Syntactical Analysis of Total Termination

M.C.F. Ferreira and H. Zantema

UU-CS-1994-28
July 1994

Utrecht University

Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

Syntactical analysis of total termination

M. C. F. Ferreira and H. Zantema
Utrecht University, Department of Computer Science
[illegible], The Netherlands
[illegible]

Abstract

Termination is an important issue in the theory of term rewriting. In general termination is undecidable. There are nevertheless several methods successful in special cases. In [5] we introduced the notion of *total termination*: basically terms are interpreted compositionally in a total well-founded order, in such a way that rewriting chains map to descending chains. Total termination is then a semantic notion. It turns out that most of the usual techniques for proving termination fall within the scope of total termination. This paper consists of two parts. In the first part we introduce a generalization of the recursive path order presenting a new proof of its well foundedness without using Kruskal's theorem. We also show that the notion of total termination covers this generalization. In the second part we present some syntactical characterizations of total termination that can be used to prove that many term rewriting systems are not totally terminating, and hence outside the scope of the usual techniques. One of these characterizations can be considered as a sound and complete description of totality of orderings on terms.

1 Introduction

Most of the usual techniques for proving termination of term rewriting systems (TRS's) make use of total term orders. In [5] this notion of *total termination* was investigated in detail, with the emphasis on the underlying ordinal theory. Here we provide a syntactical analysis of total termination. A typical property of total orders is that if f is a strictly monotone function and $f(a) > f(b)$, then $a > b$. The main topic of this paper is to characterize totality of an order by properties like this. These characterizations are useful to prove that a TRS is not totally terminating. For example, the TRS

[illegible]

is terminating. Assume it is also totally terminating. Then according to the above observation it would still be terminating if the [illegible] from the first rule and the [illegible] from the second rule were stripped, yielding

[illegible]

which is clearly not terminating. [illegible]

[illegible]

ISSN: 0924-3275

One way to define total termination is the following: a TRS is totally terminating if and only if there is a total well-founded order $>$ on ground terms closed under ground contexts such that $l\sigma > r\sigma$ for each rewrite rule $l \to r$ and each ground substitution σ. In practical applications it is very natural to require this totality: for example in Knuth-Bendix completion such a well-founded term ordering is required, and a highly desirable property is that all new critical pairs can be ordered by the ordering. Totality on non-ground terms can not be achieved since commutativity conflicts with well-foundedness; totality on ground terms is the strongest feasible requirement. The totality property is essential for unfailing completion strategies. In the case of ground AC-equational theories finitely presented, the existence of a reduction ordering AC-compatible and total on $\mathcal{T}(\mathcal{F})/=_{AC}$ ensures that such theories always admit a canonical rewrite system. For more information on AC-compatible total orders see for example [13, 15]. Additionally most of the usual techniques for proving termination of TRS's like *polynomial interpretations* [11, 1], *elementary interpretations* [12], *Knuth-Bendix order* (KBO), prove in fact total termination.

In section 2 we give some basic definitions and properties over term rewriting in general and total termination in particular. The rest of the paper can be divided into two independent parts: section 3 on precedence based orders, and sections 4 and 5 on syntactical characterization of total termination.

In section 3 we present a slightly generalized version of the *recursive path order* (RPO). For this order we give a new proof of well-foundedness which is independent of Kruskal's theorem. We also show that the class of TRS's whose termination can be proved by RPO falls within the class of totally terminating TRS's. The same holds for other precedence based orders like the Knuth-Bendix ordering.

In section 4 we describe two characterizations of total termination that are effective in the sense that they provide powerful techniques to prove that TRS's are not totally terminating. However, they are not complete characterizations: we construct systems that are not totally terminating, but can not be dealt with the techniques presented. Such systems are rather tricky, and it is unlikely that they will appear in any application. In section 5 we describe a complete characterization of total termination: a system is totally terminating if and only if its rewrite relation is contained in a strict partial order having some syntactical properties. These properties cover the characterizations of section 4. However, this new characterization is not effective any more.

2 Basic definitions and properties

Below we give some basic notions over TRS's. For more information the reader is referred to [3].

Let $\mathcal{F}$ be a *signature*, i. e. $\mathcal{F}$ is a (non-empty) set of function symbols each with a fixed arity ≥ 0, denoted by *arity*(). Let $\mathcal{X}$ denote a set of variables, such that $\mathcal{F} \cap \mathcal{X} = \emptyset$. The set of terms over $\mathcal{F}$ and $\mathcal{X}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and the set of ground terms over $\mathcal{F}$ by $\mathcal{T}(\mathcal{F})$.

A *term rewriting system* (TRS) is a tuple $(\mathcal{F}, \mathcal{X}, R)$, where R is a subset of $\mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$. The elements of R are called the rules of the TRS and are usually denoted by $l \to r$. They obey the restriction that l must be a non-variable and every variable in r must also occur in l. In the following, unless otherwise specified, we identify the TRS with R, being $\mathcal{F}$ the set of function symbols occurring in R.

Given a function symbol f with arity $n \geq 0$, its *embedding rules* are n rules of the form

$f(x_1,\ldots,x_n) \to x_i$, with $1 \le i \le n$, where $x_1,\ldots,x_n$ are pairwise different variables. We denote by $\mathcal{E}mb_{\mathcal{F}}$ all embedding rules for all function symbols occurring in R.

