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Abstract

A polyhedron P is castable if its boundary can be partitioned by a plane into two polyhedral terrains. Castable polyhedra can be manufactured easily using two cast parts, where each cast part can be removed from the object without breaking the cast part or the object. If we assume that the cast parts are each removed by a single translation, it is shown that for a simple polyhedron with n vertices, castability can be decided in $O(n^2 \log n)$ time and linear space using a simple algorithm. A more complicated algorithm solves the problem in $O(n^{3/2+\epsilon})$ time and space, for any fixed $\epsilon > 0$. In the case where the cast parts are to be removed in opposite directions, a simple $O(n^2)$ time algorithm is presented. Finally, if the object is a convex polyhedron and the cast parts are to be removed in opposite directions, a simple $O(n \log^2 n)$ algorithm is presented.

1 Introduction

A growing application area of computational geometry is in the area of automated manufacturing, where an engineer can design an object with the aid of a computer, and determine by which manufacturing process the object can be constructed. There are several types of manufacturing processes studied in computational geometry, such as gravity casting [4, 5, 13], NC machining [14], automated welding [18] and layer deposition methods such as stereolithography [3].

In this paper, we study the geometric and computational aspects of casting. Casting consists of filling the open region bounded by two or more cast parts with a material such as a liquid metal, then removing the cast parts. The requirement to remove the cast parts without breaking them imposes certain restrictions on the shape of the objects that can be constructed. For *sand casting* (see e.g. [12, 26]), only two cast parts are used. To construct the cast parts, a prototype of the object is first obtained (see Figure 1). The prototype is then divided into two parts along a plane. The facet of each

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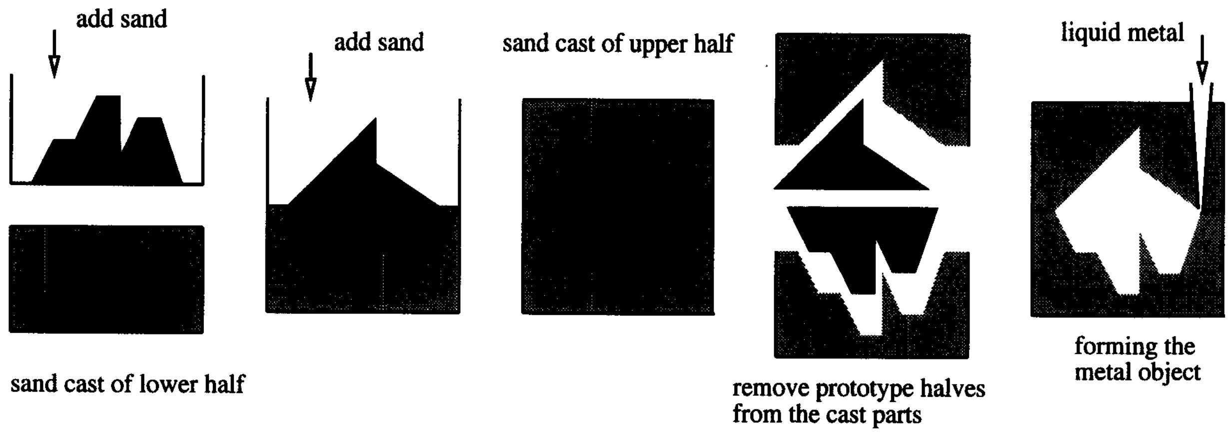

Figure 1: Construction of an object by sand casting, using two halves of the object as prototypes.

prototype part adjacent to the cutting plane is referred to as the base. The first cast part is made by placing the base of the first prototype part on a flat surface, and then adding sand around it. The part is then rotated such that the base is facing up, and the other prototype part is placed such that the bases coincide. The second cast part is built by adding sand around this prototype part while maintaining a channel into the cavity. Once the sand hardens, the cast of the prototype object is complete and the prototype parts can be removed. To build a metal rendition of the prototype object with this cast, liquid metal is poured into the opening until it fills the cavity. After the metal solidifies, the cast parts are removed from the object. The key to constructing a cast with this process is the ability to remove the prototype object without breaking the cast. This property is not restricted to casts built for manufacturing methods related to sand casting but also applies to other metal casting methods [12, 26], as well as injection molding and blow molding methods for plastics [22, 27]. The ability to remove the prototype object from the cast without breaking the cast allows one to re-use the same cast when mass-producing a particular object. Thus for several different manufacturing methods involving casting, the geometry of the object determines its feasibility of construction.

We note that more complicated objects can be made by using cores and inserts [12, 22, 26, 27]. However, their use slows down the manufacturing process and makes it more costly. Thus to be cost efficient, cores and inserts should be avoided. We do not study the extra possibilities of cores and inserts in this paper. We also omit treatment of issues related to filling the cast, such as whether air bubbles are trapped by the rising liquid. For a geometric treatment of some of these issues, see [4, 5].

An object is *castable* if it can be manufactured by casting. In other words, a cast of the object can be constructed such that each cast part can be removed from the object without breaking the object or any of the cast parts. Geometric and algorithmic issues of the castability of planar objects have been studied by Rosenbloom and Rappaport [23]. This paper addresses casting of objects modelled by polyhedra. In geometric terms, castability can be defined as follows (for a polyhedron P, ∂P denotes the boundary of P, and for a plane h, h^+ and h^- refer to the open half-spaces above and below h):

Definition 1 *A simple polyhedron P is* castable *if there exists a plane h such that $h^+ \cap \partial P$ is a weak terrain in some orientation, and $h^- \cap \partial P$ is a weak terrain in some orientation. The plane h is called the* casting plane. *(A weak terrain may contain edges and facets parallel to the orientation in which it is a terrain.)*

To manufacture a castable object (modelled as a polyhedron P), first determine a casting plane h. The plane h divides P into two cast parts. Make each cast part from the prototype halves $h^+ \cap \partial P$ and $h^- \cap \partial P$. Since P is castable, the prototype halves can be removed from the cast parts, and later the manufactured object can be removed from the cast parts. We consider three versions of the castability problem. They differ in the way the cast parts may be removed from the polyhedron P. Figure 2 shows the three versions for planar polygons.

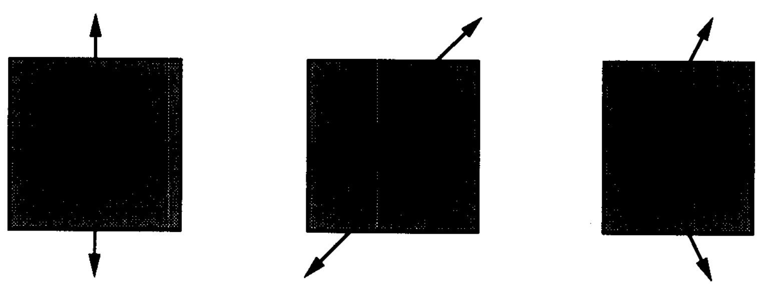

Figure 2: Three versions of the castability problem.

1. The two cast parts must be removed from P by one translation each, in opposite directions, and normal to the casting plane (orthogonal cast removal).
2. The two cast parts must be removed from P by one translation each, and in opposite directions (opposite cast removal).
3. The two cast parts must be removed from P by one translation each, in arbitrary directions (arbitrary cast removal).

Any convex polygon (in the plane) is castable in any of the three versions. In 3 dimensions, the equivalent property does not hold for convex polyhedra; in fact, some convex polyhedra are not castable in any of the three versions. In manufacturing, developing machines that perform orthogonal and opposite cast removal is much simpler than machines that perform arbitrary cast removal. In fact, opposite cast removal seems to be the most popular technique used [8, 22]. Furthermore, if orthogonal or opposite cast removal is possible, it can be determined more efficiently. We summarize the complexity of the different algorithms we developed for the casting problem. In the top half of the table, the time bounds of simple, linear space algorithms are shown. The bottom half of the table shows improvements made (in theory) by using $O(n^{3/2+\epsilon})$ storage (for any positive constant ϵ).

		orthogonal	opposite	arbitrary
linear space	convex polyhedra	$O(n \log^2 n)$	$O(n \log^2 n)$	$O(n^2 \log n)$
	simple polyhedra	$O(n^2)$	$O(n^2)$	$O(n^2 \log n)$
best results (in theory)	convex polyhedra	$O(n \log^2 n)$	$O(n \log^2 n)$	$O(n^{3/2+\epsilon})$
	simple polyhedra	$O(n^{3/2+\epsilon})$	$O(n^{3/2+\epsilon})$	$O(n^{3/2+\epsilon})$

We first derive some useful geometric properties of castable objects. These properties are the foundation upon which the algorithms are developed. The differences between the three versions of cast removal for both convex and simple polyhedra are considered (Section 2). The main approach taken by the algorithms is to compute a set of candidate casting planes, and to test each one of them. Therefore, in Section 3, we give bounds on the maximum number of distinct (candidate) casting planes. As a by-product that is interesting in its own right, we prove that the number of planes that intersect a convex polyhedron without intersecting any of its facets properly is $O(n)$, and that the total number of edges contained in these planes is $O(n \log n)$. Algorithms for cast removal in orthogonal and in opposite directions are presented in Section 4. Section 5 presents the algorithms for cast removal in arbitrary directions.

