_{mul}$ is a well-founded partial order on $\mathcal{M}(\mathcal{F})$. We conclude that $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset))$ is a terminating TRS. □

6. Other Notions of Termination

In this final section we investigate the relationship between simple termination and other restricted kinds of termination as introduced in [17]. First we recall some terminology. Let $\mathcal{F}$ be a signature. A *monotone* $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ consists of a non-empty $\mathcal{F}$-algebra $\mathcal{A}$ and a partial order $\succ$ on the carrier A of $\mathcal{A}$ such that every algebra operation is strictly monotone in all its coordinates, i.e., if $f \in \mathcal{F}$ has arity n then

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \;\succ\; f_{\mathcal{A}}(a_1, \ldots, b_i, \ldots, a_n)$$

for all $a_1, \ldots, a_n, b_i \in A$ with $a_i \succ b_i$ ($i \in \{1, \ldots, n\}$). We call a monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ *well-founded* if $\succ$ is well-founded. We define a partial order $\succ_{\mathcal{A}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ as follows: $s \succ_{\mathcal{A}} t$ if $[\alpha](s) \succ [\alpha](t)$ for all assignments $\alpha: \mathcal{V} \to A$. Here $[\alpha]$ denotes the homomorphic extension of α. Finally, a TRS $(\mathcal{F}, \mathcal{R})$ is said to be *compatible* with $(\mathcal{A}, \succ)$ if $(\mathcal{F}, \mathcal{R})$ and $\succ_{\mathcal{A}}$ are compatible.

It is not difficult to show that the relation $\succ_{\mathcal{A}}$ is a rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, for every monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$. If $(\mathcal{A}, \succ)$ is well-founded then $\succ_{\mathcal{A}}$ is a reduction order. It is also straightforward to show that a TRS $(\mathcal{F}, \mathcal{R})$ is terminating if and only if it is compatible with a well-founded monotone $\mathcal{F}$-algebra. Simple termination can be characterized semantically as follows.

DEFINITION 6.1. A monotone $\mathcal{F}$-algebra is called *simple* if it is compatible with the TRS $\mathcal{E}mb(\mathcal{F}, \succ)$ for some partial well-order $\succ$ on $\mathcal{F}$.

It is straightforward to show that a TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if it is compatible with a simple monotone $\mathcal{F}$-algebra.

DEFINITION 6.2. A TRS $(\mathcal{F}, \mathcal{R})$ is called *totally terminating* if it is compatible with a well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ such that $\succ$ is a total order on the carrier set of $\mathcal{A}$. If the carrier set of $\mathcal{A}$ is the set of natural numbers and $\succ$ is the standard order then the TRS is called *ω-terminating.* If in addition the operation $f_{\mathcal{A}}$ is a polynomial for every $f \in \mathcal{F}$, the TRS is called *polynomially terminating.*

Total termination has been extensively studied in [6]. Clearly every polynomially terminating TRS is ω-terminating and every ω-terminating TRS is totally terminating. For both assertions the converse does not hold, as can be shown by the counterexamples $\mathcal{R}_1 = \{f(g(h(x))) \to g(f(h(g(x))))\}$ and $\mathcal{R}_2 = \{f(g(x)) \to g(f(f(x)))\}$ respectively. An easy observation ([17]) shows that every totally terminating TRS over a finite signature is simply terminating. Again the converse does not hold as is shown by the well-known example $\mathcal{R}_3 = \{f(a) \to f(b), g(b) \to g(a)\}$.

Somewhat surprisingly, for infinite signatures total termination does not imply simple termination any more: we prove that the non-simply terminating TRS $(\mathcal{F}, \mathcal{R}_4)$ is even polynomially terminating. Here $\mathcal{F}$ is the signature $\{f_i, g_i \mid i \in \mathbb{N}\}$ and $\mathcal{R}_4$ consists of all rewrite rules

$$f_i(g_j(x)) \quad \to \quad f_j(g_j(x))$$

where $i, j \in \mathbb{N}$ with $i < j$. First we prove that $(\mathcal{F}, \mathcal{R}_4)$ is not simply terminating. Let $\succ$ be any PWO on $\mathcal{F}$. Consider the infinite sequence $(f_i)_{i \geqslant 1}$. Since every infinite sequence is good, we have $f_j \succ f_i$ for some $i < j$. Hence $\mathcal{E}mb(\mathcal{F}, \succ)$ contains the rewrite rule $f_j(x) \to f_i(x)$, yielding the infinite reduction sequence

$$f_i(g_j(x)) \to f_j(g_j(x)) \to f_i(g_j(x)) \to \cdots$$

in the TRS $(\mathcal{F}, \mathcal{R}_4 \cup \mathcal{E}mb(\mathcal{F}, \succ))$. Lemma 5.4 shows that $(\mathcal{F}, \mathcal{R}_4)$ is not simply terminating.

For proving polynomial termination of $(\mathcal{F}, \mathcal{R}_4)$, interpret the function symbols as the following polynomials over $\mathbb{N}$:

$$f_{i\mathcal{A}}(x) = x^3 - ix^2 + i^2x \quad \text{and} \quad g_{i\mathcal{A}}(x) = x + 2i$$

for all $i, x \in \mathbb{N}$. Let $i \in \mathbb{N}$. The interpretation $g_{i\mathcal{A}}$ of g_i is clearly strictly monotone in its single argument. The same holds for the interpretation of f_i since

$$f_{i\mathcal{A}}(x+1) - f_{i\mathcal{A}}(x) \quad = \quad (x+1-i)^2 + 2x^2 + 2x + i \quad > \quad 0$$

for all $x \in \mathbb{N}$. It remains to show that $f_{i\mathcal{A}}(g_{j\mathcal{A}}(x)) > f_{j\mathcal{A}}(g_{j\mathcal{A}}(x))$ for all $i, j, x \in \mathbb{N}$ with $i < j$. Fix i, j, x and let $y = g_{j\mathcal{A}}(x) = x + 2j$. Then

$$f_{i\mathcal{A}}(g_{j\mathcal{A}}(x)) - f_{j\mathcal{A}}(g_{j\mathcal{A}}(x)) = f_{i\mathcal{A}}(y) - f_{j\mathcal{A}}(y) = y(j-i)(y-j-i) > 0$$

since $j > i$ and $y \geqslant 2j > j + i > 0$. We conclude that $(\mathcal{F}, \mathcal{R}_4)$ is polynomially terminating.

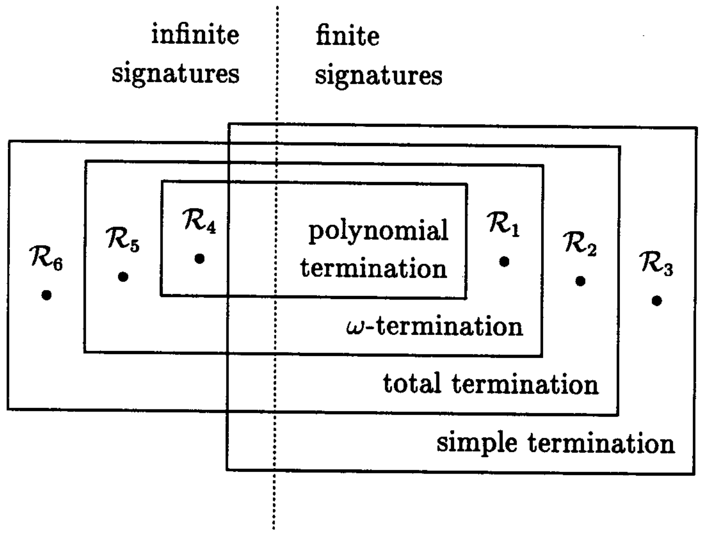

FIGURE 1.

Summarizing the relationship between the various kinds of termination we obtain Figure 1; for $\mathcal{R}_5$ and $\mathcal{R}_6$ we simply take the union of $\mathcal{R}_4$ with $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Uwe Waldmann (personal communication) was the first to prove total termination of a non-simply terminating system similar to $\mathcal{R}_4$, using a much more complicated total well-founded order.