A TRS R induces a *rewrite relation* over $\mathcal{T}(\mathcal{F},\mathcal{X})$, denoted by $\to_R$, as follows: $s \to_R t$ iff $s = C[l\sigma]$ and $t = C[r\sigma]$, for some context C, substitution σ and rule $l \to r \in R$. The transitive closure of $\to_R$ is denoted by $\to_R^+$ and its reflexive-transitive closure by $\to_R^*$. A TRS is called *terminating* (strongly normalizing or noetherian) if there exists no infinite sequence of the form $t_0 \to_R t_1 \to_R \cdots$.

We define a *well-founded monotone $\mathcal{F}$-algebra* $(A,>)$ to be an $\mathcal{F}$-algebra A for which the underlying set is provided with a well-founded order $>$ and each algebra operation is monotone[1] in all of its coordinates, more precisely: for each operation symbol $f \in \mathcal{F}$ and all $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$ for which $a_i > b_i$ for some i, and $a_j = b_j$ for all $j \neq i$, we have $f_A(a_1,\ldots,a_n) > f_A(b_1,\ldots,b_n)$.

Given a well-founded monotone $\mathcal{F}$-algebra $(A,>)$, let $A^{\mathcal{X}} = \{\sigma : \mathcal{X} \to A\}$; the interpretation function $[\![\ ]\!]_A : \mathcal{T}(\mathcal{F},\mathcal{X}) \times A^{\mathcal{X}} \to A$ is defined inductively by

$$\begin{aligned} [\![x,\sigma]\!]_A &= \sigma(x), \\ [\![f(t_1,\ldots,t_n),\sigma]\!]_A &= f_A([\![t_1,\sigma]\!]_A,\ldots,[\![t_n,\sigma]\!]_A) \end{aligned}$$

for $x \in \mathcal{X}, \sigma \in A^{\mathcal{X}}, f \in \mathcal{F}, t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$. The algebra $(A,>)$ induces an order $>_A$ over $\mathcal{T}(\mathcal{F},\mathcal{X})$ as follows: $s >_A t \iff \forall \sigma \in A^{\mathcal{X}} : [\![s,\sigma]\!] > [\![t,\sigma]\!]$. Intuitively $t >_A t'$ means that for each interpretation of the variables in A the interpreted value of t is greater than that of t'. The order $>_A$ is closed under substitutions and contexts.

A well-founded monotone $\mathcal{F}$-algebra $(A,>)$ and a TRS R are said to be *compatible* if $l >_A r$, for all rules $l \to r$ in R. From [17] we recall:

Theorem 2.1 *A TRS is terminating if and only if admits a compatible non-empty well-founded monotone algebra.*

Definition 2.2 *A TRS is called* totally terminating *if it admits a compatible non-empty well-founded monotone algebra in which the corresponding well-founded order is total.*

Theorem 2.3 *R is totally terminating if and only if $R \cup \mathcal{E}mb_{\mathcal{F}}$ is totally terminating.*

A useful characterization of total termination without referring to monotone algebras is the following.

Theorem 2.4 *Let $\mathcal{F}'$ be $\mathcal{F}$ extended with a new constant if $\mathcal{F}$ does not contain any. Then R is totally terminating if and only if there is a strict partial order $>$ on $\mathcal{T}(\mathcal{F}')$, such that*

- *$>$ is total and well-founded;*
- *$>$ is closed under ground contexts, i. e. if $C[\ ]$ is a ground context with exactly one hole, and t and s are ground terms with $s > t$ then $C[s] > C[t]$;*
- *$l\sigma > r\sigma$ for every rule $l \to r$ in R and every ground substitution σ.*

[1] By monotone we mean *strictly increasing*.

roofFirst, consider the if part. Since $>$ is total and well-founded on $\mathcal{T}(\mathcal{F}')$, we can make $(\mathcal{T}(\mathcal{F}'), >)$ a well-founded total monotone algebra over $\mathcal{F}$ by interpreting each function symbol in $\mathcal{F}$ by itself. From the properties of $>$ follows that R is compatible with this interpretation, yielding the total termination of R.

For the only-if part, first note that total termination of $(\mathcal{F}, \mathcal{X}, R)$ implies total termination of $(\mathcal{F}', \mathcal{X}, R)$ (see lemma 4.2), so we consider total monotone algebras over $\mathcal{F}'$.

The essential step in this part is the existence of any total order on the set of ground terms, well-founded and closed under contexts. To construct such an order, consider the set of function symbols $\mathcal{F}'$. By Zermelo's Theorem (see [10]) there is a total, well-founded order on $\mathcal{F}'$. Let $\succ$ be such an order, called a precedence. Consider the order $>_{lpo}$ associated with this precedence and taking lexicographic sequences from left to right. In section 3 we prove that this order has all the required properties.

Since R is totally terminating, we know that R is compatible with a (non-empty) monotone $\mathcal{F}'$-algebra $(A, >)$, with $>$ total and well-founded. Again let $[t]$ be the interpretation in A of a ground term t.