2 Preliminaries

All polygons and polyhedra in this paper are simple (see [21] for a definition). For a polyhedron P, denote by V, E and F the set of its vertices, edges and facets. Edges that bound two parallel facets are not allowed; they can be removed without changing the shape of the polyhedron. Edges and facets are open, the closure of an edge e or facet f is denoted by $cl(e)$ or $cl(f)$. The open interior of a polyhedron P is denoted by $int(P)$, the open exterior by $ext(P)$, and the boundary of P by ∂P. Although technically the object to be constructed is the interior of P, and the boundary of P is part of the cast, with a slight abuse of notation, we nevertheless state that P is castable or not castable.

A polyhedral surface S is called a *weak terrain with respect to a direction* $\vec{d}$ if any line with orientation $\vec{d}$ intersects S in a point or a line segment. A polyhedron P is called a *weak terrain with respect to a facet* Q *and a direction* $\vec{d}$ if $\partial P - Q$ is a weak terrain with respect to $\vec{d}$. In the rest of this paper we use *terrain* to mean *weak terrain*

For a non-vertical plane h, we denote by h^+ and h^- the open half-spaces above and below h. If h is vertical but does not contain a line parallel to the y-axis, then h^+ and h^- denote the open half-spaces bounded by h that contain the points $(0, \infty, 0)$ and $(0, -\infty, 0)$, respectively. If h is vertical and contains a line parallel to the y-axis then h^+ and h^- denote the open half-spaces bounded by h that contain the points $(\infty, 0, 0)$ and $(-\infty, 0, 0)$, respectively. We use h_0^+ and h_0^- to denote h^+ and h^- translated so that the bounding plane intersects the origin. Given direction $\vec{d}$ and facet f, we say that f is *compatible* with $\vec{d}$ if the inner product between $\vec{d}$ and the outward normal of facet f is non-negative (i.e. $\vec{d}$ makes an angle of at most $\pi/2$ radians with the outward normal of f). We say that f is *incompatible* with $\vec{d}$ if it is not compatible.

Observation 1 *Let* P *be a polyhedron and let* h *be a plane that intersects* P. *The surface* $\partial P \cap cl(h^+)$ *is a terrain for direction* $\vec{d}$ *if and only if every facet of* P *that intersects* h^+ *is compatible with* $\vec{d}$.

Therefore, castability with respect to a plane h is only determined by the facets of P that intersect h^+ and the ones that intersect h^-. If h is a casting plane for P, then h can be perturbed if this does not involve new facets intersecting h. In case of orthogonal cast removal, the only perturbation allowed is translation.

Observation 2 *For castability with orthogonal cast removal, we may assume that the casting plane contains at least one vertex of* P. *For opposite and arbitrary cast removal, we may assume that the casting plane contains at least three vertices of* P.

For two polyhedra P and Q whose interiors lie on different sides of a plane h, and which are both bounded by the same facet f that lies inside h, we define the *union* of P and Q as the polyhedron with all vertices of P and Q, with all facets of P and Q except f, and with all edges of P and Q except the ones contained in h that bound two parallel facets.

2.1 The sphere of directions

We represent the space of all directions in 3-space by the points on the surface of a sphere. Let $\mathcal{S}$ be the unit sphere centered at the origin o. Any point p on $\mathcal{S}$ represents the direction $\vec{op}$. Let *north* and *south* denote the points on $\mathcal{S}$ that represent the $\vec{z}$ and $-\vec{z}$ directions. Let $\mathcal{E}$ denote the equator (the set of points $p \in \mathcal{S}$, such that $\vec{op} \cdot \vec{z} = 0$). For any point $p \in \mathcal{S}$, let $H(p)$ denote the open hemisphere representing all directions that make an angle less than $\pi/2$ radians with $\vec{op}$, and $cl(H(p))$ is the closure of this hemisphere.

Let P be a convex polyhedron, let h be a casting plane and let $\vec{d_1}$ and $\vec{d_2}$ be the two cast removal directions, represented by points d_1 and d_2 on the sphere of directions.

We re-orient P and h such that *north* is normal to h, thus d_1 and d_2 cannot both lie in the upper hemisphere or the lower hemisphere. Without loss of generality, let $d_1 \in cl(H(north))$ and $d_2 \in cl(H(south))$.

Observation 3 *If a facet f of P intersects h^+, and f has its outward normal represented by a point q on $\mathcal{S}$, then $q \in cl(H(d_1))$. Similarly, if f intersects h^-, then $q \in cl(H(d_2))$. Therefore, if f intersects the casting plane h, then $q \in cl(H(d_1)) \cap cl(H(d_2))$ (recall that f is open).*

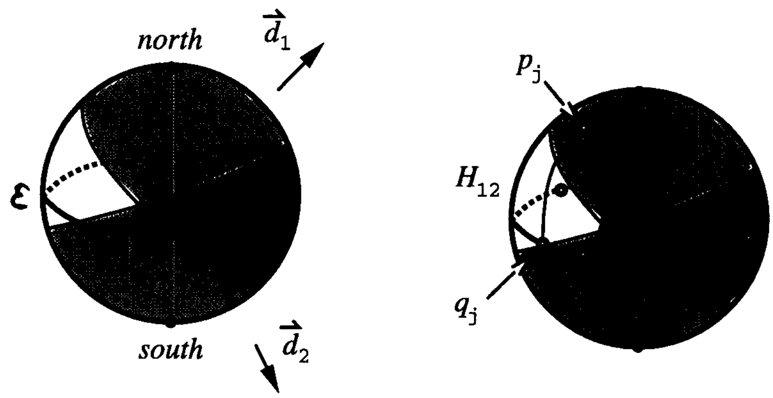

Figure 3: The sphere of directions. The shaded hemispheres are $H(d_1)$ and $H(d_2)$, and the darker shaded region is their intersection.

Define $C(d_1)$ and $C(d_2)$ to be the great circles that bound $H(d_1)$ and $H(d_2)$. If $\vec{d_1}$ and $\vec{d_2}$ are opposite, then $C(d_1) = C(d_2)$, otherwise, $C(d_1) \cap C(d_2)$ consists of a pair of antipodal points on $\mathcal{S}$ different from *north*, and *south*.

For any point $p \in \mathcal{S} - \{north, south\}$, define $\lambda(p)$ to be the nearest point on the equator (i.e., the intersection point of the equator $\mathcal{E}$ with the great circle through *north* and p nearest to p). By definition, we have

$$\overrightarrow{op} \cdot \overrightarrow{o\lambda(p)} \geq 0\,. \tag{1}$$

Furthermore, p and $\lambda(p)$ lie to the same side of any great circle through *north* and *south*.

Assume that $\vec{d_1}$ and $\vec{d_2}$ are non-opposite in the following (see Figure 3). Define C_{12} to be the great circle containing *north*, *south* and the points of $C(d_1) \cap C(d_2)$. Note that $cl(H(d_1)) \cap cl(H(d_2))$ does not intersect one of the (open) hemispheres defined by C_{12}. Let H_{12} be this open hemisphere. By the above observation, any facet that has its outward normal in H_{12} cannot be intersected by the casting plane. We use this fact in the following lemma.

Lemma 1 *If a simple polyhedron P is castable in non-opposite directions with casting plane h, then h contains an edge of P.*

Proof: Let $Q = P \cap h$. If Q consists of more than one connected component, or if Q has holes, then h cannot be a casting plane for P. Therefore, Q is a simple polygon. Let $e_1, \ldots, e_m$ be the clockwise sequence of edges bounding Q and let $q_1, \ldots, q_m$ be the points on $h \cap \mathcal{S}$ that represent the outward normals of $e_1, \ldots, e_m$. Since h is chosen to be horizontal, $q_1, \ldots, q_m \in \mathcal{E}$. Every open half-circle in $\mathcal{E}$ contains at least one point of $q_1, \ldots, q_m$, because Q is a simple polygon.

Given that P is castable with respect to non-opposite directions $\vec{d_1}$ and $\vec{d_2}$, assume that every e_i is the intersection of a facet f_i of P with the casting plane (i.e. no edge of Q is an edge of P). Let C_{12} and H_{12} be as defined above, and let e_j be an edge of Q such that $q_j \in \mathcal{E} \cap H_{12}$ (see Figure 3). Let p_j be the point on $\mathcal{S}$ that represents the outward normal of f_j. Then $q_j = \lambda(p_j)$, and by (1), we know p_j lies in H_{12}. However, H_{12} does not contain any point in $cl(H(d_1)) \cap cl(H(d_2))$, so by Observation 3 the facet f_j cannot intersect the casting plane, which is a contradiction. Thus h contains an edge of P. ∎

Lemma 2 *If a simple polyhedron P is castable with casting plane h and in non-opposite directions, then h contains an edge of the convex hull of P.*

Proof: Let P be castable with respect to non-opposite directions $\vec{d_1}$ and $\vec{d_2}$. If the cast of $P \cap cl(h^+)$ can be removed in a direction $\vec{d_1}$, then the convex hull of $P \cap cl(h^+)$ can also be removed in the direction $\vec{d_1}$. The same statement holds for direction $\vec{d_2}$ and the cast of $P \cap cl(h^-)$.