The class of simply terminating TRSs is properly included in the class of all TRSs that are compatible with a well-founded rewrite order having the subterm property. Nevertheless, it's quite big. For instance, it includes all TRSs whose termination can be shown by means of the recursive path order (Dershowitz [2]) and its variants. This can be seen as follows. It is known that $\succ_{rpo}$ is a rewrite order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the subterm property (cf. [2]). It is not difficult to show that $\succ_{rpo}$ extends $\succ_{emb}$, for any precedence $\succ$ on the signature $\mathcal{F}$. Hence $\succ_{rpo}$ is a simplification order whenever the precedence $\succ$ is a PWO. In particular, if the signature is finite then every $\succ_{rpo}$ is a simplification order. If $\succ$ is a well-founded precedence on an arbitrary signature then $\succ_{rpo}$ is included in a simplification order (and hence well-founded). This follows from the *incrementality* of the recursive path order (i.e., if $\succ \subseteq \sqsupset$ then $\succ_{rpo} \subseteq \sqsupset_{rpo}$) and the well-known fact that every well-founded partial order can be extended to a total well-founded partial order. Hence every TRS $(\mathcal{F}, \mathcal{R})$ that is compatible with $\succ_{rpo}$ for some well-founded precedence $\succ$ on $\mathcal{F}$ is simply terminating.

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ABSTRACT

In this paper we investigate the concept of simple termination. A term rewriting system is called simply terminating if its termination can be proved by means of a simplification order. The basic ingredient of a simplification order is the subterm property, but in the literature two different definitions are given: one based on (strict) partial orders and another one based on preorders (or quasi-orders). In the first part of the paper we argue that there is no reason to choose the second one, while the first one has certain advantages.

Simplification orders are known to be well-founded orders on terms over a finite signature. This important result no longer holds if we consider infinite signatures. Nevertheless, well-known simplification orders like the recursive path order are also well-founded on terms over infinite signatures, provided the underlying precedence is well-founded. We propose a new definition of simplification order, which coincides with the old one (based on partial orders) in case of finite signatures, but which is also well-founded over infinite signatures and covers orders like the recursive path order.

1. Introduction

One of the main problems in the theory of term rewriting is the detection of termination: for a fixed system of rewrite rules, determine whether there exist infinite reduction sequences or not. Huet and Lankford [9] showed that this problem is undecidable in general. However, there are several methods for deciding termination that are successful for many special cases. A well-known method for proving termination is the recursive path order (Dershowitz [2]). The basic idea of such a path order is that, starting from a given order (the so-called *precedence*) on the operation symbols, in a recursive way a well-founded order on terms is defined. If every reduction step in a term rewriting system corresponds to a decrease according this order, one can conclude that the system is terminating. If the order is closed under contexts and substitutions then the decrease only has to be checked for the rewrite rules instead of all reduction steps. The bottleneck of this kind of method is how to prove that a relation defined recursively on terms is indeed a well-founded order. Proving irreflexivity and transitivity often turns out to be feasible, using some induction and case analysis. However, when stating an arbitrary recursive definition of such an order, well-foundedness is very hard to prove directly. Fortunately, the powerful *Tree Theorem* of Kruskal implies that if the order satisfies some simplification property, well-foundedness is obtained for free. An order satisfying this property is called a *simplification order*. This notion of simplification comprises two ingredients:

- a term decreases by removing parts of it, and
- a term decreases by replacing an operation symbol with a smaller (according to the precedence) one.

If the signature is infinite, both of these ingredients are essential for the applicability of Kruskal's Tree Theorem. It is amazing, however, that in the term rewriting literature the notion of simplification order is motivated by the applicability of Kruskal's Tree Theorem but only covers the first ingredient. For infinite signatures one easily defines non-well-founded orders that are simplification orders according to that definition. Therefore, the usual definition of simplification order is only helpful for proving termination of systems over finite signatures. Nevertheless, it is well-known that simplification orders like the recursive path order are also well-founded on terms over infinite signatures (provided the precedence on the signature is well-founded).

In this paper we propose a definition of a simplification order that matches exactly the requirements of Kruskal's Tree Theorem, since that is the basic motivation for the notion of simplification order. According to this new definition all simplification orders are well-founded, both over finite and infinite signatures. For finite signatures the new and the old notion of simplification order coincide. A term rewriting system is called *simply terminating* if there is a simplification order that orients the rewrite rules from left to right. It is immediate from the definition that every recursive path order over a well-founded precedence can be extended to a simplification order, and hence it is well-founded. Even if one is only interested in finite term rewriting systems this is of interest: *semantic labelling* ([18]) often succeeds in proving termination of a finite but "difficult" (non-simply terminating) system by transforming it into an infinite system over an infinite signature to which the recursive path order readily applies.

In the literature simplification orders are sometimes based on preorders (or quasi-orders) instead of (strict) partial orders. A main result of this paper is that there are no compelling reasons for doing so. We prove (constructively) that every term rewriting system which can be shown to be terminating by means of a simplification order based on preorders, can be shown to be terminating by means of a simplification order (based on partial orders). Since basing the notion of simplification order on preorders is more susceptive to mistakes and results in stronger proof obligations, simplification orders should be based on partial orders. (As explained in Section 3 these remarks already apply to finite signatures.) As a consequence, we prefer the partial order variant of *well-quasi-orders*, the so-called *partial well-orders*, in case of infinite signatures. By choosing partial well-orders instead of well-quasi-orders a great part of the theory is not affected, but another part becomes cleaner. For instance, in Section 5 we prove a useful result stating that a term rewriting system is simply terminating if and only if the union of the system and a particular system that captures simplification is terminating. Based on well-quasi-orders a similar result does not hold.

A useful notion of termination for term rewriting systems is *total termination* (see [6, 17]). For finite signature one easily shows that total termination implies simple termination. In Section 6 we show that for infinite signatures this does not hold any more: we construct an infinite term rewriting system whose termination can be proved by a polynomial interpretation, but which is not simply terminating.

2. Termination

In order to fix our notations and terminology, we start with a very brief introduction to term rewriting. Term rewriting is surveyed in Dershowitz and Jouannaud [4] and Klop [11].

A *signature* is a set $\mathcal{F}$ of *function symbols*. Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ be the set of all terms built from $\mathcal{F}$ and a countably infinite set $\mathcal{V}$ of *variables*, disjoint from $\mathcal{F}$. The set of variables occurring in a term t is denoted by $\mathcal{V}ar(t)$. A term t is called *ground* if $\mathcal{V}ar(t) = \varnothing$. The set of all ground terms is denoted by $\mathcal{T}(\mathcal{F})$.

We introduce a fresh constant symbol $\square$, named *hole*. A *context* C is a term in $\mathcal{T}(\mathcal{F} \cup \{\square\},\mathcal{V})$ containing precisely one hole. The designation *term* is restricted to members of $\mathcal{T}(\mathcal{F},\mathcal{V})$. If C is a context and t a term then $C[t]$ denotes the result of replacing the hole in C by t. A term s is a *subterm* of a term t if there exists a context C such that $t = C[s]$. A subterm s of t is *proper* if $s \neq t$. We assume familiarity with the *position* formalism for describing subterm occurrences. A *substitution* is a map σ from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ with the property that the set $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. If σ is a substitution and t a term then $t\sigma$ denotes the result of applying σ to t. We call $t\sigma$ an *instance* of t. A binary relation R on terms is *closed under contexts* if $C[s]\ R\ C[t]$ whenever $s\ R\ t$, for all contexts C. A binary relation R on terms is *closed under substitutions* if $s\sigma\ R\ t\sigma$ whenever $s\ R\ t$, for all substitutions σ. A *rewrite relation* is a binary relation on terms that is closed under contexts and substitutions.