In $\mathcal{T}(\mathcal{F}')$ we define the order $\sqsupset$ by

$$s \sqsupset t \iff ([s] > [t]) \text{ or } ([s] = [t] \text{ and } s >_{lpo} t)$$

Irreflexivity and transitivity of $\sqsupset$ follows from irreflexivity and transitivity of both $>$ and $>_{lpo}$. Given any two ground terms s, t then either $[s] > [t]$ or $[t] > [s]$ or $[t] = [s]$, since $>$ is total. In the first two cases we conclude $s \sqsupset t$ or $t \sqsupset s$ respectively. In the last case, since $>_{lpo}$ is total we know that either $s >_{lpo} t$ or $t >_{lpo} s$ or $s = t$, hence the order $\sqsupset$ is total. Well-foundedness of $\sqsupset$ follows directly from well-foundedness of both $>$ and $>_{lpo}$. The same holds for closedness under ground contexts.

If $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}')$ is any ground substitution and $l \to r$ is a rule in R, then $[l\sigma] > [r\sigma]$, since $(A, >)$ is compatible with R, and therefore $l\sigma \sqsupset r\sigma$, concluding the proof.

3 Precedence based orderings

In [8], Hofbauer proved that for a finite TRS proved terminating by recursive path order with only multiset status, a proof of total termination can be given in the natural numbers with primitively recursive operations. In this section we show that orders like RPO or KBO, even in their most general form, actually prove total termination, i. e. if a TRS R is proven terminating by RPO (or KBO), then R is totally terminating. The reverse is not true; the system

$$f(g(x)) \ \to \ g(f(f(x)))$$

is totally terminating (see [5]), but it cannot be proven terminating by RPO or KBO.

We introduce some needed definitions; mainly conventions and notations of [2, 16] will be followed.

Given a poset $(S, >)$ we consider two useful extensions of $>$, namely *lexicographic extension* (denoted by $>_{lex}$) defined as usual over sequences of elements of S, and *multiset extension* (denoted by $>_{mul}$) and defined over $M(S)$, the finite multisets over S (see [4, 16]).

Quasi-orders over a set S are transitive and reflexive relations over S. They will be denoted in general by $\succeq$. Any quasi-order defines an equivalence relation, namely $\succeq \cap \preceq$, and a partial order, namely $\succeq \setminus \preceq$ (or vice-versa). We usually denote the equivalence relation by

$\sim$. Conversely, given a partial order $\succ$ and an equivalence $\sim$, their union does not always define a quasi-order (the transitive closure of their union does). However if $\succ$ and $\sim$ satisfy

$$(\sim \cap \succ = \emptyset) \text{ and } (\sim \circ \succ \circ \sim) \subseteq \succ \tag{1}$$

where $\circ$ represents composition, then $\succ \cup \sim$ is a quasi-order, of which $\succ$ is the strict part and $\sim$ the equivalence part.

From now on if we characterize a quasi-order via $\succ \cup \sim$, we assume that the conditions of (1) are satisfied. Also we take as partial order defined by a quasi-order $\succeq$ the relation $\succ = \succeq \setminus \preceq$.

Given a quasi-order $\succeq$ over S, the quotient $S/\sim$ consists of the equivalence classes of $\sim$; such classes are denoted by $\langle\ \rangle$. We can extend $\succ$ to $S/\sim$ in a natural way, namely $\langle s \rangle \sqsupset \langle t \rangle$ iff $s \succ t$. Since $\succ$ and $\sim$ satisfy condition (1), the relation $\sqsupset$ does not depend on the class representative and thus is well-defined. Furthermore $\sqsupset$ is a partial order over $S/\sim$. When this extension is well-defined we abusively write $\succ$ instead of $\sqsupset$.

Given two quasi-orders $\succeq$ and $\succeq'$ over the same set, we say that $\succeq'$ extends $\succeq$ iff $\succ \subseteq \succ'$ and $\sim \subseteq \sim'$.

For any quasi-order $\succeq$, $\succeq_{lex}$ and $\succeq_{mul}$ denote its lexicographic and multiset extensions, respectively. These quasi-orders are defined as in the partial order case, with equality being replaced by the more general equivalence $\sim$.

Lexicographic and multiset extensions preserve well-foundedness, more precisely:

Lemma 3.1 *$\succeq$ is well-founded over a set A iff $\succeq_{mul}$ is well-founded over $M(A)$.*

Lemma 3.2 *$\succeq$ is well-founded over a set A iff $\succeq_{lex}$ is well-founded over A^n, the set of sequences over A of size at most n, for a fixed $n \geq 1$.*

To each function $f \in \mathcal{F}$ we associate a status $\tau(f)$. Status indicates how the arguments of the function symbol are to be taken. We consider two possible cases:

- $\tau(f) = mul$; indicates that, for the purpose of ordering, the arguments of f are to be taken as a multiset.
- $\tau(f) = lex_\pi$, where π is a permutation of the set $\{1, \ldots, arity(f)\}$; indicates that, for the purpose of ordering, the arguments are to be taken as a lexicographic sequence whose order is given by π.

Given the set of function symbols $\mathcal{F}$, let $\trianglerighteq$ denote a quasi-order over $\mathcal{F}$ usually called a *quasi-precedence*. We reserve the term *precedence* to partial orders over $\mathcal{F}$.