Let $Q = P \cap h$. The convex hull of Q is the closure of a facet bounding both $CH(P \cap h^+)$ and $CH(P \cap h^-)$ (note that the convex hull is defined as a *closed* set). As in the proof of the previous lemma, there exists an edge e_j of the convex hull of Q where the outward normal of the edge on plane h lies in H_{12}. We need to prove that e_j is also an edge of $CH(P)$. Let f_1 be the facet of $CH(P \cap h^+)$ incident to e_j and not in h. Define f_2 analogously for $CH(P \cap h^-)$. Let q_j, p_1 and p_2 be the points on $h \cap \mathcal{S}$ and $\mathcal{S}$ that represent the outward normals of e_j, f_1 and f_2, respectively. Since f_1 and f_2 are incident to e_j, we have $\lambda(p_1) = \lambda(p_2) = q_j$, so p_1, p_2 and q_j lie on a half-circle between *north* and *south* and in H_{12}. Since $p_1 \in H(d_1)$ and $p_2 \in H(d_2)$ are both contained in H_{12}, the half-circle through *north*, *south*, p_1 and p_2 must contain a point r that is not in $cl(H(d_1))$ nor in $cl(H(d_2))$. The plane h' with normal $\vec{or}$ and containing e_j has $CH(P \cap h^+)$ completely to the one side, with the exception of $cl(e_j)$. Similarly, $CH(P \cap h^-)$ lies completely to the one side of h' with the exception of $cl(e_j)$. Since these convex hulls lie to the same side, it follows that P lies completely to the one side of h' with the exception of the endpoints of e_j, and possibly e_j itself (if e_j is an edge of P). Therefore, e_j is an edge of $CH(P)$. ∎

Notice that the above two lemmas imply that if a polyhedron is castable, but not with opposite cast removal, then the casting plane contains both an edge of P and an edge of the convex hull of P (this might be the same edge). This will aid considerably to determine castability with arbitrary cast removal.

2.2 Relation to linear programming

Let P be a polyhedron and let h be a plane. The plane h partitions the set V of vertices of P into three subsets V_h, V_h^+ and V_h^- of vertices in, above and below h, respectively. Similarly, h partitions the set E of edges of P in four subsets E_h, $E_h^\times$, E_h^+ and E_h^- of edges contained in h, intersecting h, above h and below h, respectively. The set F of facets is partitioned in the same way. For any facet $f \in F$, denote by $\Psi(f)$ the closed half-space bounded by a plane supporting f, and such that for any point in f, $\Psi(f)$ does not intersect the interior of P in an ϵ-neighborhood of the point. Denote by $\Psi_0(f)$ the same half-space, but translated such that the bounding plane contains the origin. We define

$$\xi^+(h) \equiv cl(h_0^+) \cap \left\{ \bigcap_{f \in F_h^+ \cup F_h^\times} \Psi_0(f) \right\} \quad \text{and} \quad \xi^-(h) \equiv cl(h_0^-) \cap \left\{ \bigcap_{f \in F_h^- \cup F_h^\times} \Psi_0(f) \right\}.$$

The intersection of a set of half-spaces is called *non-trivial* if it contains more than a single point. Denote by *refl*(b) the reflection of an object b through the origin (i.e. every point in b is negated). We make the following observations.

Observation 4 *The plane h is a casting plane for polyhedron P for arbitrary cast removal if and only if $\xi^+(h)$ and $\xi^-(h)$ are both non-trivial.*

Observation 5 *The plane h is a casting plane for polyhedron P for opposite cast removal if and only if $\xi^+(h) \cap refl(\xi^-(h))$ is non-trivial.*

Observation 6 *Let h be a plane and let ℓ be a line perpendicular to h and through the origin. The plane h is a casting plane for polyhedron P for orthogonal cast removal if and only if $\ell \cap \xi^+(h) \cap refl(\xi^-(h))$ is non-trivial.*

With the above observations, we can test efficiently whether a given plane h is a casting plane for P. Since the casting problem for a plane h and a polyhedron P can be transformed in linear time to a linear programming problem in 3 dimensions, the test requires only linear time [19, 20, 25].

Lemma 3 *Given a polyhedron P and a plane h, one can test in linear time whether h is a casting plane for P in any of the three versions for removing the cast.*

Similarly, given a polyhedron and two cast removal directions (but not a casting plane), one can test using linear programming whether the polyhedron is castable with respect to those cast removal directions.

Lemma 4 *Given a polyhedron P and two cast removal directions, one can test in linear time whether there exists a casting plane h that allows removing the cast parts in the given directions.*

Proof: Let the two cast removal directions be $\vec{d_1}$ for $\partial P \cap h^+$ and $\vec{d_2}$ for $\partial P \cap h^-$. For every facet f of P, one can determine whether f should lie above the casting plane h (is compatible only with $\vec{d_1}$), below h (is compatible only with $\vec{d_2}$), may intersect h (is compatible with both $\vec{d_1}$ and $\vec{d_2}$) or is incompatible with the cast removal directions. If there is a facet of P that is incompatible, then there does not exist any casting plane for directions $\vec{d_1}$ and $\vec{d_2}$.

The classification of the facets as "above", "below", and "intersect" imposes a classification of the edges. Any edge is classified either as "above/above" (a/a), "above/below" (a/b), "above/intersect" (a/i), "below/below" (b/b), "below/intersect" (b/i) or "intersect/intersect" (i/i), corresponding to the classification of the two facets incident to that edge.

Similarly, the classification of an edge determines where both endpoints of the edge must lie. For example, if an edge is classified as (a/a) then both endpoints must lie in $h^+ \cup h$. We summarize the implications that the classification of the edges has on their endpoints in the table below.

edge class.	endpoints
(a/a)	$h^+ \cup h$
(a/i)	$h^+ \cup h$
(b/b)	$h^- \cup h$
(b/i)	$h^- \cup h$
(a/b)	h
(i/i)	anywhere

The classification of the endpoints of edges, in turn, determines where the vertices of P must lie. Since every vertex is adjacent to at least 3 edges, no vertex can be adjacent to only (i/i) edges. Hence, one can decide for every vertex whether it must be contained in h, lie in $h^+ \cup h$, or lie in $h^- \cup h$. We dualize the vertices to planes, consider the half-spaces to the appropriate side of these planes, based on the classification, and obtain a linear programming problem to decide whether a plane h exists that has the appropriate location with respect to the vertices of P. ∎

2.3 Antipodality properties

For opposite cast removal, we prove that if a casting plane intersects a facet, then it intersects the boundary of that facet in antipodal pairs (note that this also holds for orthogonal cast removal). This is an important property that is used to bound the number of distinct casting planes.

Lemma 5 *If the casting plane h intersects a facet f of a convex polyhedron P, and also two vertices u and v in the closure of f, then for opposite cast removal, vertices u and v must be antipodal in $cl(f)$.*

Proof: Let u, v be two vertices in $cl(f) \cap h$, and assume that they are not antipodal. Let h_f be the plane that contains f. Since u and v are not antipodal, there are two edges e_u and e_v in $cl(f)$ incident to u and v, respectively, which lie on the same side of h and diverge in the plane h_f (when directed away from h). Suppose without loss of generality that $e_u, e_v \in h^+$.

We again represent the space of all possible directions in 3-space as a sphere of directions with the casting plane as horizontal and *north* $\in h^+$. Since f lies partially above and partially below h, we know by Observation 3 that $\vec{d}$ must correspond to a point p_d on $C(p_f) \cap h^+$, where $\vec{d}$ is the removal direction for the cast part above h.

Because P is convex and $\vec{d}$ is parallel to the facet f, the condition that h is casting plane for P has implications for a 2-dimensional casting problem, namely, f must be castable with $h \cap h_f$ as the casting line. Since u and v are not antipodal and e_u and e_v diverge in h_f, it follows that f is not castable in h_f with respect to the casting line $h \cap h_f$, which implies that P is not castable with respect to h, a contradiction. ∎

Corollary 1 *Let h be a casting plane for a polyhedron P which intersects a facet f properly, and assume opposite cast removal. If h intersects a vertex v and properly intersects an edge e in the closure of f, then v is antipodal to both endpoints of e. If h properly intersects two edges in the closure of f, then they are parallel.*

2.4 Convexity properties

In this subsection we derive some additional geometric properties of convex polyhedra that form the basis of faster algorithms. We also establish an important property that relates the castability of a simple polyhedron to that of its convex hull.

If P is a convex polyhedron, then the linear programming problems defined by P and a candidate casting plane h need not consider all facets of F, but only those intersecting h and those adjacent to h. We make this more precise. For the subset E_h of the edges of P contained in h, let $F^+(E_h)$ denote the subset of F^+ of facets that contain at least one edge of E_h in their closure. Define $F^-(E_h)$ analogously. Furthermore, we define

$$\phi^+(h) \equiv cl(h_0^+) \cap \left\{ \bigcap_{f \in F_h^\times \cup F^+(E_h)} \Psi_0(f) \right\} \quad \text{and} \quad \phi^-(h) \equiv cl(h_0^-) \cap \left\{ \bigcap_{f \in F_h^\times \cup F^-(E_h)} \Psi_0(f) \right\}.$$

Lemma 6 *If P is convex, $\xi^+(h) = \phi^+(h)$ and $\xi^-(h) = \phi^-(h)$.*

Proof: We only prove that $\xi^+(h) = \phi^+(h)$; the other proof is similar. Furthermore, $\xi^+(h) \subseteq \phi^+(h)$ is trivial, so we prove $\phi^+(h) \subseteq \xi^+(h)$.