A *rewrite rule* is a pair (l, r) of terms such that the left-hand side l is not a variable and variables which occur in the right-hand side r occur also in l, i.e., $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. Since we are interested in (simple) termination in this paper, these two restrictions rule out only trivial cases. Rewrite rules (l,r) will henceforth be written as $l \to r$.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F},\mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. We often present a TRS as a set of rewrite rules, without making explicit its signature, assuming that the signature consists of the function symbols occurring in the rewrite rules. The smallest rewrite relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that contains $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$. So $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $l \to r$ in $\mathcal{R}$, a substitution σ, and a context C such that $s = C[l\sigma]$ and $t = C[r\sigma]$. The subterm $l\sigma$ of s is called a *redex* and we say that s rewrites to t by *contracting* redex $l\sigma$. We call $s \to_{\mathcal{R}} t$ a *rewrite* or *reduction step.* The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to^+_{\mathcal{R}}$ and $\to^*_{\mathcal{R}}$ denotes the transitive and reflexive closure of $\to_{\mathcal{R}}$. If $s \to^*_{\mathcal{R}} t$ we say that s *reduces* to t. The converse of $\to^*_{\mathcal{R}}$ is denoted by $\leftarrow^*_{\mathcal{R}}$.

A TRS $(\mathcal{F}, \mathcal{R})$ is called *terminating* if there are no infinite reduction sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. In order to simplify matters, we assume throughout this paper that the signature $\mathcal{F}$ contains a constant symbol. Hence a TRS is terminating if and only if there do not exist infinite reduction sequence involving only ground terms.

A (strict) *partial order* $\succ$ is a transitive and irreflexive relation. The reflexive closure of $\succ$ is denoted by $\succcurlyeq$. The converse of $\succcurlyeq$ is denoted by $\preccurlyeq$. A partial order $\succ$ on a set A is *well-founded* if there are no infinite descending sequences $a_1 \succ a_2 \succ \cdots$ of elements of A. A partial order $\succ$ on A is *total* if for all different elements $a, b \in A$ either $a \succ b$ or $b \succ a$. A *preorder* (or *quasi-order*) $\succsim$ is a transitive and reflexive relation. The converse of $\succsim$ is denoted by $\precsim$. The *strict part* of a preorder $\succsim$ is the partial order $\succ$ defined as $\succsim \setminus \precsim$. Every preorder $\succsim$ induces an equivalence relation $\sim$ defined as $\succsim \cap \precsim$. It is easy to see that $\succ \, = \, \succsim \setminus \sim$. A preorder is said to be well-founded if its strict part is a well-founded partial order.

A rewrite relation that is also a partial order is called a *rewrite order.* A well-founded rewrite order is called a *reduction order.* We say that a TRS $(\mathcal{F}, \mathcal{R})$ and a partial order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are *compatible* if $\mathcal{R}$ is contained in $\succ$, i.e., $l \succ r$ for every rewrite rule $l \to r$ of $\mathcal{R}$. It is easy to show that a TRS is terminating if and only if it is compatible with a reduction order.

DEFINITION 2.1. We say that a binary relation R on terms has the *subterm property* if $C[t] \; R \; t$ for all contexts $C \neq \Box$ and terms t.

The task of showing that a given *transitive* relation R has the subterm property amounts to verifying $f(t_1, \ldots, t_n) \; R \; t_i$ for all function symbols f of arity $n \geqslant 1$, terms $t_1, \ldots, t_n$, and $i \in \{1, \ldots, n\}$. This observation will be used freely in the sequel.

DEFINITION 2.2. Let $\mathcal{F}$ be a signature. The TRS $\mathcal{E}mb(\mathcal{F})$ consists of all rewrite rules

$$f(x_1, \ldots, x_n) \quad \to \quad x_i$$

with $f \in \mathcal{F}$ a function symbol of arity $n \geqslant 1$ and $i \in \{1, \ldots, n\}$. Here $x_1, \ldots, x_n$ are pairwise different variables. We abbreviate $\to^+_{\mathcal{E}mb(\mathcal{F})}$ to $\rhd_{emb}$ and $\leftarrow^*_{\mathcal{E}mb(\mathcal{F})}$ to $\unlhd_{emb}$. The latter relation is called *embedding.*

The following easy result relates the subterm property to embedding.

LEMMA 2.3. *A rewrite order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has the subterm property if and only if it is compatible with the TRS $\mathcal{E}mb(\mathcal{F})$.* $\Box$

PROOF.

- $\Rightarrow$ The subterm property yields $f(x_1, \ldots, x_n) \succ x_i$ and hence $l \succ r$ for every rewrite rule $l \to r \in \mathcal{E}mb(\mathcal{F})$.
- $\Leftarrow$ We have to show $f(t_1, \ldots, t_n) \succ t_i$ for all function symbols f of arity $n \geqslant 1$, terms $t_1, \ldots, t_n$, and $i \in \{1, \ldots, n\}$. By assumption $f(x_1, \ldots, x_n) \succ x_i$. Closure under substitutions yields $f(t_1, \ldots, t_n) \succ t_i$.

$\Box$

3. Simple Termination — Finite Signatures

Throughout this section we are dealing with *finite* signatures only.

DEFINITION 3.1. A *simplification order* is a rewrite order with the subterm property. A TRS $(\mathcal{F},\mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$.

Since we are only interested in signatures consisting of function symbols with fixed arity, we have no need for the *deletion property* (cf. [2]). It should also be noted that many authors (e.g. [1, 2, 3, 7, 10, 16]) do not require that simplification orders are closed under substitutions. Since we don't really want to check whether a simplification order orients *all instances* of rewrite rules from left to right in order to conclude termination, and concrete simplification orders like the recursive path order are closed under substitutions, closure under substitutions should be part of the definition. Moreover, it is easy to show that the class of simply terminating TRSs is not affected by imposing closure under substitutions. Dershowitz [1, 2] showed that every simply terminating TRS is terminating. The proof is based on the beautiful Tree Theorem of Kruskal [12].

DEFINITION 3.2. An infinite sequence $t_1, t_2, t_3, \ldots$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$ is *self-embedding* if there exist $1 \leqslant i < j$ such that $t_i \trianglelefteq_{emb} t_j$.

THEOREM 3.3 (KRUSKAL'S TREE THEOREM—FINITE VERSION). *Every infinite sequence of ground terms is self-embedding.* □

We refrain from proving Theorem 3.3 since it is a special case of the general version of Kruskal's Tree Theorem, which is presented and proved in Section 4.

THEOREM 3.4. *Every simply terminating TRS is terminating.*

PROOF. Suppose there exists a simply terminating TRS $(\mathcal{F},\mathcal{R})$ that is not terminating. So $(\mathcal{F},\mathcal{R})$ is compatible with a simplification order $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ and there exists an infinite reduction sequence $t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} t_3 \rightarrow_{\mathcal{R}} \cdots$ involving only ground terms. From Kruskal's Tree Theorem we learn the existence of $1 \leqslant i < j$ such that $t_i \trianglelefteq_{emb} t_j$. From Lemma 2.3 we easily obtain $t_j \succcurlyeq t_i$. However, since $(\mathcal{F},\mathcal{R})$ is compatible with $\succ$, $t_i \rightarrow_{\mathcal{R}}^{+} t_j$ implies $t_i \succ t_j$. Hence we have a contradiction with the fact that $\succ$ is a partial order. We conclude that $(\mathcal{F},\mathcal{R})$ is terminating. □

The following well-known result is especially useful for showing that a given TRS is *not* simply terminating, see [17].

LEMMA 3.5. *A TRS $(\mathcal{F},\mathcal{R})$ is simply terminating if and only if $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F}))$ is terminating.*

PROOF.

- ⇒ Let $(\mathcal{F},\mathcal{R})$ be compatible with the simplification order $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$. From Lemma 2.3 we learn that $\succ$ is compatible with the TRS $\mathcal{E}mb(\mathcal{F})$. Hence the TRS $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F}))$ is simply terminating. Theorem 3.4 yields the termination of $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F}))$.
- ⇐ Let $\succ$ be the rewrite order associated with $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F}))$ (i.e., the transitive closure of its rewrite relation). Clearly $\succ$ is compatible with $\mathcal{E}mb(\mathcal{F})$. Lemma 2.3 shows that it is a simplification order. Since also the TRS $(\mathcal{F},\mathcal{R})$ is compatible with $\succ$, it is simply terminating.

□

In the term rewriting literature the notion of simplification order is sometimes based on preorders instead of partial orders. Dershowitz [2] obtained the following result.