From now on we assume that a quasi-precedence over $\mathcal{F}$ is given as well as a status function τ, under the following restriction: lexicographic and multiset status cannot be mixed, i. e.

$$\text{if } f \sim g \text{ and } \tau(f) = mul \text{ then } \tau(g) = mul \tag{2}$$

Write $>_{rpo}^{=}$ for recursive path order with status as it appears in [16]. This definition is not suitable to our purposes. We need to define a total well-founded monotone algebra $(A, >)$ and a good candidate is $(\mathcal{T}(\mathcal{F}), >_{rpo}^{=})$. If we define the congruence $\simeq$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: $s \simeq t$ iff $s = t$ or $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, $f \sim g$, $m = n$ and either

- $\tau(f) = \tau(g) = mul$ and there is a permutation π of $\{1, \ldots m\}$ such that $s_i \simeq t_{\pi(i)}$, for any $1 \leq i \leq m$;

- $\tau(f) = lex_{\pi_f}$ and $\tau(g) = lex_{\pi_g}$ and $s_{\pi_f(i)} \simeq t_{\pi_g(i)}$ for all $1 \leq i \leq m$.

Then if for ground terms s, t, $s \simeq t$ and $s \neq t$, both $s \not>^{=}_{rpo} t$ and $t \not>^{=}_{rpo} s$. So $(\mathcal{T}(\mathcal{F}), >^{=}_{rpo})$ is not total and it seems reasonable to take $A = \mathcal{T}(\mathcal{F})/\simeq$. But unfortunately the natural extension of $>^{=}_{rpo}$ to the congruence classes of $\mathcal{T}(\mathcal{F},\mathcal{X})/\simeq$ is not well-defined even for total precedences (condition (1) does not hold). This can be repaired by extending the definition of $>^{=}_{rpo}$ to $>_{rpo}$, namely replace equality by $\simeq$.

Definition 3.3 (RPO with status) *Given two terms s, t we say that $s \sim_{rpo} t$ if $s \simeq t$, and $s >_{rpo} t$ iff $s = f(s_1, \ldots, s_m)$ and either*

1. *$t = g(t_1, \ldots, t_n)$ and*
   (a) *$f \rhd g$ and $s >_{rpo} t_i$, for all $1 \leq i \leq n$, or*
   (b) *$f \sim g$ and $(s_1, \ldots, s_m) >_{rpo,\tau} (t_1, \ldots, t_n)$ and $s >_{rpo} t_i$, for all $1 \leq i \leq n$; or*
2. *$\exists 1 \leq i \leq m :\ s_i >_{rpo} t$ or $s_i \sim_{rpo} t$.*

It can be seen by straightforward induction proofs that $>_{rpo}$ and $\sim_{rpo}$ have the following properties:

- $>_{rpo}$ is a strict partial order and $\sim_{rpo}$ is an equivalence, both defined over $\mathcal{T}(\mathcal{F},\mathcal{X})$. Furthermore $>_{rpo}$ and $\sim_{rpo}$ satisfy condition (1).
- $>_{rpo}$ and $\sim_{rpo}$ are closed under contexts and substitutions and $>_{rpo}$ has the subterm property, i. e. $C[t] >_{rpo} t$, for any term t and non-trivial context $C[\ ]$.
- $\succeq_{rpo}$ is monotone with respect to quasi-precedences, i. e. if $\unrhd, \unrhd'$ are quasi-precedences over $\mathcal{F}$ such that $\unrhd'$ extends $\unrhd$, then $>_{rpo}$ associated with $\unrhd'$ extends $>_{rpo}$ associated with $\unrhd$. Consequently $>_{rpo}$ extends $>^{=}_{rpo}$, for any fixed quasi-precedence and status.
- If $\unrhd$ is total over $\mathcal{F}$ then $>_{rpo}$ is total over $\mathcal{T}(\mathcal{F})/\sim_{rpo}$.
- If all function symbols have *lex* status then $>_{rpo}$ coincides with Kamin and Lévy's ([9]) *lexicographic path order* (that we denote by $>_{lpo}$). If $\rhd$ is total and $\sim$ is syntactical equality then, as a consequence of the previous remark, we have that $>_{lpo}$ is total over $\mathcal{T}(\mathcal{F})$.

In order for $>_{rpo}$ to be useful for proving termination of term rewriting systems, the order has to be well-founded. Unfortunately, well-foundedness of $\unrhd$ alone is not sufficient to guarantee well-foundedness of $>_{rpo}$ as the following example shows. Let $\mathcal{F}$ consist of two constants $a \rhd b$ and function symbols f_i, $i \geq 1$, such that f_i has arity i, $\tau(f_i) = lex_{Id}$ and $f_i \sim f_j$, for any i, j. Then we have the following infinite descending chain

$$f_1(a) >_{rpo} f_2(b,a) >_{rpo} f_3(b,b,a) >_{rpo} f_4(b,b,b,a) >_{rpo} \cdots$$

The problem stems from the fact that lexicographic sequences of unbounded size are not well-founded.[2] Kamin and Lévy ([9]) proved that $>_{lpo}$ is well-founded provided that equivalent function symbols have the same arity. In the following we prove that this restriction can be

[2]Note that even if $\unrhd$ would be total or $\mathcal{F}$ finite, with a function symbol f allowing different arities, the same problem would arise.

weakened. It is enough to require that for every equivalence class of function symbols with lexicographic status, there is a natural number bounding the arities of the function symbols in the class. That is

$$\forall f \in \mathcal{F} : \tau(f) = lex_\pi \Rightarrow (\exists n \geq 0 : \forall g \in \langle f \rangle : arity(g) \leq n) \tag{3}$$

Before proving well-foundedness of $>_{rpo}$, we need some additional definitions and results from [7].