If $\phi^+(h)$ only contains the origin then so does $\xi^+(h)$. Otherwise, let r be a half-line originating at the origin and inside $\phi^+(h)$. If $r \notin \xi^+(h)$, then there is a facet $f \in F_h^+ \setminus F^+(E_h)$ for which $r \notin \Psi_0(f)$. Let $\overline{\Psi}(f)$ denote the (closed) half-space supporting f distinct from $\Psi(f)$. Since P is convex,

$$f \subset cl(h^+) \cap \left\{ \bigcap_{f \in F_h^\times \cup F^+(E_h)} \overline{\Psi}(f) \right\}.$$

Since $r \in \phi^+(h)$, it follows that the projection of any point in f parallel to r onto h will lie in $h \cap P$. But since $r \notin \Psi_0(f)$, the line segment connecting a point in f with this projection will be (partially) outside P, namely, in the neighborhood of f. This contradicts the convexity of P. ■

With Lemma 6, we conclude the following:

Lemma 7 *The plane h is a casting plane for a convex polyhedron P for opposite cast removal if and only if $\phi^+(h) \cap refl(\phi^-(h))$ is non-trivial.*

Lemma 8 *Let h be a plane and let ℓ be a line perpendicular to h through the origin. The plane h is a casting plane for a convex polyhedron P for orthogonal cast removal if and only if $\ell \cap \phi^+(h) \cap refl(\phi^-(h))$ is non-trivial.*

The following theorem forms the crucial link between simple polyhedra and convex polyhedra in terms of castability.

Theorem 1 *If a simple polyhedron P is castable, then the convex hull of P is also castable using the same casting plane and cast removal directions.*

To prove the theorem, we first establish a few important lemmas.

Lemma 9 *A convex polyhedron P is a terrain with respect to a facet Q and a direction $\vec{d}$ if and only if the vertices of P project into $cl(Q)$ when projected in direction $-\vec{d}$ onto the supporting plane of Q.*

Proof:
($\Rightarrow$) If P is a terrain with respect to a direction $\vec{d}$ and a facet Q, then every point of P projects into $cl(Q)$ in direction $-\vec{d}$.

($\Leftarrow$) Suppose every vertex of P projects into $cl(Q)$ in direction $-\vec{d}$. Since P is convex, the line segment from every vertex v to Q in direction $-\vec{d}$ must be inside P. It follows that a ray with direction $\vec{d}$ from every vertex is outside P. By Observation 1, P is a terrain with respect to $\vec{d}$ and Q. ■

Lemma 10 *If a polyhedron P is a terrain with respect to a direction $\vec{d}$ and facet Q then $CH(P)$ is a terrain with respect to $\vec{d}$ and $CH(Q)$.*

Proof: Every vertex of P is on one side of the plane induced by Q; it follows that the convex hull of Q must be a facet of $CH(P)$. Since every vertex of $CH(P)$ is a vertex of P, every vertex of $CH(P)$ must project into $CH(Q)$ in direction $\vec{d}$. By Lemma 9, P is a terrain with respect $\vec{d}$ and $CH(Q)$. ■

Lemma 11 *Let h be plane, let C_1 and C_2 be convex polygons in h such that $C_1 \subseteq C_2$, and let S be a set of points entirely contained in one of the half-spaces bounded by h. If $CH(C_1 \cup S)$ is a terrain with respect to a direction $\vec{d}$ and facet C_1, then $CH(C_2 \cup S)$ is a terrain with respect to $\vec{d}$ and C_2.*

Proof: Suppose that $CH(C_1 \cup S)$ is a terrain with respect to $\vec{d}$ and C_1. By Lemma 9, S projects inside C_1 in direction $-\vec{d}$. Since $C_1 \subseteq C_2$, S also projects inside C_2 in direction $-\vec{d}$. By Lemma 9, $CH(C_2 \cup S)$ is a terrain with respect to direction $\vec{d}$ and facet C_2. ■

Proof: (of Theorem 1)

Let P be a simple polyhedron, and let h be a casting plane for P with casting directions $\vec{d_1}$ for the cast part of $P \cap cl(h^+)$ and $\vec{d_2}$ for the cast part of $P \cap cl(h^-)$. The polyhedron $CH(P \cap h^+) \cup CH(P \cap h^-)$ is also castable for casting plane h and directions $\vec{d_1}$ and $\vec{d_2}$ by Lemma 10. Denote $P^+ = CH(P \cap h^+)$ and $P^- = CH(P \cap h^-)$.

We need to show that $P_H = CH(P)$ is castable with casting plane h and casting directions $\vec{d_1}$ and $\vec{d_2}$. Let $P_H^+ = CH(P_H \cap h^+)$ and $P_H^- = CH(P_H \cap h^-)$. Since P^+ is contained in P_H^+ and P^- is contained in P_H^-, the theorem follows from Lemma 11. ■

3 The number of distinct casting planes

Given a polyhedron P with vertex set V, two planes h_1 and h_2 are *(combinatorially) distinct* if the partitioning of the facets into F^+, F^-, $F^{\subset}$ and $F^{\times}$ they define is different. By Observation 2, a trivial upper bound on the number of distinct casting planes for a polyhedron with n vertices is $O(n^3)$.

This section gives a linear upper bound on the maximum number of distinct casting planes for convex polyhedra in case of orthogonal and opposite cast removal as well as a quadratic upper bound for arbitrary cast removal. The proofs are constructive, i.e., sets of *candidate casting planes* of linear or quadratic size are defined which contain all distinct casting planes. In the following sections we will use these sets of candidate casting planes to determine castability efficiently.

3.1 Orthogonal and opposite cast removal

Observe that for orthogonal cast removal, a casting plane h can intersect a polyhedron P as follows (these properties follow from the previous section):

1. A facet f that intersects h properly is perpendicular to h.
2. An edge that intersects h properly is perpendicular to h (because otherwise one of the incident facets cannot be perpendicular).
3. Two vertices in the closure of a facet f and in h are antipodal in $cl(f)$. Any vertex and edge in the closure of f and intersecting h are antipodal in $cl(f)$. (See Lemma 5).

For opposite cast removal, we have the following properties of intersections of a casting plane h and a polyhedron P:

1. The facets of $F^{\times}$ that intersect h properly have their outward normals such that when translated to the origin, they span a plane or part of it (since $\bigcap\{\Psi_0(f) \mid f \in F^{\times}\}$ must contain a line through o).
2. All edges that intersect h properly are parallel (otherwise the incident facets span more than a plane).
3. Any two vertices in the closure of a facet f and in h are antipodal in $cl(f)$. Any vertex and edge in the closure of f and intersecting h are antipodal in $cl(f)$. (See Lemma 5.)

Let P be a convex polyhedron with n vertices. Since a linear upper bound on the number of distinct casting planes in case of opposite cast removal implies the same result for orthogonal cast removal, we only prove the opposite case.

Lemma 12 *Given a convex polyhedron P, the number of distinct casting planes that intersect some edge of P properly is at most linear in the number of vertices of P for opposite cast removal.*

Proof: Let E' be a maximal subset of parallel edges of P, and of which at least one edge is properly intersected by some casting plane. By convexity of P, such a casting plane must intersect the closure of all edges of E', because no such closure of an edge can be strictly above or below the casting plane. The cast removal directions are parallel to the edges of E', and by the classification defined in the proof of Lemma 4, for every vertex v of P it is specified that either $v \in h \cup h^+$ or $v \in h \cup h^-$ or $v \in h$, for any casting plane h. Let V^+, V^- and $V^{\subset}$ be these three subsets of vertices, respectively. If $V^{\subset}$ contains three or more vertices, then at most one distinct casting plane is possible for this direction. Otherwise, we consider the following three cases. Note that since P is convex, by Lemma 6 we only need to consider the facets that intersect h and those adjacent to an edge of P in h.

Case 1: $V^{\subset}$ is empty. In this case, the facets that intersect h are all the facets adjacent to the edges of E'. Let G^+ be the endpoints of E' contained in V^+ and let G^- be the endpoints of E' contained in V^-. For a plane to intersect the closure of all edges of E', it must separate G^+ from G^-. Since we are considering opposite cast removal, a casting plane must contain at least three vertices. The vertices that the casting plane may contain must come from the set $G = G^+ \cup G^-$, since $V^{\subset}$ is empty. Therefore, to bound the number of distinct casting planes that intersect an edge of E' properly, we must count the number of planes that separate G^+ from G^- and contain at least three vertices from the set G.

To do this, we dualize the set of vertices G^+ to a set of planes $\mathcal{D}^+$ and the set of vertices G^- to a set of planes $\mathcal{D}^+$. Let I be the convex polytope that lies below all planes in $\mathcal{D}^+$ and above all planes in $\mathcal{D}^-$. The vertices of I are precisely the duals of the planes. Therefore, there are $O(|E_i|)$ distinct planes.

Case 2: $V^{\subset}$ contains one vertex. Argument similar to case 1. Simply include the vertex in $V^{\subset}$ in the sets G^+ and G^-.

Case 3: $V^{\subset}$ contains two vertices. Same argument as case 2.

Thus, we see that the number of distinct casting planes that intersect an edge of E' properly is bounded by $O(|E'|)$. Since every edge of P contributes to only one subset E' of parallel edges, the lemma follows by Euler's formula. ∎

The following lemma is the basis of an inductive argument to prove a linear bound on the number of distinct casting planes that intersect no edge properly.