THEOREM 3.6. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS. Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which is closed under contexts and has the subterm property. If $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ then $(\mathcal{F}, \mathcal{R})$ is terminating.* □

A preorder that is closed under contexts and has the subterm property is sometimes called a *quasi-simplification order*. Observe that we require *$l\sigma \succ r\sigma$ for all substitutions σ* in Theorem 3.6. It should be stressed that this requirement cannot be weakened to the compatibility of $(\mathcal{F}, \mathcal{R})$ and $\succ$ (i.e., $l \succ r$ for all rules $l \to r \in \mathcal{R}$) if we additionally require that $\succsim$ is closed under substitutions, as is incorrectly done in Dershowitz and Jouannaud [4]. For instance, the relation $\to_{\mathcal{R}}^*$ associated with the TRS

$$\mathcal{R} = \begin{cases} f(g(x)) & \to & f(f(x)) \\ f(g(x)) & \to & g(g(x)) \\ f(x) & \to & x \\ g(x) & \to & x \end{cases}$$

is a rewrite relation with the subterm property (because $\mathcal{R}$ contains $\mathcal{E}mb(\{f, g\})$). Moreover, $l \to_{\mathcal{R}}^* r$ but not $r \to_{\mathcal{R}}^* l$, for every rewrite rule $l \to r \in \mathcal{R}$. So $\mathcal{R}$ is included in the strict part of $\to_{\mathcal{R}}^*$. Nevertheless, $\mathcal{R}$ is not terminating:

$$f(g(g(x))) \to_{\mathcal{R}} f(f(g(x)) \to_{\mathcal{R}} f(g(g(x))) \to_{\mathcal{R}} \cdots.$$

The point is that the strict part of $\to_{\mathcal{R}}^*$ is not closed under substitutions. Hence to conclude termination from compatibility with $\succsim$ it is essential that $\succ$ is closed under substitutions. A simpler TRS illustrating the same point, due to Enno Ohlebusch (personal communication), is $\{f(x) \to f(a),\ f(x) \to x\}$.

Dershowitz [2] writes that Theorem 3.6 generalizes Theorem 3.4. We have the following result.

THEOREM 3.7. *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that is closed under contexts, has the subterm property, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ.* □

The proof is given in Section 5, where the above theorem is generalized to TRSs over arbitrary, not necessarily finite, signatures.

So every TRS whose termination can be shown by means of Theorem 3.6 is simply terminating, i.e., its termination can be shown by a simplification order. Since it is easier to check $l \succ r$ for finitely many rewrite rules $l \to r$ than $l\sigma \succsim r\sigma$ but not $r\sigma \succsim l\sigma$ for finitely many rewrite rules $l \to r$ and infinitely many substitutions σ, there is no reason to base the definition of simplification order on preorders.

4. Partial Well-Orders

Theorem 3.4 does not hold if we allow infinite signatures. Consider for instance the TRS $(\mathcal{F}, \mathcal{R})$ consisting of infinitely many constants a_i and rewrite rules $a_i \to a_{i+1}$ for all $i \geqslant 1$. The rewrite order $\to_{\mathcal{R}}^+$ vacuously satisfies the subterm property, but $(\mathcal{F}, \mathcal{R})$ is not terminating:

$$a_1 \to_{\mathcal{R}} a_2 \to_{\mathcal{R}} a_3 \to_{\mathcal{R}} \cdots$$

So in case $\mathcal{F}$ is infinite, compatibility with $\mathcal{E}mb(\mathcal{F})$ does not ensure termination. In the next section we will see that the results of the previous section can be recovered by suitably extending the TRS $\mathcal{E}mb(\mathcal{F})$.

DEFINITION 4.1. Let $\succ$ be a partial order on a signature $\mathcal{F}$. The TRS $\mathcal{E}mb(\mathcal{F},\succ)$ consists of all rewrite rules of $\mathcal{E}mb(\mathcal{F})$ together with all rewrite rules

$$f(x_1,\ldots,x_n) \;\to\; g(x_{i_1},\ldots,x_{i_m})$$

with f an n-ary function symbol in $\mathcal{F}$, g an m-ary function symbol in $\mathcal{F}$, $n \geqslant m \geqslant 0$, $f \succ g$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. Here $x_1,\ldots,x_n$ are pairwise different variables. We abbreviate $\to^+_{\mathcal{E}mb(\mathcal{F},\succ)}$ to $\succ_{emb}$ and $\leftarrow^*_{\mathcal{E}mb(\mathcal{F},\succ)}$ to $\preccurlyeq_{emb}$. The latter relation is called *homeomorphic embedding*.

Since $\mathcal{E}mb(\mathcal{F},\varnothing) = \mathcal{E}mb(\mathcal{F})$, homeomorphic embedding generalizes embedding. Consider for instance the signature $\mathcal{F}$ consisting of constants a and b, a unary function symbol g, and binary functions symbols f and h. Define the partial order $\succ$ on $\mathcal{F}$ by $a \succ b \succ f \succ g \succ h$. In the TRS

$$\mathcal{E}mb(\mathcal{F},\succ) = \mathcal{E}mb(\mathcal{F}) \cup \left\{ \begin{array}{rcl} a & \to & b \\ f(x,y) & \to & g(x) \\ f(x,y) & \to & g(y) \\ f(x,y) & \to & h(x,y) \end{array} \right\}$$

we have the reduction sequence $f(h(a,b),g(a)) \to f(a,g(a)) \to f(a,a) \to f(a,b)$, hence the term $f(a,b)$ is homeomorphically embedded in $f(h(a,b),g(a))$. Since there is no reduction sequence in the TRS $\mathcal{E}mb(\mathcal{F})$ from $f(h(a,b),g(a))$ to $f(a,b)$, the term $f(a,b)$ is not embedded in $f(h(a,b),g(a))$.

In the next section we show that all results of the previous section carry over to infinite signatures if we require compatibility with $\mathcal{E}mb(\mathcal{F},\succ)$, provided the partial order $\succ$ satisfies a stronger property than well-foundedness. This property is explained below.

DEFINITION 4.2. Let $\succ$ be a partial order on a set A.

- An infinite sequence $(a_i)_{i\geqslant 1}$ over A is called *good* if there exist indices $1 \leqslant i < j$ with $a_i \preccurlyeq a_j$, otherwise it is called *bad*.
- An infinite sequence $(a_i)_{i\geqslant 1}$ over A is called a *chain* if $a_i \preccurlyeq a_{i+1}$ for all $i \geqslant 1$. We say that $(a_i)_{i\geqslant 1}$ contains a chain if it has a subsequence that is a chain.
- An infinite sequence $(a_i)_{i\geqslant 1}$ over A is called an *antichain* if neither $a_i \preccurlyeq a_j$ nor $a_j \preccurlyeq a_i$, for all $1 \leqslant i < j$.

LEMMA 4.3. *Let $\succ$ be a partial order on a set A. The following statements are equivalent.*
(1) *Every partial order that extends $\succ$ (including $\succ$ itself) is well-founded.*
(2) *Every infinite sequence over A is good.*
(3) *Every infinite sequence over A contains a chain.*
(4) *The partial order $\succ$ is well-founded and does not admit antichains.*

PROOF.

(1) $\Rightarrow$ (2) Suppose $(a_i)_{i\geqslant 1}$ is a bad sequence. Define $\succ' = (\succ \cup \{(a_i, a_{i+1}) \mid i \geqslant 1\})^+$. Assume $a \succ' a$ for some $a \in A$. Since $\succ$ is irreflexive there is a non-empty sequence of numbers $i_1, \ldots, i_n$ such that

$$a \succcurlyeq a_{i_1},\ a_{i_1+1} \succcurlyeq a_{i_2},\ a_{i_2+1} \succcurlyeq a_{i_3},\ \ldots,\ a_{i_{n-1}+1} \succcurlyeq a_{i_n},\ a_{i_n+1} \succcurlyeq a.$$

Since $(a_i)_{i\geqslant 1}$ is bad $a_i \succcurlyeq a_j$ is only possible for $i \leqslant j$. Hence we obtain the impossible

$$i_1 < i_1 + 1 \leqslant i_2 < i_2 + 1 \leqslant i_3 < \cdots < i_{n-1} + 1 \leqslant i_n < i_n + 1 \leqslant i_1.$$

We conclude that $\succ'$ is irreflexive. By definition it is transitive, hence it is a partial order extending $\succ$. However, since $a_1 \succ' a_2 \succ' a_3 \succ' \cdots$, it is not well-founded.