Definition 3.4 *A quasi-order* $\succeq$ *over a set* S *is a* well quasi-order, *abbreviated to* wqo, *iff every quasi-order extending it (including* $\succeq$ *itself) is well-founded.*

There are several equivalent characterizations of *wqo*'s. We also use the following (see [7]): "Every infinite sequence $(s_i)_{i\geq 0}$ of elements of S contains some infinite subsequence $(s_{\phi(i)})_{i\geq 0}$ such that $s_{\phi(i+1)} \succeq s_{\phi(i)}$, for all $i \geq 0$".

A traditional way of proving well-foundedness of $>_{rpo}$ is via Kruskal's theorem. Given our extended definition of $>_{rpo}$, we cannot apply Kruskal's theorem in a straightforward way. This is so because $>_{rpo}$ no longer contains the embedding relation. Let us elaborate some more here. Given a quasi-order $\trianglerighteq$ over $\mathcal{F}$, the *embedding relation* $>_{emb}$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is defined as follows ([7]). Either:

- $f(t_1, \ldots, t_n) \geq_{emb} g$ iff $f \trianglerighteq g$; or
- $f(\ldots, t, \ldots) \geq_{emb} t$; or
- $f(s_1, \ldots, s_m) \geq g(t_1, \ldots, t_n)$ iff $f \trianglerighteq g$, $n \leq m$ and there are integers $j_1, \ldots, j_n$ such that $1 \leq j_1 < \ldots < j_n \leq m$ and $s_{j_i} \geq_{emb} t_i$, for all $1 \leq i \leq n$.

Kruskal's theorem states that if $\trianglerighteq$ is a *wqo* on $\mathcal{F}$ then $\geq_{emb}$ is also a *wqo* on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Consequently any relation containing the embedding relation is well-founded. Previous versions of $>_{rpo}$ fall within this category. For definition 3.3 this does no longer hold: in the example above we have $f_2(b, a) >_{emb} f_1(a)$, however $f_2(b, a) \not>_{rpo} f_1(a)$.

A way of dealing with orders for which Kruskal's theorem is not applicable is given in [6]. Well-foundedness of $>_{rpo}$ can be derived from results presented there. Nevertheless here we present a proof of well-foundedness of $>_{rpo}$ inspired by the proof of Kruskal's theorem itself as presented in [7, 14] and closely following [6]. We should emphasize that the proof given does not rely on Kruskal's theorem and is therefore simpler if you consider the degree of difficulty involved in Kruskal's theorem itself.

Theorem 3.5 *Let* $\trianglerighteq$ *be a quasi-precedence over* $\mathcal{F}$ *and* τ *a status function such that conditions (2) and (3) are satisfied. Then* $>_{rpo}$ *is well-founded over* $\mathcal{T}(\mathcal{F}, \mathcal{X})$ *iff* $\trianglerighteq$ *is well-founded over* $\mathcal{F}$.

roofFor the if part, let $\trianglerighteq$ be a well-founded quasi-precedence over $\mathcal{F}$ and τ a status function such that conditions (2) and (3) are satisfied. We first extend $\trianglerighteq$ to a total well-founded quasi-order $\trianglerighteq'$ such that $\sim' = \sim$. This is done in the "usual" way: using Zorn's Lemma we extend the well-founded partial order $\triangleright^3$ over $\mathcal{F}/\sim$ to a total well-founded partial order $>'$ over $\mathcal{F}/\sim$. Then $\triangleright'$ and $\sim$ are compatible and $\trianglerighteq'$ (with $\sim' = \sim$), is total and well-founded

[3] Itself a natural extension of $\triangleright$ to $\mathcal{F}/\sim$ that we abusively denote equally.

over $\mathcal{F}$, where as expected $\rhd'$ is defined as $\forall f, g \in \mathcal{F} : f \rhd' g \iff \langle f \rangle >' \langle g \rangle$. The reason why we require that $\sim' = \sim$ is to avoid problems with the status of equivalent symbols, i. e. to guarantee that conditions (2) and (3) still hold for the extended quasi-precedence.

Since $\rhd'$ is total and well-founded, every extension of it is well-founded, hence $\unrhd'$ is a *wqo* over $\mathcal{F}$. Suppose now that $>_{rpo}$ taken over this total well-founded quasi-precedence, is not well-founded. Take then an infinite descending chain

$$t_0 >_{rpo} t_1 >_{rpo} t_2 >_{rpo} \cdots$$

minimal in the following sense:

- $|t_0| \leq |s_0|$, for all infinite chains $s_0 >_{rpo} s_1 >_{rpo} \cdots$
- $|t_{i+1}| \leq |s_{i+1}|$, for all infinite chains $s_0 >_{rpo} s_1 >_{rpo} \cdots$, such that $t_j = s_j$ for $0 \leq j < i+1$.

where $|t|$ represents the number of function symbols occurring in t.