Lemma 13 *Given a convex polyhedron P, there exists a vertex v with constant degree such that v participates in a constant number of antipodal pairs on the incident facets.*

Proof: Let $\tilde{V}, \tilde{E}, \tilde{F}$ be the number of vertices, edges and facets of P. The summed degree of all vertices $D = 2\tilde{E} \leq 6\tilde{V} - 12$. Every vertex has at least degree 3, thus there must be at least $\tilde{V}/2 + 1$ vertices of degree at most 8. The total number of antipodal pairs, summed over all facets, is at most $3\tilde{F}/2 \leq 3\tilde{V} - 6$, which implies that the total vertex contribution in antipodal pairs, A, satisfies $A \leq 6\tilde{V} - 12$ [21]. Observe that every vertex of P participates in at least 3 antipodal pairs; at least one in each incident facet. If all $\tilde{V}/2+1$ vertices of degree at most 8 are in at least 9 antipodal pairs on the incident facets, then $A \geq 9(\tilde{V}/2 + 1) + 3(\tilde{V}/2 - 1) = 6\tilde{V} + 6$, a contradiction. Hence, there exists a vertex which is in at most 8 antipodal pairs and with degree at most 8. ∎

Let h be a candidate casting plane of P, and let $Q = h \cap P$. If Q contains three consecutive vertices u, v, w that are also vertices of P, then each of u and w is either an endpoint of an edge incident to v, or a vertex antipodal to v on the closure of a facet f incident to v. We say that the plane through u, v, w is *generated by* v. It follows that the set of candidate casting planes generated by v has size $\binom{d+a}{2}$, where d is the degree of v and a is the number of vertices antipodal to v in the closures of the facets incident to v. Every casting plane h that does not intersect any edge properly contains at least three vertices that are consecutive in $h \cap P$, and therefore, every such casting plane is generated by some vertex of P.

Theorem 2 *Given a convex polyhedron P with n vertices, the maximum number of distinct casting planes for P is $O(n)$, assuming opposite removal of the cast parts.*

Proof: First, assume that the casting plane h intersects some edge e of P properly. By Lemma 12, there are $O(n)$ distinct casting planes of this type.

Next, we show that the number of casting planes that do not intersect any edge properly is linear. For such a casting plane h, all vertices of the intersection polygon $Q = h \cap P$ are also vertices of P.

The proof is by induction. Let v be a vertex of P of degree at most 8 and which participates in at most 8 antipodal pairs (see Lemma 13). The number of casting planes containing v which do not intersect any edges properly is bounded from above by the number of planes generated by v, and hence, is constant. We remove v from P and continue the count on the convex hull of the remaining vertices. We have counted all distinct casting planes that contain v. Since any casting plane of P that does not contain v and does not intersect any edge incident to v properly is also a casting plane of CH(vertices of $P - v$), the lemma follows by induction. ∎

There is another interesting combinatorial bound on the complexity of the intersection of all distinct casting planes with a convex polyhedron. Referring to the proof of Lemma 12, we notice that two distinct casting planes h_1 and h_2 that intersect an edge of E' properly are *similar*, because they define the same cast removal directions, and they intersect the same closure of edges and facets. In other words, if h_1 and h_2 each intersect edges properly that are parallel, there cannot be two vertices u, v strictly to the one side of h_1 and strictly to different sides of h_2. We use the term *weakly equivalent* for two such planes. Two planes are *strongly distinct* if they are not weakly equivalent. There are $O(n)$ strongly distinct casting planes for any convex polyhedron P with n vertices. We analyze the combinatorial complexity of $h \cap P$, summed over all strongly distinct casting planes h. This quantity is well-defined for opposite cast removal, since two weakly equivalent casting planes have an equal-size intersection with P (although they may intersect different facets, edges and vertices). We prove a bound of $O(n \log n)$ on the summed complexity. Note that when the sum is over all distinct casting planes (not *strongly* distinct), the summed complexity can be $\Theta(n^2)$ if P has a set of $\Omega(n)$ parallel edges. The bound makes use of a hierarchical decomposition of P that closely resembles the hierarchy of Dobkin and Kirkpatrick [10]. It is the basis of the $O(n \log^2 n)$ time algorithms for casting of convex polyhedra with opposite cast removal.

Lemma 14 *Given a convex polyhedron P with n vertices, there exists a subset V' of the vertices V of size $\Omega(n)$, such that each $v \in V'$ has degree at most 8 and is antipodal to at most 12 vertices in facets incident to v.*

Proof: Similar to Lemma 13, one can prove that there are at least $\tilde{V}/5$ vertices of degree at most 8 and in at most 12 antipodal pairs. (Otherwise, $A \geq 13(\frac{1}{2} - \frac{1}{5})\tilde{V} + 3(\frac{1}{2} + \frac{1}{5})\tilde{V} = 6\tilde{V}$, a contradiction.) ■

The following hierarchical decomposition of P generates a set of planes that contains all the candidate casting planes that do not intersect an edge properly. The correctness follows from the proof of Theorem 2.

Algorithm 1: *Compute all generated planes*

1. Set $i = 1$.
2. Compute the antipodal pairs of the facets of P.
3. Select a subset V_i of V as in Lemma 14. For every vertex $v \in V_i$, generate all planes through u, v, w. For every vertex $v \in V_i$, the number of generated planes is at most $\binom{12+8}{2} = 190$, thus $O(n)$ for the whole subset.
4. Recompute the convex hull of the vertices of P minus the vertices of V_i.
5. Repeat at step 2 with $i = i + 1$ unless P has no vertices left.

The number of generated planes is linear since each vertex generates a constant number. Antipodal pairs computations take $O(n)$ time and convex hull computations take $O(n \log n)$ time, see e.g. [11, 21]. The total time taken by Algorithm 1 is given by the recurrence $T(n) \leq T((1 - \alpha)n) + O(n \log n)$ where $\alpha \geq 1/5$ is the constant in the $\Omega(n)$ of Lemma 14. This recurrence solves to $T(n) = O(n \log n)$.

Theorem 3 *Given a convex polyhedron P with n vertices, the total complexity of $h \cap P$, summed over all strongly distinct casting planes h for P, is $O(n \log n)$ for opposite cast removal.*

Proof: In the following proof, we make a distinction between planes that are *generated*, and other planes that can be casting planes. Planes of the second type intersect some edge properly.

Consider a hierarchical decomposition of the vertices of P into sets $V_1, \ldots, V_m$ as described above. Observe that $m \in O(\log n)$.

Let h be any plane, and let $v_1, \ldots, v_k$ be the sequence of vertices in $h \cap P$. We first show that every consecutive subsequence $v_i, \ldots, v_{i+2m-1}$ of vertices that also are vertices of P (no proper intersections of edges of P with h) contains a vertex that generates h. To this end, observe that v_j generates h if and only if v_j is in a vertex set V_s with lower or equivalent index as its neighbors, thus if $v_{j-1} \in V_r$ and $v_{j+1} \in V_t$, then $r \geq s$ and $t \geq s$. Since there are only m vertex sets, any consecutive sequence of $2m$ vertices contains at least one that that generates the plane h.

Consider the vertices v_i that are proper intersections of h and an edge of P. Any edge e gives rise to at most one strongly distinct casting plane, and therefore, the total number of these vertices in $h \cap P$, summed over all strongly distinct casting planes, is linear.

Summarizing, the sequences of $h \cap P$ summed over all strongly distinct casting planes contain $O(n)$ vertices that generate a casting plane, $O(n)$ vertices that are proper intersections of edges with a casting plane, and at most $2m-1$ vertices in between. It follows that the total complexity of the intersections is $O(nm) = O(n \log n)$. ∎

Corollary 2 *Given a convex polyhedron P with n vertices, the number of planes that intersect the interior of P but do not intersect any facets of P is $O(n)$, and the number of edges of P contained in these planes, summed over all planes, is $O(n \log n)$.*

3.2 Arbitrary cast removal

We have shown that the number of casting planes that also allow opposite cast removal is linear. For the other casting planes, we know from Lemma 1 that they contain an edge of P. Since we may also assume that they contain a third vertex, we immediately conclude:

Theorem 4 *Given a convex polyhedron P with n vertices, the number of distinct casting planes for P is $O(n^2)$, assuming arbitrary removal of the cast parts.*

4 Algorithms for orthogonal and opposite cast removal

In this section and the next, algorithms are presented for the computation of casting planes, and hence, determining whether a given polyhedron is castable. This section focuses on orthogonal and opposite cast removal.

4.1 A simple algorithm for simple polyhedra

We compute $O(n)$ candidate casting planes as follows. By Theorem 1, we need only consider the casting planes of the convex hull of P. We first compute the candidate casting planes that intersect some edge properly, and then the ones that are generated. We only consider opposite cast removal; the case of orthogonal cast removal only requires some straightforward changes.

Let $E_1, \ldots, E_k$ be a partitioning of E into maximal sets of parallel edges. For each E_i, let V_i^+ denote the upper endpoints of E_i, V_i^- the lower endpoints of E_i, and V_i^C the set of vertices that must be contained in the casting plane for the cast removal direction parallel to the edges of E_i. We compute all planes that contain the vertices of V_i^{C}, separate V_i^+ from V_i^-, and contain at least three vertices of $V_i^{\mathsf{C}} \cup V_i^+ \cup V_i^-$ by intersecting the corresponding set of half-spaces in dual space, as in Lemma 12. Each vertex of the resulting polyhedron in dual space corresponds to a plane with the desired properties. This gives $O(|E_i|)$ candidate casting planes. The intersection of $|E_i|$ half-spaces in 3-dimensional space can be computed in $O(|E_i| \log |E_i|)$ time, see e.g. [11, 21]. Summed over all subsets $E_1, \dots, E_k$, we obtain $O(n)$ candidate casting planes in $O(n \log n)$ time.