(2) $\Rightarrow$ (3) Let $(a_i)_{i\geqslant 1}$ be any infinite sequence over A. Consider the subsequence consisting of all elements a_i with the property that $a_i \preccurlyeq a_j$ holds for no $j > i$. If this subsequence is infinite then it is a bad sequence, contradicting (2). Hence it is finite, and thus there exists an index $N \geqslant 1$ such that for every $i \geqslant N$ there exists a $j > i$ with $a_i \preccurlyeq a_j$. Define inductively

$$\phi(i) = \begin{cases} N & \text{if } i = 1, \\ \min\{j \mid j > \phi(i-1) \text{ and } a_{\phi(i-1)} \preccurlyeq a_j\} & \text{if } i > 1. \end{cases}$$

Now $a_{\phi(1)}, a_{\phi(2)}, a_{\phi(3)}, \ldots$ is a chain.

(3) $\Rightarrow$ (4) If $\succ$ is not well-founded then there exists an infinite sequence $a_1 \succ a_2 \succ \cdots$. Clearly $a_i \preccurlyeq a_j$ doesn't hold for any $1 \leqslant i < j$. Hence this sequence doesn't contain a chain. If $\succ$ admits an antichain then this antichain is an infinite sequence not containing a chain.

(4) $\Rightarrow$ (1) For a proof by contradiction, let $\succ$ be a well-founded partial order not satisfying (1). Then there is an extension $\succ'$ of $\succ$ that is not well-founded. So there exists an infinite sequence $a_1 \succ' a_2 \succ' \cdots$. Since $\succ$ is well-founded, the sequence $(a_i)_{i\geqslant 1}$ contains an element a_i with the property that for no $j > i$ $a_i \succ a_j$ holds. Actually, $(a_i)_{i\geqslant 1}$ contains infinitely many such elements. We claim that the infinite subsequence $(a_{\phi(i)})_{i\geqslant 1}$ consisting of those elements is an antichain (with respect to $\succ$). Let $1 \leqslant i < j$. By construction $a_{\phi(i)} \succ a_{\phi(j)}$ is impossible. If $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$ then also $a_{\phi(i)} \preccurlyeq' a_{\phi(j)}$, contradicting $a_{\phi(i)} \succ' a_{\phi(j)}$. Hence $\succ$ admits a anti-chain.

□

DEFINITION 4.4. A partial order $\succ$ on a set A is called a *partial well-order* (PWO for short) if it satisfies one of the four equivalent assertions of Lemma 4.3.

Using the terminology of PWOs, Theorem 3.3 can now be read as follows: if $\mathcal{F}$ is a finite signature then $\rhd_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$.

By definition every PWO is a well-founded order, but the reverse does not hold. For instance, the empty relation on an infinite set is a well-founded order but not a PWO. Clearly every total well-founded order (or well-order) is a PWO. Any partial order extending a PWO is a PWO. The following lemma states how new PWOs can be obtained by restricting existing PWOs.

LEMMA 4.5. *Let $\succ$ be a PWO on a set A and let $\sqsupset$ be a PWO on a set B. Let $\varphi\colon A \to B$ be any function. The partial order $\succ'$ on A defined by $a \succ' b$ if and only if $a \succ b$ and $\varphi(a) \sqsupseteq \varphi(b)$ is a PWO.*

PROOF. Let $(a_i)_{i\geqslant 1}$ be any infinite sequence over A. Since $\succ$ is a PWO this sequence admits a chain

$$a_{\phi(1)} \preccurlyeq a_{\phi(2)} \preccurlyeq a_{\phi(3)} \preccurlyeq \cdots.$$

Since $\sqsupset$ is a PWO on B there exist $1 \leqslant i < j$ with $\varphi(a_{\phi(i)}) \sqsubseteq \varphi(a_{\phi(j)})$. Transitivity of $\preccurlyeq$ yields $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$. Hence $a_{\phi(i)} \preccurlyeq' a_{\phi(j)}$, while $\phi(i) < \phi(j)$. We conclude that $(a_i)_{i\geqslant 1}$ is a good sequence with respect to $\succ'$, so $\succ'$ is a PWO. □

COROLLARY 4.6. *The intersection of two PWOs on a set A is a PWO on A.*

PROOF. Choose the function φ in Lemma 4.5 to be the identity on A. □

THEOREM 4.7 (KRUSKAL'S TREE THEOREM—GENERAL VERSION). *If $\succ$ is a PWO on a signature $\mathcal{F}$ then $\succ_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$.* □

For the sake of completeness, below we present a proof of this beautiful theorem, even though it is very similar to the proof of the Kruskal's Tree Theorem formulated in terms of *well-quasi-orders* (see e.g. Gallier [7]). First we show a related result for strings, known as *Higman's Lemma* (Higman [8]).

DEFINITION 4.8. Let $\succ$ be a partial order on a set A. We define a relation $\succ^*$ on A^* as follows: if $w_1 = a_1a_2\cdots a_n$ and $w_2 = b_1b_2\cdots b_m$ are elements of A^* then $w_1 \succ^* w_2$ if and only if $w_1 \neq w_2$ and either

- $m = 0$, or
- $n \geqslant m > 0$ and there exist indices $i_1,\ldots,i_m$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and $a_{i_j} \succcurlyeq b_j$ for all $j \in \{1,\ldots,m\}$.

The next result can be viewed as an alternative definition of $\succ^*$.

LEMMA 4.9. *Let $\succ$ be a partial order on a set A. The relation $\succ^*$ is the least partial order $\sqsupset$ on A^* satisfying the following two properties:*

(1) $w_1aw_2 \sqsupset w_1w_2$ *for all* $w_1, w_2 \in A^*$ *and* $a \in A$,

(2) $w_1aw_2 \sqsupset w_1bw_2$ *for all* $w_1, w_2 \in A^*$ *and* $a, b \in A$ *with* $a \succ b$.

PROOF. First we show that $\succ^*$ is a partial order. Irreflexivity is obvious. Let $w_1 = a_1\cdots a_n$, $w_2 = b_1\cdots b_m$, and $w_3 = c_1\cdots c_l$ be elements of A^* such that $w_1 \succ^* w_2 \succ^* w_3$. If $l = 0$ then $m > 0$ (because $w_2 \neq w_3$) and $n \geqslant m > 0$. Hence $w_1 \succ^* w_3$. Suppose $l > 0$. We have $n \geqslant m \geqslant l$. There exist indices $i_1,\ldots,i_l$ and $j_1,\ldots,j_m$ such that $1 \leqslant i_1 < \cdots < i_l \leqslant m$, $b_{i_k} \succcurlyeq c_k$ for all $k \in \{1,\ldots,l\}$, $1 \leqslant j_1 < \cdots < j_m \leqslant n$, and $a_{j_k} \succcurlyeq b_k$ for all $k \in \{1,\ldots,m\}$. Since $1 \leqslant j_{i_1} < \cdots < j_{i_l} \leqslant n$ and $a_{j_{i_k}} \succcurlyeq b_{i_k} \succcurlyeq c_k$ for all $k \in \{1,\ldots,l\}$, we have $w_1 \succ^* w_3$. This concludes the proof of the transitivity of $\succ^*$. It is very easy to see that $\succ^*$ satisfies properties (1) and (2). Conversely, let $\sqsupset$ be any partial order on A^* that satisfies properties (1) and (2). We will show that $\succ^* \subseteq \sqsupset$. Suppose $w_1 = a_1\cdots a_n \succ^* b_1\cdots b_m = w_2$. If $m = 0$ then $n > 0$ and hence the sequence $w_1 = a_1\cdots a_n \sqsupset a_2\cdots a_n \sqsupset \cdots \sqsupset a_n \sqsupset \varepsilon = w_2$ is non-empty, showing that $w_1 \sqsupset w_2$. If $n \geqslant m > 0$ then there exist indices $i_1,\ldots,i_m$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and $a_{i_j} \succcurlyeq b_j$ for all $j \in \{1,\ldots,m\}$. Let $w_3 = a_{i_1}\cdots a_{i_m}$. We have $w_1 \sqsupseteq w_3$ by successively removing elements a_i from w_1 whose index i does not belong to the set $\{i_1,\ldots,i_m\}$. (Clearly $w_1 = w_3$ if and only if $n = m$.) We have $w_3 \sqsupseteq w_2$ by replacing a_{i_j} with b_j whenever $a_{i_j} \succ b_j$. Therefore $w_1 \sqsupseteq w_2$ and since $w_1 \neq w_2$ we obtain $w_1 \sqsupset w_2$. □