We remark that no proper subterm of a term t_i, $i \geq 0$, in the above chain, can initiate an infinite descending chain; for, suppose u^i_j is such a subterm, then the chain

$$t_0 >_{rpo} \cdots >_{rpo} t_{i-1} >_{rpo} u^i_j >_{rpo} u_1 >_{rpo} \cdots$$

will be an infinite descending chain contradicting the minimality of $(t_i)_{i \geq 0}$ (since $|u^i_j| < |t_i|$).

Let $root(t)$ be the head function symbol of the term t. We see that there is no infinite subsequence $(t_{\phi(i)})_{i \geq 0}$ of $(t_i)_{i \geq 0}$ such that $root(t_{\phi(i)}) \sim root(t_{\phi(j)})$, for all $i, j \geq 0$. Suppose it is not so and let $(t_{\phi(i)})_{i \geq 0}$ be such a subsequence. Due to condition (2), all root symbols in this sequence have the same status (either *mul* or *lex*). By definition of $>_{rpo}$, and since $t_{\phi(i)} >_{rpo} t_{\phi(i+1)}$, for all $i \geq 0$, we must have

$$args(t_{\phi(0)}) >_{rpos,\tau} args(t_{\phi(1)}) >_{rpos,\tau} \cdots$$

where $args(t)$ are the proper subterms of t. From lemma 3.1 or 3.2, we conclude that $>_{rpo}$ is not well-founded over $\bigcup_{i \geq 0} Args(t_{\phi(i)})$ (where $Args(t)$ is the set of proper subterms of t), contradicting the minimality of $(t_i)_{i \geq 0}$.

Consider the sequence $(root(t_i))_{i \geq 0}$. This sequence is infinite and since $\unrhd'$ is a *wqo* over $\mathcal{F}$, an infinite subsequence $(root(t_{\phi(i)}))_{i \geq 0}$ of $(root(t_i))_{i \geq 0}$ exists such that $root(t_{\phi(i+1)}) \unrhd' root(t_{\phi(i)})$, for all $i \geq 0$. But since every $\sim$-equivalence class appears only finitely many times in the sequence $(root(t_i))_{i \geq 0}$, we can say without loss of generality that the subsequence $(root(t_{\phi(i)}))_{i \geq 0}$ fulfils $root(t_{\phi(i+1)}) \rhd' root(t_{\phi(i)})$, for all $i \geq 0$[4]. But $t_{\phi(i)} >_{rpo} t_{\phi(i+1)}$ (for all $i \geq 0$), then, by definition of $>_{rpo}$, both $t_{\phi(i)}$ and $t_{\phi(i+1)}$ are not constants and we must have $u_{\phi(i)} >_{rpo} t_{\phi(i+1)}$ or $u_{\phi(i)} \sim_{rpo} t_{\phi(i+1)}$, for some $u_{\phi(i)} \in Args(t_{\phi(i)})$. In both cases a contradiction with the minimality of $(t_i)_{i \geq 0}$ arises.

Well-foundedness of $>_{rpo}$ over the original quasi-precedence $\unrhd$ follows from the fact that $>_{rpo}$ is monotone with respect to precedences (since $\unrhd'$ is an extension of $\unrhd$).

For the only-if part, suppose that $>_{rpo}$ is well-founded over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and that $\unrhd$ is not well-founded on $\mathcal{F}$. Let $f_0 \rhd f_1 \rhd \cdots$ be an infinite descending sequence in $\mathcal{F}$. This sequence does not contain an infinite subsequence consisting only of constants, since if $(f_{\phi(i)})_{i \geq 0}$ would be such a sequence, we would have $f_{\phi(0)} >_{rpo} f_{\phi(1)} >_{rpo} \cdots$, contradicting the well-foundedness

[4] Strictly speaking there is a subsequence of $(root(t_{\phi(i)}))_{i \geq 0}$ with this property.

of $>_{rpo}$. Let then $(f_{\phi(i)})_{i\geq 0}$ be an infinite subsequence of $(f_i)_{i\geq 0}$ such that $arity(f_{\phi(i)}) \geq 1$, for all $i \geq 0$. Let x be any variable. By definition of $>_{rpo}$, we conclude that

$$f_{\phi(0)}(x, \ldots, x) >_{rpo} f_{\phi(1)}(x, \ldots, x) >_{rpo} \cdots$$

contradicting the well-foundedness of $>_{rpo}$.

Another approach to prove well-foundedness of our version of $>_{rpo}$ is the following. Every function symbol with status *lex* has its arity augmented to the maximal arity associated with its equivalence class. The new arguments are filled with a dummy constant. By this construction all function symbols in the same equivalence class are forced to have the same arity, hence the old version of $>_{rpo}$ is applicable, provided we change the status function consistently. Well-foundedness of our version of $>_{rpo}$ then follows from well-foundedness of previous $>_{rpo}$ versions. However the classical proof of this well-foundedness makes use of Kruskal's theorem.