Second, we compute the other candidate casting planes in $O(n \log n)$ time by Algorithm 1. We conclude:

Lemma 15 *Given a polyhedron P with n vertices, one can compute in $O(n \log n)$ time a set Γ of $O(n)$ planes such that any casting plane h that contains at least three vertices of P is contained in Γ, assuming opposite cast removal.*

Theorem 5 *Given a polyhedron P with n vertices, one can decide in $O(n^2)$ time and linear space whether P is castable when the cast parts must be removed in orthogonal or opposite directions.*

Proof: If P is a convex polyhedron, the theorem follows immediately from Lemmas 3, 4 and 15. If P is a simple polyhedron, we additionally apply Theorem 1. ∎

4.2 Walking around convex polyhedra

For convex polyhedra, the above result can be improved as follows. By Lemma 7 determining whether a plane h is a casting plane for P can be done by only considering the facets intersected by h and the facets incident to the edges that are contained in h (this only holds for convex polyhedra). A linear program on this set of facets tells us whether h is a casting plane. We also know, by Theorem 3, that the total number of facets that we check, for all $O(n)$ candidate casting planes, is only $O(n \log n)$. This will lead to an $O(n \log^2 n)$ time algorithm for a convex polyhedron P with n vertices. The algorithm is split up in two parts, each of which walks around the polyhedron to find the relevant facets. The first algorithm tests each class of weakly equivalent planes that intersect some edge properly. The second tests all remaining planes that are *generated*, in the terminology of Theorem 3.

Each edge defines a class of weakly equivalent casting planes. The traversal of $h \cap P$ is performed for a generic (i.e. partially specified) plane h in this class. If any plane in the weak equivalence class is a valid casting plane, the linear program constructed by the traversal will find it. By Corollary 1 we know that any valid casting plane must intersect a facet in antipodal faces. In the next algorithm we take advantage of the fact that the casting direction is know given that a specified edge is properly intersected. We preprocess the polyhedron for Algorithm 2 as follows:

1. With Algorithm 1, compute a hierarchical decomposition of P into $O(\log n)$ vertex sets $V_1, \ldots, V_m$, as in Theorem 3. Store with each vertex v all $O(1)$ planes generated by v.

2. For every facet f, store the outward normals of the facets that are incident to an edge in the closure of f in a list in cyclic order around the facet. We perform binary search in the list only if the candidate cast removal direction $\vec{d}$ is parallel to f. In such a case, the cyclic list can be split at two places. One sublist contains the facets compatible with $\vec{d}$ and the other sublist contains the facets compatible with $-\vec{d}$.

3. For every vertex, store the outward normals of its incident facets in a list in cyclic order around the vertex. Again, using binary search, the list can be split at two places.

These preprocessing steps can be done in $O(n \log n)$ time.

Algorithm 2: *Test weak equivalence classes of planes that intersect an edge properly.*

for every edge $e_1 \in E$

if e_1 is untreated **then**

(* Trace $h \cap P$ for a generic casting plane h that intersects all edges parallel to e_1, resulting in a generic sequence $\tilde{e}_1, \tilde{v}_1, \tilde{e}_2, \tilde{v}_2, \ldots$ describing the edges and vertices that define $h \cap P$. Here $\tilde{e}_i$ is a facet of P or an edge of P contained in h, and $\tilde{v}_i$ is a vertex of P or an edge of P intersecting h. *)

Let $\vec{d}$ be a direction parallel to e_1, and let f_1 be a facet incident to e_1.

$\tilde{e}_1 \leftarrow f_1, \quad \tilde{v}_1 \leftarrow e_1, \quad i \leftarrow 1$

repeat

(* check $\tilde{v}_i$ and discover $\tilde{e}_{i+1}$ *)

if $\tilde{v}_i$ is an edge **then**

If $\tilde{v}_i$ is not parallel to e_1, fail, otherwise, mark $\tilde{v}_i$ as treated and let $\tilde{e}_{i+1}$ be the facet adjacent to $\tilde{v}_i$ distinct from the facet $\tilde{e}_i$.

else $\tilde{v}_i$ is a vertex

Check (in constant time) if $\tilde{v}_i$ is coplanar with every other vertex discovered; if not then fail.

If $(\tilde{v}_{i-2}, \tilde{v}_{i-1}, \tilde{v}_i)$ is a generated triple of vertices, mark it as treated.

Find by binary search the facet or edge $\tilde{e}_{i+1}$ distinct from $\tilde{e}_i$ that splits the facets incident to the vertex $\tilde{v}_i$ into those compatible with $\vec{d}$ and those incompatible with $\vec{d}$.

end if

(* check $\tilde{e}_{i+1}$ discover $\tilde{v}_{i+1}$ *)

if $\tilde{e}_{i+1}$ is a facet **then**

If $\tilde{e}_{i+1}$ is not parallel to e_1, fail.

Otherwise, find by binary search the edge or vertex $\tilde{v}_{i+1}$ of facet $\tilde{e}_{i+1}$ distinct from $\tilde{v}_i$ that splits the edges of $cl(\tilde{e}_{i+1})$ into those where neighboring facet is compatible with $\vec{d}$ and those where it is incompatible.

else $\tilde{e}_i$ is an edge contained in h

$\tilde{v}_{i+1} \leftarrow$ the endpoint of $\tilde{e}_i$ different from $\tilde{v}_i$.

$i \leftarrow i + 1$

until $\tilde{v}_i = e_1$ or h has failed

if the walk returns to e_1 **then**

Determine by linear programming whether a plane exists that intersects the closure of the edges discovered on the walk, and also the discovered vertices (dualize the endpoints of the edges and the vertices as in the proof of Lemma 12 to obtain the constraints).

If the LP is feasible, polyhedron P is castable with cast removal directions $\vec{d}$ and $-\vec{d}$, and the plane corresponding to the feasible solution of the LP.

end if

end if

next Edge

We now need to test those candidate casting planes that intersect no edge properly. The key observation for the next algorithm is that any casting plane that intersects no edge properly must be generated. For Algorithm 3, we carry out the additional preprocessing steps:

1. For every vertex v of P, store the edges adjacent to v in clockwise order, so that it is possible to determine for any query plane h containing v, the facets or edges incident to v that h intersects by binary search.
2. For every facet f of P, store the vertices in the closure of f in clockwise order, so that it is possible to determine for any query plane h which edges or vertices in the boundary of f intersect h by binary search.

Each of these preprocessing steps can be carried out in $O(n \log n)$ time, so the total preprocessing time is $O(n \log n)$.

For a given candidate casting plane h, we use v_i to denote the i-th vertex of $Q = h \cap P$ discovered, and F_h to to denote the set of facets that intersect h properly or are incident on an edge of P contained in h. It should be noted that triples marked as treated in Algorithm 2 remain marked at the beginning of Algorithm 3.

Algorithm 3: *Test all candidate planes that do not intersect an edge properly.*

for every generated triple (u, v_1, v_2)

  if (u, v_1, v_2) has not been treated **then**

    Let h be the plane through u, v_1, v_2. Mark (u, v_1, v_2) as treated. $i \leftarrow 2$, $F_h \leftarrow \emptyset$.

    while we have not walked all the way around to v_1 or failed

      Determine by binary search the edge or facet $\tilde{e}_{i+1}$ that h intersects clockwise from v_i.

      if $\tilde{e}_{i+1}$ is an edge $e = (v_i, v)$ **then**

        $v_{i+1} \leftarrow v$

        Add both facets adjacent to e to F_h.

      else $\tilde{e}_{i+1}$ is a facet

        Add $f = \tilde{e}_{i+1}$ to F_h.

        Determine by binary search what other vertex or edge $\tilde{v}$ in the boundary of f intersects h.

        If $\tilde{v}$ is an edge, fail, since h was tested with Algorithm 2, otherwise, $v_{i+1} \leftarrow \tilde{v}$

      end if

      if (v_{i-1}, v_i, v_{i+1}) is generated **then**

        If (v_{i-1}, v_i, v_{i+1}) has already been treated, fail, otherwise, mark (v_{i-1}, v_i, v_{i+1}) as treated.

      end if

      $i \leftarrow i + 1$

    next Step

    If the triple didn't fail, construct $\phi^+(h)$ and $\phi^-(h)$ from F_h. Test by linear programming if $\phi^+(h) \cap \mathit{refl}(\phi^-(h))$ is non-trivial. If so, accept h as a casting plane, with the casting directions given by the solution to the LP and the opposite thereof.

  end if

next Triple

Theorem 6 *Given a convex polyhedron P with n vertices, one can decide in $O(n \log^2 n)$ time and linear space whether P is castable when the cast parts must be removed in orthogonal or opposite directions.*

Proof: The above algorithms attain the claimed time bound. This can be seen as follows. The total preprocessing time is $O(n \log n)$. Let us count the total number of steps walking around the polyhedron in Algorithm 2. We charge each go through the loop either to the last encountered edge parallel to e_1 that was properly intersected, or to the last encountered generated triple (whichever came last). By the proof of Theorem 3, we know that there are $O(n)$ generated triples and that every $2m = O(\log n)$ consecutive vertices contain at least one generated triple. Furthermore, every edge is encountered during at most one walk. It follows that $O(n \log n)$ steps are taken during all walks. Since each walking step takes $O(\log n)$ time, Algorithm 2 takes $O(n \log^2 n)$ time to generate linear programs.