LEMMA 4.10 (HIGMAN'S LEMMA). *If $\succ$ is a PWO on a set A then $\succ^*$ is a PWO on A^*.*

PROOF. The following proof is essentially due to Nash-Williams [14]. We have to show that there are no bad sequences over A^*. Suppose to the contrary that there exist bad sequences over A^*. We construct a *minimal bad sequence* $(w_i)_{i\geqslant 1}$ as follows:

> Suppose we already chose the first $n-1$ strings $w_1, \ldots, w_{n-1}$. Define w_n to be a shortest string such that there are bad sequences that start with $w_1, \ldots, w_n$.

Because $\varepsilon \preccurlyeq^* w$ for all $w \in A^*$, we have $w_i \neq \varepsilon$ for all $i \geqslant 1$. Hence we may write $w_i = a_i v_i$ ($i \geqslant 1$). Since $\succ$ is a PWO on A, the infinite sequence $(a_i)_{i\geqslant 1}$ contains a chain, say $(a_{\phi(i)})_{i\geqslant 1}$. Because $v_{\phi(1)}$ is shorter than $w_{\phi(1)}$, the sequence

$$w_1,\ \ldots,\ w_{\phi(1)-1},\ v_{\phi(1)},\ v_{\phi(2)},\ \ldots$$

must be good. Clearly $w_i \preccurlyeq^* w_j$ ($1 \leqslant i < j \leqslant \phi(1)-1$) is impossible as $(w_i)_{i\geqslant 1}$ is bad. Likewise, $w_i \preccurlyeq^* v_{\phi(j)}$ ($1 \leqslant i \leqslant \phi(1)-1$ and $1 \leqslant j$) contradicts the badness of $(w_i)_{i\geqslant 1}$ since $v_{\phi(j)} \preccurlyeq^* w_{\phi(j)}$ and therefore $w_i \preccurlyeq^* w_{\phi(j)}$. Hence we must have $v_{\phi(i)} \preccurlyeq^* v_{\phi(j)}$ for some $1 \leqslant i < j$. Combining this with $a_{\phi(i)} \preccurlyeq a_{\phi(j)}$ easily yields $w_{\phi(i)} = a_{\phi(i)}v_{\phi(i)} \preccurlyeq^* a_{\phi(j)}v_{\phi(j)} = w_{\phi(j)}$, contradicting the badness of $(w_i)_{i\geqslant 1}$. We conclude that there are no bad sequences over A^*. □

PROOF OF KRUSKAL'S TREE THEOREM—GENERAL VERSION. The proof, essentially due to Nash-Williams [14], has the same structure as the proof of Higman's Lemma. We have to show that there are no bad sequences of terms in $\mathcal{T}(\mathcal{F})$. Suppose to the contrary that there exist bad sequences of ground terms. We construct a minimal bad sequence $(t_i)_{i\geqslant 1}$ as follows:

> Suppose we already chose the first $n-1$ terms $t_1, \ldots, t_{n-1}$. Define t_n to be a smallest (with respect to size) term such that there are bad sequences that start with $t_1, \ldots, t_n$.

For every $i \geqslant 1$, let f_i be the root symbol of t_i and let A_i be the set of arguments of t_i (if t_i is a constant then $A_i = \varnothing$). Moreover, let w_i be the string of arguments (from left to right) of t_i. Finally, let $A = \bigcup_{i\geqslant 1} A_i$.

We claim that $\succ_{emb}$ is a PWO on the subset A of $\mathcal{T}(\mathcal{F})$. For a proof by contradiction, suppose $(a_i)_{i\geqslant 1}$ is a bad sequence over A. Let $a_1 \in A_k$. Since $A' = \bigcup_{i=1}^{k-1} A_i$ is a finite set and all elements of $(a_i)_{i\geqslant 1}$ are different, only finitely many elements of $(a_i)_{i\geqslant 1}$ belong to A'. Thus there exists an index $l > 1$ such that $a_i \in A\backslash A'$ for all $i \geqslant l$. Because a_1 is a proper subterm of t_k, the sequence

$$t_1,\ \ldots,\ t_{k-1},\ a_1,\ a_l,\ a_{l+1},\ \ldots$$

must be good. Clearly $t_i \preccurlyeq_{emb} t_j$ ($1 \leqslant i < j \leqslant k-1$) is impossible as $(t_i)_{i\geqslant 1}$ is bad. Likewise, $t_i \preccurlyeq_{emb} a_j$ ($1 \leqslant i \leqslant k-1$ and $j = 1$ or $l \leqslant j$) contradicts the badness of $(t_i)_{i\geqslant 1}$ since $a_j \preccurlyeq_{emb} t_m$ for some $m \geqslant k$—recall that a_1 is a proper subterm of t_k and if $j \geqslant l$ then $a_j \in A\backslash A'$—and thus $t_i \preccurlyeq_{emb} t_j$. Hence we must have $a_i \preccurlyeq_{emb} a_j$ for some $1 \leqslant i < j$ (and $i, j \notin \{2, \ldots, l-1\}$), contradicting the badness of $(a_i)_{i\geqslant 1}$. Hence $\succ_{emb}$ is a PWO on A. From Higman's Lemma we infer that $\succ_{emb}^*$ is a PWO on A^*.

Since $\succ$ is a PWO on $\mathcal{F}$, the infinite sequence $(f_i)_{i\geqslant 1}$ contains a chain, say $(f_{\phi(i)})_{i\geqslant 1}$. Consider the infinite sequence $(w_{\phi(i)})_{i\geqslant 1}$ over A^*. Since $\succ_{emb}^*$ is a PWO on A^*, we have $w_{\phi(i)} \preccurlyeq_{emb}^* w_{\phi(j)}$ for some $1 \leqslant i < j$. A straightforward case analysis reveals that $f_{\phi(i)} \preccurlyeq f_{\phi(j)}$ and $w_{\phi(i)} \preccurlyeq_{emb}^* w_{\phi(j)}$ imply $t_{\phi(i)} \preccurlyeq_{emb} t_{\phi(j)}$. Hence we obtained a contradiction with the badness of $(t_i)_{i\geqslant 1}$. We conclude that there are no bad sequences over $\mathcal{T}(\mathcal{F})$. □

PWOs are closely related to the more familiar concept of well-quasi-order.

DEFINITION 4.11. A *well-quasi-order* (WQO for short) is a preorder that contains a PWO.

The above definition is equivalent to all other definitions of WQO found in the literature. Kruskal's Tree Theorem is usually presented in terms of WQOs. This is not more powerful than the PWO version: notwithstanding the fact that the strict part of a WQO is not necessarily a PWO, it is very easy to show that the WQO version of Kruskal's Tree Theorem is a corollary of Theorem 4.7, and vice-versa.