The following TRS's

$$\begin{array}{lcl} f(1, x) & \rightarrow & g(0, x, x) \\ g(x, 1, y) & \rightarrow & f(x, 0) \end{array}$$

and

$$\begin{array}{lcl} a & \rightarrow & g(c) \\ g(a) & \rightarrow & b \\ f(g(x), b) & \rightarrow & f(a, x) \end{array}$$

are totally terminating. Just take quasi-precedences $\trianglerighteq$ and status function τ satisfying $1 \triangleright 0$, $f \sim g$, $\tau(f) = \tau(g) = lex_{Id}$, for the first TRS, and $a \triangleright g$, $a \triangleright c$, $a \sim b$ and $\tau(f) = mul$, for the second TRS. Earlier versions or $>_{rpo}$ fail to prove termination of these TRS's: for the first TRS we cannot choose $f \triangleright g$ nor $g \triangleright f$ nor uncomparability of f and g, and if $f \sim g$, the status of these symbols cannot be the multiset status.

Theorem 3.6 *Given a TRS R, suppose $\trianglerighteq$ is a well-founded quasi-precedence over $\mathcal{F}$ and τ is a status function such that conditions (2) and (3) are satisfied. If $l >_{rpo} r$ for every rule $l \rightarrow r \in R$ then R is totally terminating.*

Proof We give a sketch of the proof. In order to establish total termination of R we need to define a total well-founded monotone algebra. For that we choose $\mathcal{T}(\mathcal{F})/\sim_{rpo}$, where $\sim_{rpo}$ is the congruence associated with $>_{rpo}$. If $\mathcal{F}$ does not contain any constant, we introduce one to force $\mathcal{T}(\mathcal{F})$ to be non-empty. With respect to the quasi-precedence $\trianglerighteq$, the relative order of this new element is irrelevant and does not influence the behaviour of $>_{rpo}$. We extend $\trianglerighteq$ to a total well-founded quasi-precedence $\trianglerighteq^t$ such that the equivalence part remains the same (done using Zorn's lemma as described in the proof of theorem 3.5) and consider $>_{rpo}$ over this extended quasi-precedence. By theorem 3.5, we know that $>_{rpo}$ is well-founded, and as remarked before $>_{rpo}$ extended to $\mathcal{T}(\mathcal{F})/\sim_{rpo}$ is total and well-founded. In $\mathcal{A} = (\mathcal{T}(\mathcal{F})/\sim_{rpo}, >_{rpo})$ we interpret the function symbols of $\mathcal{F}$ by $f_A(\langle s_1 \rangle, \ldots, \langle s_{arity(f)} \rangle) = \langle f(s_1, \ldots, s_{arity(f)}) \rangle$. Since $\sim_{rpo}$ is a congruence $f_{\mathcal{A}}$ is well-defined. The interpretation function $[\![\;]\!] : \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{A}^{\mathcal{X}} \rightarrow \mathcal{A}$ is given as usual.

Since $\mathcal{A}$ is total and well-founded, the only condition we need to check to establish total termination is compatibility with the rules of R. It can be seen, by induction on t, that

$$\forall t \in \mathcal{F}\ \forall \tau \in \mathcal{A}^{\mathcal{X}} : \ [\![t, \tau]\!] = \langle t\sigma \rangle$$

where σ is any ground substitution satisfying $\sigma(x) \in \tau(x)$, for all $x \in \mathcal{X}$. Note that the class $\langle t\sigma \rangle$ does not depend on the choice of σ. Let $l \to r$ be a rule in R and let $\tau : \mathcal{X} \to \mathcal{A}$ be an assignment. Let σ be a ground substitution satisfying $\sigma(x) \in \tau(x)$ for all $x \in \mathcal{X}$. Since $>_{rpo}$ is monotone with respect to quasi-precedences and by hypothesis $l >_{rpo} r$, with $>_{rpo}$ taken over $\trianglerighteq$, we also have $l >_{rpo} r$, where now the $>_{rpo}$ is based on the total quasi-precedence $\trianglerighteq^t$. Consequently $\langle l, \sigma \rangle >_{rpo} \langle r, \sigma \rangle$, thus $[\![l, \tau]\!] >_{rpo} [\![r, \tau]\!]$, and we conclude that R is totally terminating , with $\mathcal{T}(\mathcal{F})/\!\sim_{rpo}$ as total well-founded monotone algebra. □

The Knuth-Bendix order uses the concept of *weight function*.
Let $\phi : \mathcal{F} \cup \mathcal{X} \to \mathbb{N}$ be a function such that

$$\phi(f) \text{ is } \begin{cases} = \phi_0 > 0 & \text{if } f \in \mathcal{X} \\ \geq \phi_0 & \text{if arity}(f) = 0 \\ > 0 & \text{if arity}(f) = 1 \end{cases}$$

We extended ϕ to terms as follows: $\phi(f(s_1, \ldots, s_m)) = \phi(f) + \sum_{i=1}^{m} \phi(s_i)$.

Let $\#_x(t)$ denote the number of occurrences of variable x in term t. The Knuth-Bendix order with status is defined as follows ([16]).

Definition 3.7 (KBO with status) *We say that $s >_{kbo} t$ iff $\forall x \in \mathcal{X} : \#_x(s) \geq \#_x(t)$ and*

1. $\phi(s) > \phi(t)$ *or*
2. $\phi(s) = \phi(t)$, $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$ *and*
   (a) $f \rhd g$ *or*
   (b) $f \sim g$ *and* $s_1, \ldots, s_m >_{kbos,\tau} t_1, \ldots, t_n$

Knuth-Bendix order has properties similar to $>_{rpo}$ (see [16]), namely it is a partial order closed under substitutions and contexts and monotone with respect to quasi-precedences.