For Algorithm 3, the time bound follows in a similar way; each step that discovers a vertex is charged to the most recently encountered generated triple. It follows that the second algorithm also takes $O(n \log n)$ steps and $O(n \log^2 n)$ time to generate linear programs.

By Theorem 3, the total complexity of all linear programs generated by both algorithms is $O(n \log n)$, hence the total time to test all candidate planes is $O(n \log n)$. ■

4.3 A data structuring approach

The second, $O(n \log^2 n)$ time solution described above only applies to convex polyhedra. By using data structures, we will improve upon the quadratic time results of Theorem 5 for simple polyhedra as well. Unfortunately, the storage requirements increase with the data structuring method. The idea is to test every candidate casting plane by querying with it in a data structure (instead of applying linear programming). The query determines whether h really is a valid casting plane for the polyhedron P. The preprocessing of the data structure should be less than quadratic and the query time should be less than linear in order to beat the quadratic time bound. The previous subsection showed how to find the $O(n)$ candidate casting planes in $O(n \log n)$ time, so we only describe the data structure and the query algorithm. It turns out that the data structure is exactly the same for the three versions of removing the cast; only the query algorithms are different.

Let P be a polyhedron. For any vertex $v \in V$, define $F(v)$ as the set of facets incident to v, and for any subset $V' \subseteq V$, define $F(V')$ as the set of facets incident to at least one vertex of V'. We make the following observation:

Observation 7 *For any plane h, we have $F_h^+ \cup F_h^\times = F(V_h^+)$.*

We store P in a 2-level data structure T. The primary tree is a 3-dimensional partition tree that stores the set V of vertices of P, see Matoušek [16, 17] and Agarwal and Sharir [2] for example. They show that for any constant $\epsilon > 0$, a structure of size and preprocessing time $O(n^{3/2+\epsilon})$ exists, such that for any query plane h, the subset $V_h^+ \subseteq V$ of vertices above h can be retrieved in $O(n^{1/2+\epsilon})$ canonical nodes in $O(n^{1/2+\epsilon})$ time. For any node δ of T, corresponding to a canonical subset V_δ, the secondary structure at δ stores V_δ as follows. Recall from Subsection 2.2 that for a facet f, $\Psi(f)$ is the closed half-space which supports f and locally does not contain P, and $\Psi_0(f)$ is $\Psi(f)$ translated such that the bounding plane contains the origin. Let $F(V_\delta)$ be the set of facets incident to at least one vertex of V_δ. Let B_δ be the cone $\Psi_0(F(V_\delta))$ with apex at the origin. The secondary structure is simply an array or balanced binary tree that stores the facets of B_δ in cyclic order around the apex. The secondary structure allows for queries with a half-line starting at the origin, to determine whether the half-line is contained in B_δ. This query is in fact a 2-dimensional query to determine whether a point lies in a convex polygon.

Suppose that we wish to determine whether h is a valid casting plane for orthogonal cast removal. Then we search with h in T and determine the canonical nodes of T with respect to h, in particular, the set $\Delta^+ = \{\delta_1, \ldots, \delta_t\}$ of nodes of which the associated sets $V_{\delta_1}, \ldots, V_{\delta_t}$ are a partition of V_h^+. Let ℓ_h^+ be the upward half-line normal to h starting at the origin. For each of the nodes δ_i, we query with ℓ_h^+ in the secondary structure to determine by binary search whether $\ell_h^+ \in B_{\delta_i}$. If the answer is positive for all nodes $\delta_1, \ldots, \delta_t$, then $P \cap cl(h^+)$ is a terrain with respect to the direction normal to h (and parallel to ℓ_h^+). The query is repeated for h^-, to determine whether $P \cap cl(h^-)$ is a terrain with respect to the direction normal to h. If this is also the case, then h is a casting plane of P for orthogonal cast removal. Since there are $O(n^{1/2+\epsilon})$ cones (canonical nodes), the query time is $O(n^{1/2+\epsilon} \log n)$.

Next we consider cast removal in opposite directions. The query is a variation to the previous solution. We determine both sets Δ^+ and Δ^- of canonical nodes for the queries with h^+ and h^-, respectively. Let $\Delta^+ = \{\delta_1, \ldots, \delta_t\}$ and $\Delta^- = \{\delta'_1, \ldots, \delta'_s\}$. We wish to determine whether the common intersection of the t cones $B_{\delta_1}, \ldots, B_{\delta_t}$, intersected with the reflection in the origin of the common intersection of the s cones $B_{\delta'_1}, \ldots, B_{\delta'_s}$, is non-trivial, see Observation 5. One can decide whether the common intersection is non-trivial using the algorithm of Reichling [24]. He shows how to find an extremal point in the common intersection of k convex n-gons in $O(k \log^2 n)$ time. In our case, we 'reflect' all operations on the second set of cones. (Alternatively, we could store both the normal and the reflected cones explicitly at every node and choose the appropriate set, but this is not necessary.) The query time is $O((s+t)\log^2 n) = O(n^{1/2+\epsilon} \log^2 n)$ time.

Thirdly, we consider the query in the same structure to solve arbitrary cast removal. We remark that for arbitrary cast removal, we can determine using $O(n \log n)$ queries whether a casting plane exists, even though we do not have a subquadratic bound on the number of casting planes in this case. This will be shown in the next section.

Let h be the plane of which we wish to determine whether it is a casting plane. We determine the set Δ^+ as before; let $B_{\delta_1}, \ldots, B_{\delta_t}$ be the cones that are stored at the set Δ^+ canonical nodes. Now we have to determine whether the common intersection of these cones is non-trivial. Any half-line starting at the origin and in the common intersection of the cones represents a direction with respect to which $P \cap cl(h^+)$ is a terrain. A half-line in the common intersection of the t cones can be determined in $O(t \log^2 n) = O(n^{1/2+\epsilon} \log^2 n)$ time using Reichling's algorithm. If no such half-line exists, then $P \cap cl(h^+)$ is not a terrain with respect to any direction. If $P \cap cl(h^+)$ is a terrain for some direction, we repeat the query to test whether $P \cap cl(h^-)$ is a terrain.

Theorem 7 *For any constant $\epsilon > 0$ and any simple polyhedron P with n vertices, a data structure of size and preprocessing time $O(n^{3/2+\epsilon})$ exists, such that for any query plane h, one can determine in $O(n^{1/2+\epsilon})$ time whether h is a casting plane for P in any of the versions for removing the cast.*

Remark: In fact, we have also shown that for any query half-space h^+, we can determine within the same bounds whether $P \cap h^+$ is a terrain in some direction. Furthermore, by choosing a different partition tree for the primary tree, we can trade query time for storage space, see e.g. Chazelle, Sharir and Welzl [6], Matoušek [17] and Agarwal and Sharir [2]. For any $n \leq N \leq n^3$, a data structure of size and preprocessing time $O(N^{1+\epsilon})$ exists with query time $O(n^{1+\epsilon}/N^{1/3})$. The theorem above states the version we need for the casting problem.

Corollary 3 *For any constant $\epsilon > 0$ and any simple polyhedron P with n vertices, one can determine in $O(n^{3/2+\epsilon})$ time and space whether P is castable when the cast parts must be removed orthogonal to the casting plane, or in opposite directions.*

Proof: Generate the $O(n)$ candidate casting planes. Construct the data structure of Theorem 7, and query with the candidate casting planes. The result follows. ∎

Remark: De Berg [9] noted that the result for orthogonal cast removal can be improved to $O(n^{4/3+\epsilon})$ time and space. Conceptually, the improved data structure reverses the tests done in the main and the secondary structure to determine castability, when compared to the previously described solution. Use a two-level data structure of which the main tree is a 2-dimensional partition tree on the planes bounding $\Psi_0(f)$. Since these planes all pass through the origin, a 2-dimensional partition tree is indeed sufficient. It allows one to select all planes below and all planes above a given query ray (normal to the query plane) starting at the origin in $O(n^{1/3+\epsilon})$ canonical subsets [2, 6, 17]. For any canonical subset of a node δ, the vertices in the closures of the facets f for which the half-space $\Psi_0(f)$ appears in that canonical subset are further preprocessed into a secondary data structure with δ for half-space emptyness queries, see for example Clarkson and Shor [7]. The whole data structure requires $O(n^{4/3+\epsilon})$

space and preprocessing time. This improvement only applies to orthogonal cast removal, because the cast removal direction must be known in advance.

5 Algorithms for arbitrary cast removal

In this section we study the most general casting problem of this paper: determine whether a simple polyhedron P is castable when the cast parts may be removed in arbitrary directions. Using results of the previous sections, we can obtain an $O(n^{9/4+\epsilon})$ time algorithm: we devise a data structure as in the previous section to test all candidate $O(n^2)$ casting planes. We will do better in this section. Using Lemmas 1 and 2 and one more observation on arbitrary cast removal, we first obtain a simple $O(n^2 \log n)$ time and linear space algorithm, and then a more complicated $O(n^{3/2+\epsilon})$ time and space algorithm.

Let P be a polyhedron. We first test whether P admits opposite cast removal using the simple $O(n^2)$ time algorithm of Theorem 5. If so, we are done. Otherwise, if P is convex, then, by Lemma 1, we only have to consider casting planes that contain some edge of P. If P is non-convex, then, by Lemma 2, we only have to consider casting planes that contain an edge of the convex hull of P.