Let $\succ$ be a PWO on a signature $\mathcal{F}$. A natural question is whether we can restrict $\succ_{emb}$ while retaining the property of being a PWO on $\mathcal{T}(\mathcal{F})$. In particular, do we really need all rewrite rules in $\mathcal{E}mb(\mathcal{F},\succ)$? In case there is a uniform bound on the arities of the function symbols in $\mathcal{F}$, we can greatly reduce the set $\mathcal{E}mb(\mathcal{F},\succ)$. That is, suppose there exists an $N \geqslant 0$ such that all function symbols in $\mathcal{F}$ have arity less than or equal to N. Now we can apply Lemma 4.5: choose φ to be the function that assigns to every function symbol its arity and take $\sqsupset$ to be the empty relation on $\{1,\ldots,N\}$. Hence the partial order $\succ'$ on $\mathcal{F}$ defined by $f \succ' g$ if and only if f and g have the same arity and $f \succ g$ is a PWO. The corresponding set $\mathcal{E}mb(\mathcal{F},\succ')$ consists, besides all rewrite rules of the form $f(x_1,\ldots,x_n) \rightarrow x_i$, of all rewrite rules $f(x_1,\ldots,x_n) \rightarrow g(x_1,\ldots,x_n)$ with f and g n-ary function symbols such that $f \succ g$. This construction does not work if the arities of function symbols in $\mathcal{F}$ are not uniformly bounded. Consider for instance a signature $\mathcal{F}$ consisting of a constant a and n-ary function symbols f_n for every $n \geqslant 1$ (and let $\succ$ be any PWO on $\mathcal{F}$). The sequence

$$f_1(a),\ f_2(a,a),\ f_3(a,a,a),\ \ldots$$

is bad with respect to $\succ'_{emb}$. Finally, one may wonder whether the restriction to all rewrite rules $f(x_1,\ldots,x_n) \rightarrow g(x_{i+1},\ldots,x_{i+m})$ with f an n-ary function symbol, g an m-ary function symbol, $n \geqslant m \geqslant 0$, $n-m \geqslant i \geqslant 0$, and $f \succ g$ is sufficient. This is also not the case, as can be seen by extending the previous signature with a constant b and considering the sequence

$$f_2(b,b),\ f_3(b,a,b),\ f_4(b,a,a,b),\ \ldots .$$

Of course, if the signature $\mathcal{F}$ is finite then the rules of $\mathcal{E}mb(\mathcal{F})$ are sufficient since the empty relation is a PWO on any finite set.

5. Simple Termination — Infinite Signatures

Kurihara and Ohuchi [13] were the first to use the terminology simple termination. They call a TRS $(\mathcal{F},\mathcal{R})$ simply terminating if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$. Since compatibility with a simplification order doesn't ensure the termination of TRSs over infinite signatures, see the example at the beginning of the previous section, this definition of simple termination is clearly not the right one. Ohlebusch [15] and others call a TRS $(\mathcal{F},\mathcal{R})$ simply terminating if it is compatible with a *well-founded* simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$. This is a very artificial way to ensure that every simply terminating is terminating, more precisely, termination of simply terminating TRSs has nothing to do with Kruskal's Tree Theorem; simply terminating TRSs are terminating by definition. We propose instead to bring the definition of simple termination in accordance with (the general version of) Kruskal's Tree Theorem.

DEFINITION 5.1. A *simplification order* is a rewrite order on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that contains $\succ_{emb}$ for some PWO $\succ$ on $\mathcal{F}$. A TRS $(\mathcal{F},\mathcal{R})$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$.

This definition coincides with the one in Section 3 in case of finite signatures:

LEMMA 5.2. *A rewrite order $\sqsupset$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ with $\mathcal{F}$ finite is a simplification order if and only if it has the subterm property, i.e., $\rhd \subseteq \sqsupset$.*

PROOF.

$\Rightarrow$ By definition there exists a PWO $\succ$ on $\mathcal{F}$ such that $\succ_{emb} \subseteq \sqsupset$. Since $\rhd \subseteq \succ_{emb}$, $\sqsupset$ has the subterm property.

$\Leftarrow$ The empty relation $\varnothing$ is a PWO on any finite set. The subterm property yields $\varnothing_{emb} = \rhd \subseteq \sqsupset$. Hence $\sqsupset$ is a simplification order.

□

THEOREM 5.3. *Every simply terminating TRS is terminating.*

PROOF. Let $(\mathcal{F},\mathcal{R})$ be compatible with a simplification order $\sqsupset$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$. Let $\succ$ be any PWO such that $\succ_{emb}$ is included in $\sqsupset$. Theorem 4.7 shows that the restriction of $\succ_{emb}$ to ground terms is a PWO. Hence the extension $\sqsupset$ of $\succ_{emb}$ is well-founded on ground terms. Therefore $(\mathcal{F},\mathcal{R})$ is terminating. □

The following result extends the very useful Lemma 3.5 to arbitrary TRSs. In the proof of Theorem 5.9 below and in the final example of Section 6 we make use of this result.

LEMMA 5.4. *A TRS $(\mathcal{F},\mathcal{R})$ is simply terminating if and only if the TRS $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is terminating for some PWO $\succ$ on $\mathcal{F}$.*

PROOF.

$\Rightarrow$ Let $(\mathcal{F},\mathcal{R})$ be compatible with the simplification order $\sqsupset$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$. By definition there exists a PWO $\succ$ on $\mathcal{F}$ such that $\succ_{emb} \subseteq \sqsupset$. If $l \to r \in \mathcal{E}mb(\mathcal{F},\succ)$ then $l \succ_{emb} r$ and therefore $l \sqsupset r$. Hence $\mathcal{E}mb(\mathcal{F},\succ)$ is also compatible with $\sqsupset$. So $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is simply terminating. Theorem 5.3 shows that $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is terminating.

$\Leftarrow$ Suppose $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$ is terminating for some PWO $\succ$ on $\mathcal{F}$. Let $\sqsupset$ be the rewrite order associated with the TRS $(\mathcal{F},\mathcal{R}\cup\mathcal{E}mb(\mathcal{F},\succ))$. Clearly $\succ_{emb} \subseteq \sqsupset$. Hence $\sqsupset$ is a simplification order. Since $(\mathcal{F},\mathcal{R})$ is compatible with $\sqsupset$, we conclude that it is simply terminating.

□

It should be stressed that there is no equivalent to the above lemma if we base the definition of simplification order on WQOs. This is one of the reasons why we favor PWOs.

In the remainder of this section we generalize Theorem 3.7 (and hence Theorem 3.6) to arbitrary TRSs. Our proof is based on the elegant proof sketch of Theorem 3.6 given by Plaisted [16]. The proof employs *multiset extensions* of preorders. A *multiset* is a collection in which elements are allowed to occur more than once. If A is a set then the set of all finite multisets over A is denoted by $\mathcal{M}(A)$. The *multiset extension* of a partial order $\succ$ on A is the partial order $\succ_{mul}$ defined on $\mathcal{M}(A)$ defined as follows: $M_1 \succ_{mul} M_2$ if $M_2 = (M_1 - X) \uplus Y$ for some multisets $X, Y \in \mathcal{M}(A)$ that satisfy $\varnothing \neq X \subseteq M_1$ and for all $y \in Y$ there exists an $x \in X$ such that $x \succ y$. Using Higman's Lemma, it is quite easy to show that multiset extension preserves PWO. From this we infer that the multiset extension of a well-founded partial order is well-founded, using the well-known facts that (1) every well-founded partial order can be extended to a total well-founded order (in particular a PWO) and (2) multiset extension is monotonic (i.e., if $\succ \subseteq \sqsupset$ then $\succ_{mul} \subseteq \sqsupset_{mul}$). Using König's Lemma, Dershowitz and Manna [5] gave a direct proof that multiset extension preserves well-founded partial orders.

DEFINITION 5.5. Let $\succsim$ be a preorder on a set A. For every $a \in A$, let $[a]$ denote the equivalence class with respect to the equivalence relation $\sim$ containing a. Let $A\backslash\!\sim\, = \{[a] \mid a \in A\}$ be the set of all equivalence classes of A. The preorder $\succsim$ on A induces a partial order $\succ$ on $A\backslash\!\sim$ as follows: $[a] \succ [b]$ if and only if $a \succ b$. (The latter $\succ$ denotes the strict part of the preorder $\succsim$.) For every multiset $M \in \mathcal{M}(A)$, let $[M] \in \mathcal{M}(A\backslash\!\sim)$ denote the multiset obtained from M by replacing every element a by $[a]$. We now define the *multiset extension* $\succsim_{mul}$ of the preorder $\succsim$ as follows: $M_1 \succsim_{mul} M_2$ if and only if $[M_1] \succ^{=}_{mul} [M_2]$ where $\succ^{=}_{mul}$ denotes the reflexive closure of the multiset extension of the partial order $\succ$ on $A\backslash\!\sim$.