Observation 8 *Let P be a polyhedron and h be a plane that contains an edge e of the convex hull of P. Assume without loss of generality that e is horizontal and that a vertical plane exists which supports e and has $P - cl(e)$ completely to the one side.*

- *If $P \cap cl(h^+)$ is a terrain and $P \cap cl(h^-)$ is not a terrain, then no plane μ containing e for which $P \cap h^- \subset \mu^-$ is a casting plane.*
- *If $P \cap cl(h^+)$ is not a terrain and $P \cap cl(h^-)$ is a terrain, then no plane μ containing e for which $P \cap h^+ \subset \mu^+$ is a casting plane.*
- *If $P \cap cl(h^+)$ and $P \cap cl(h^-)$ are both not a terrain, then no plane containing e is a casting plane.*

The above observation sets up a binary search for a casting plane that contains some edge e of the convex hull of P. First, compute the convex hull of P. For any edge e of the convex hull, rotate P such that e is as in the observation. Consider the $n-2$ vertices that are not endpoints of e, and sort them by the order in which a vertical plane supporting e encounters them if the plane starts rotating about e. (The plane h can rotate in two directions about e. It is not important which direction is chosen, as long as this choice is made consistently.) Assume without loss of generality that the order is $v_1, \ldots, v_{n-2}$. We test whether the plane h supporting e and also containing $v_{n/2-1}$ is a casting plane by determining whether $P \cap cl(h^+)$ is a terrain and $P \cap cl(h^-)$ is a terrain. By the above observation, we can stop considering e if both are not terrains. If both are terrains, we can also stop and h is a casting plane. Otherwise, if only $P \cap cl(h^+)$

is a terrain, we continue the binary search on $v_{n/2}, \ldots, v_{n-2}$, and if only $P \cap cl(h^-)$ is a terrain, we continue the binary search on $v_1, \ldots, v_{n/2-2}$. After at most $\lceil \log_2(n-2) \rceil$ steps, we have determined whether there exists a casting plane that contains e. This leads to:

Theorem 8 *Given a simple polyhedron P with n vertices, one can determine in $O(n^2 \log n)$ time and linear space whether a casting plane for P exists, when the cast parts can be removed in arbitrary directions.*

Proof: To decide whether opposite cast removal is possible we first apply Theorem 5 and use $O(n^2)$ time. The computation of the convex hull of P requires $O(n \log n)$ time. There are $O(n)$ edges about which a plane is rotated. The sorting of the vertices $v_1, \ldots, v_{n-2}$ takes $O(n \log n)$ time, and each step of the binary search takes linear time by Lemma 3. Hence, the above procedure takes $O(n^2 \log n)$ time. ■

As in Subsection 4.3, we can preprocess P into a data structure such that any casting plane can be tested in $O(n^{1/2+\epsilon})$ time. Since $O(n \log n)$ casting planes are tested by the above procedure, the test part can be improved to $O(n^{3/2+\epsilon})$ time. However, how can we obtain the order of $v_1, \ldots, v_{n-2}$ without sorting? Again, the solution lies in data structuring. Notice first that the order of $v_1, \ldots, v_{n-2}$ is not needed explicitly. In the first step, the vertex that is the median $v_{n/2-1}$ must be determined, and in the following steps a median in one of the two halves.

The data structure that is needed preprocesses P for the following query problem: Given a query edge e such that there exists a plane h containing e that has the interior of P completely to the one side, and an integer k, find the k-th vertex of P that is encountered by h when it rotates about e (see Figure 4). Dualization yields a more familiar query problem: preprocess a set of n planes (dual to the vertices of P), such that for any given query ray, the k-th intersection point with the planes can be determined. The query ray is contained in the line dual to the line supporting e, and the starting point of the ray is any point dual to a plane containing e that has the interior of P completely to the one side.

Let $\mathcal{D}(V) = \{\mathcal{D}(v) \mid v \in V\}$ be the set of planes dual to the set V of vertices of P, preprocess them into a data structure for line segment intersection counting, as given by Agarwal and Matoušek [1]. They show that for any $\epsilon > 0$, a structure of size and preprocessing time $O(n^{3/2+\epsilon})$ exists, such that segment intersection counting queries can be answered in $O(n^{1/2+\epsilon})$ time. Furthermore, they show how the same structure can be used to find the k-th intersection point of a query ray with $\mathcal{D}(V)$ in $O(n^{1/2+\epsilon})$ time. Let the query ray be parameterized by $q + \lambda \cdot \vec{d}$, $\lambda \geq 0$, where q is a point and $\vec{d}$ is a vector in 3-dimensional space. In our application, if the k-th intersection point does not exist, and the last intersection point is the j-th, then the query should be continued in 'wrap-around' mode: find the $(k-j)$-th plane for the query ray along the same line and in the same direction, but with q translated in direction $-\vec{d}$ to infinity, see Figure 4. When this happens, the plane rotating about e in primal space rotates

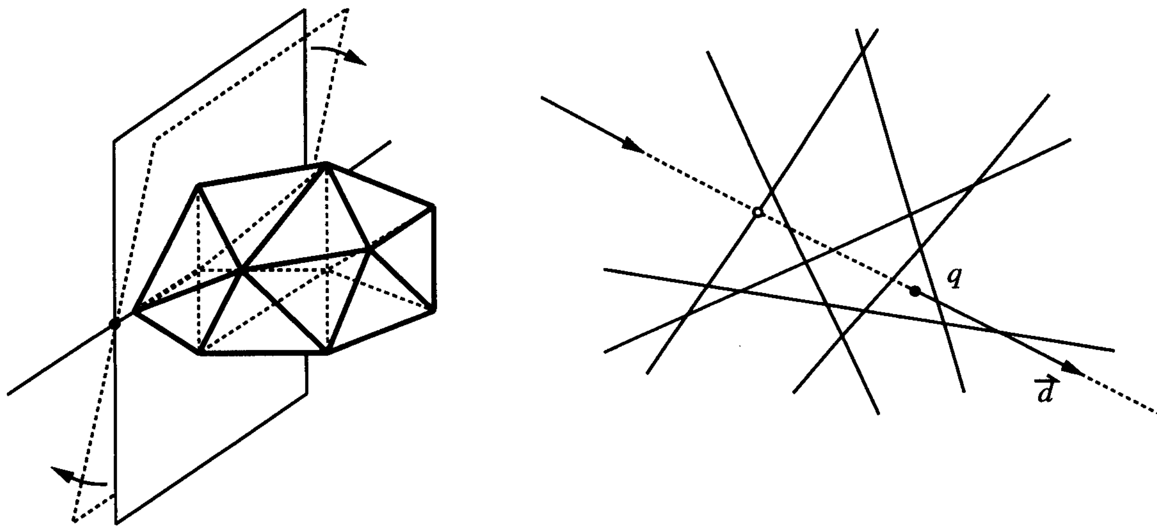

Figure 4: Left, rotating a plane about an edge through P. Right, the dual problem (in 2 dimensions), where the third plane intersecting the ray $q + \lambda \cdot \vec{d}$ is found after wrap-around.

past a vertical orientation. These adaptations to the query algorithm can easily be made within the same asymptotic time bounds. Hence, we conclude:

Theorem 9 *For any simple polyhedron P with n vertices and any constant $\epsilon > 0$, one can determine in $O(n^{3/2+\epsilon})$ time and space whether P is castable, when the cast parts can be removed in arbitrary directions.*

6 Conclusions and open problems

This paper studied the geometric version of the problem of determining whether a simple polyhedron can be manufactured using casting. It was assumed that there are two cast parts, and each has to be removed with a single translation. We presented simple algorithms that use $O(n^2)$ or $O(n^2 \log n)$ time and linear space which are based on linear programming. Furthermore, we showed that theoretically, better results can be obtained using partition trees and their variants. This leads to an $O(n^{3/2+\epsilon})$ time and space algorithm. Using the partition trees of Matoušek [17], the bound is in fact $O(n^{3/2}\,\mathrm{polylog}(n))$. Finally, as a by-product, we obtained a combinatorial bound on the number of planes intersecting a polyhedron in edges only, and on the summed number of edges in these planes.

Manufacturing applications have not been studied much in computational geometry. There are quite a large number of open problems to be solved in this area. We first list some open problems related to this paper, and then list a few others in cast design that deserve attention.

1. What is the maximum number of distinct casting planes in case of arbitrary cast removal? This paper shows $O(n^2)$, whereas the only lower bound we have is linear.

2. For a convex polyhedron P, what is the maximum summed complexity of the intersection of all distinct casting planes with P? This paper shows $O(n \log n)$ in case of opposite cast removal, but the trivial lower bound is linear.

3. Give simple algorithms for casting that improve our simple $O(n \log^2 n)$, $O(n^2)$ and $O(n^2 \log n)$ time algorithms. Find algorithms that improve upon our $O(n^{3/2+\epsilon})$ time algorithms.

4. Suppose that the casts may be removed with any motion. Give algorithms to determine whether a polyhedron is castable in this case.

5. Suppose that we wish to determine castability of an object with non-linear boundaries. Give (simple) algorithms that solve this problem.

6. Suppose that more cast parts are allowed. Determine for a polyhedron how many cast parts are necessary.

7. For some casting processes, is not necessary that the cast parts must be separated by a plane. In these cases, every convex polyhedron is castable. However, no algorithms are known for cast removal of simple polyhedra.

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