It is easy to show that $\succsim_{mul}$ is a preorder on $\mathcal{M}(A)$. The associated equivalence relation $\sim_{mul} = \succsim_{mul} \cap \precsim_{mul}$ can be characterized in the following simple way: $M_1 \sim_{mul} M_2$ if and only if $[M_1] = [M_2]$. Likewise, its strict part $\succnsim_{mul} = \succsim_{mul} \backslash \precsim_{mul} = \succsim_{mul} \backslash \sim_{mul}$ has the following simple characterization: $M_1 \succnsim_{mul} M_2$ if and only if $[M_1] \succ_{mul} [M_2]$. Observe that we denote the strict part of $\succsim_{mul}$ by $\succnsim_{mul}$ in order to avoid confusion with the multiset extension $\succ_{mul}$ of the strict part $\succ$ of $\succsim$, which is a smaller relation.

The above definition of multiset extension of a preorder can be shown to be equivalent to the more operational ones in Dershowitz [3] and Gallier [7], but since we define the multiset extension of a preorder in terms of the well-known multiset extension of a partial order, we get all desired properties basically for free. In particular, using the fact that multiset extension preserves well-founded partial orders, it is very easy to show that the multiset extension of a well-founded preorder is well-founded.

DEFINITION 5.6. If $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $S(t) \in \mathcal{M}(\mathcal{T}(\mathcal{F}, \mathcal{V}))$ denotes the finite multiset of all subterm occurrences in t and $F(t) \in \mathcal{M}(\mathcal{F})$ denotes the finite multiset of all function symbol occurrences in t. Formally,

$$M(t) = \begin{cases} \{t\} & \text{if } t \text{ is a variable,} \\ \{t\} \uplus \biguplus_{i=1}^{n} M(t_i) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

$$F(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \{f\} \uplus \biguplus_{i=1}^{n} F(t_i) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

LEMMA 5.7. *Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the subterm property. If $s \succ t$ then $S(s) \succnsim_{mul} S(t)$.*

PROOF. We show that $s \succ t'$ for all $t' \in S(t)$. This implies $\{s\} \succnsim_{mul} S(t)$ and hence also $S(s) \succnsim_{mul} S(t)$. If $t' = t$ then $s \succ t'$ by assumption. Otherwise t' is a proper subterm of t and hence $t \succsim t'$ by the subterm property. Combining this with $s \succ t$ yields $s \succ t$. □

LEMMA 5.8. *Let $\succsim$ be a preorder on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which is closed under contexts. Suppose $s \succsim t$ and let C be an arbitrary context.*

- *If $S(s) \succnsim_{mul} S(t)$ then $S(C[s]) \succnsim_{mul} S(C[t])$.*
- *If $S(s) \succsim_{mul} S(t)$ then $S(C[s]) \succsim_{mul} S(C[t])$.*

PROOF. Let $S_1 = S(C[s]) - S(s)$ and $S_2 = S(C[t]) - S(t)$. For both statements it suffices to prove that $S_1 \succsim_{mul} S_2$. Let $p \in \mathcal{P}os(C[s])$ be the position of the displayed s in $C[s]$. There is a one-to-one correspondence between terms in S_1 (S_2) and positions in $\mathcal{P}os(C) - \{p\}$. Hence it suffices to show that $s' \succsim t'$ where $s' = C[s]_{|q}$ and $t' = C[t]_{|q}$ are the to position q corresponding terms in S_1 and S_2, for all $q \in \mathcal{P}os(C) - \{p\}$. If p and q are disjoint positions then $s' = t'$. Otherwise $q < p$ and there exists a context C' such that $s' = C'[s]$ and $t' = C'[t]$. By assumption $s \succsim t$. Closure under contexts yields $s' \succsim t'$. We conclude that $S_1 \succsim_{mul} S_2$. □

After these two preliminary results we are ready for the generalization of Theorem 3.7 to arbitrary TRSs.

THEOREM 5.9. *A TRS $(\mathcal{F}, \mathcal{R})$ is simply terminating if and only if there exists a preorder $\succsim$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that is closed under contexts, contains the relation $\sqsupset_{emb}$ for some PWO $\sqsupset$ on $\mathcal{F}$, and satisfies $l\sigma \succ r\sigma$ for every rewrite rule $l \to r \in \mathcal{R}$ and substitution σ.*

PROOF. The "only if" direction is obvious since the reflexive closure $\succcurlyeq$ of the simplification order $\succ$ used to prove simple termination is a preorder with the desired properties. For the "if" direction it suffices to show that $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset))$ is a terminating TRS, according to Lemma 5.4. First we show that either $S(s) \succnsim_{mul} S(t)$ or $S(s) \sim_{mul} S(t)$ and $F(s) \sqsupset_{mul} F(t)$ whenever $s \to t$ is a reduction step in the TRS $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset))$. So let $s = C[l\sigma]$ and $t = C[r\sigma]$ with $l \to r \in \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset)$. We distinguish three cases.

- If $l \to r \in \mathcal{R}$ then $l\sigma \succ r\sigma$ by assumption and $S(l\sigma) \succnsim_{mul} S(r\sigma)$ according to Lemma 5.7. The first part of Lemma 5.8 yields $S(s) \succnsim_{mul} S(t)$.
- If $l \to r \in \mathcal{E}mb(\mathcal{F})$ then $l\sigma = f(t_1, \dots, t_n)$ and $r\sigma = t_i$ for some $i \in \{1, \dots, n\}$. Therefore $S(l\sigma) \succnsim_{mul} S(r\sigma)$ since $S(t_i)$ is properly contained in $S(f(t_1, \dots, t_n))$. Clearly $l\sigma \sqsupset_{emb} r\sigma$ and thus also $l\sigma \succsim r\sigma$. An application of the first part of Lemma 5.8 yields $S(s) \succnsim_{mul} S(t)$.
- If $l \to r \in \mathcal{E}mb(\mathcal{F}, \sqsupset) - Emb(\mathcal{F})$ then $l\sigma = f(t_1, \dots, t_n)$ and $r\sigma = g(t_{i_1}, \dots, t_{i_m})$ with $f \sqsupset g$, $n \geqslant m \geqslant 0$, and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$. We have of course $l\sigma \sqsupset_{emb} r\sigma$ and thus also $l\sigma \succsim r\sigma$. Since the multiset $\{t_{i_1}, \dots, t_{i_m}\}$ is contained in the multiset $\{t_1, \dots, t_n\}$, we obtain $S(l\sigma) \succsim_{mul} S(r\sigma)$ and $F(l\sigma) \sqsupset_{mul} F(r\sigma)$. The second part of Lemma 5.8 yields $S(s) \succsim_{mul} S(t)$. We obtain $F(s) \sqsupset_{mul} F(t)$ from $F(l\sigma) \sqsupset_{mul} F(r\sigma)$.

Kruskal's Tree Theorem shows that $\sqsupset_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$. Hence $\succsim$ is a well-founded preorder on $\mathcal{T}(\mathcal{F})$. Since multiset extension preserves well-founded preorders, $\succsim_{mul}$ is a well-founded preorder on $\mathcal{M}(\mathcal{T}(\mathcal{F}))$. Because $\sqsupset$ is a PWO on the signature $\mathcal{F}$ it is a well-founded partial order. Hence its multiset extension $\sqsupset_{mul}$ is a well-founded partial order on $\mathcal{M}(\mathcal{F})$. We conclude that $(\mathcal{F}, \mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \sqsupset))$ is a terminating TRS. □

6. Other Notions of Termination

In this final section we investigate the relationship between simple termination and other restricted kinds of termination as introduced in [17]. First we recall some terminology. Let $\mathcal{F}$ be a signature. A *monotone* $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ consists of a non-empty $\mathcal{F}$-algebra $\mathcal{A}$ and a partial order $\succ$ on the carrier A of $\mathcal{A}$ such that every algebra operation is strictly monotone in all its coordinates, i.e., if $f \in \mathcal{F}$ has arity n then

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) \ \succ \ f_{\mathcal{A}}(a_1, \dots, b_i, \dots, a_n)$$