The order $>_{kbo}$ can be used to define a congruence $\sim_{kbo}$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: $s \sim_{kbo} t$ iff $s = t$ or $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, $f \sim g$, $m = n$, $\phi(s) = \phi(t)$ and either

- $\tau(f) = mult$ and $s_i \sim_{kbo} t_{\pi(i)}$, for any $1 \leq i \leq m$, where π is a permutation of $\{1, \ldots m\}$;
- $\tau(f) = lex_{\pi_f}$, $\tau(g) = lex_{\pi_g}$ and $s_{\pi_f(i)} \sim_{kbo} t_{\pi_g(i)}$ for all $1 \leq i \leq m$.

It can be seen that $\sim_{kbo}$ is indeed a congruence i. e. a reflexive, symmetric and transitive relation, closed under contexts. Further $\sim_{kbo}$ is also closed under substitutions and it is not difficult to see that $>_{kbo}$ and $\sim_{kbo}$ are compatible, so we can extend $>_{kbo}$ to $\mathcal{T}(\mathcal{F}, \mathcal{X})/\!\sim_{kbo}$ in the usual way. As with $>_{rpo}$, given a total quasi-precedence over $\mathcal{F}$, $>_{kbo}$ is total over $\mathcal{T}(\mathcal{F})/\!\sim_{kbo}$. As for well-foundedness we have

Theorem 3.8 *Let $\trianglerighteq$ be a well-founded quasi-precedence over $\mathcal{F}$ and τ a status function such that condition (2) is satisfied. Then $>_{kbo}$ is well-founded over $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

This theorem can be proven in a way similar to theorem 3.5. Notice that condition (3) is not necessary since the use of the weight function ensures that the lexicographic extension is well-founded.

Theorem 3.9 *Given a TRS R, suppose $\trianglerighteq$ is a well-founded quasi-precedence over $\mathcal{F}$ and τ is a status function such that condition (2) is satisfied. Let ϕ be a weight function. If $l >_{kbo} r$ for every rule $l \to r \in R$ then R is totally terminating.*

Proof **(Sketch)** We proceed in a manner similar as for $>_{rpo}$. Namely we extend the well-founded quasi-precedence $\trianglerighteq$ to a total one whose underlying equivalence is the same, and take $>_{kbo}$ over this total well-founded quasi-precedence. As total well-founded monotone algebra we choose $\mathcal{T}(\mathcal{F})/\sim_{kbo}$[5] and interpret the function symbols of $\mathcal{F}$ in the same way. It is not difficult to see that all requirements of total termination are met. □

4 Proving non-total termination

From theorem 2.3 we know that a TRS R is totally terminating if and only if $R \cup \mathcal{E}mb_{\mathcal{F}}$ is totally terminating. So if $R \cup \mathcal{E}mb_{\mathcal{F}}$ is non-terminating then R is not totally terminating. A next step is context removal: if $C[t] \to_R^+ C[u]$ then R is totally terminating if and only if $R \cup \{t \to u\}$ is totally terminating.

A first rough attempt to characterize total termination resulted in the following definition.

Definition 4.1 *Given a TRS R we define the relation $\rhd \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ as follows: $s \rhd t$ iff $s \neq t$ and $(R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{t \to s\})$ is not terminating.*

It is not difficult to see that $\rhd$ has the following properties:

- if $C[s] \rhd C[t]$, for any ground context $C[\,]$, then $s \rhd t$.
- $\to^+_{R \cup \mathcal{E}mb_{\mathcal{F}}} \subseteq \rhd$
- $\rhd$ is in general not transitive.

Given a binary relation θ over a set A, not necessarily transitive, we say that θ is well-founded if there is no infinite chain $(a_i)_{i \in \mathbb{N}}$ such that $a_i \theta a_{i+1}$, for all $i \in \mathbb{N}$.

The connection between this relation and total termination is given below. First we need an auxiliary result.

Lemma 4.2 *$(\mathcal{F}, \mathcal{X}, R)$ is totally terminating if and only if $(\mathcal{F} \cup \{\bot\}, \mathcal{X}, R)$ is totally terminating, where $\bot$ is a constant not occurring in R.*

Proof For the if part, since $(\mathcal{F} \cup \{\bot\}, \mathcal{X}, R)$ is totally terminating there is a total monotone algebra compatible with $(\mathcal{F} \cup \{\bot\}, \mathcal{X}, R)$. The same algebra is obviously compatible with R.

For the only-if part, we take a total monotone algebra compatible with R and define the interpretation of $\bot$ to be an arbitrary element of the algebra. The interpretations of the other symbols do not change. It follows that this algebra is compatible with $(\mathcal{F} \cup \{\bot\}, \mathcal{X}, R)$, proving its total termination. □

Theorem 4.3 *If a TRS R is totally terminating then $\rhd$ is well-founded.*

[5] If $\mathcal{F}$ is empty, we add a dummy constant to it and assign weight ϕ_0 to that constant.

Proof Suppose R is a totally terminating TRS. By theorem 2.3, $R \cup \mathcal{E}mb_{\mathcal{F}}$ is also totally terminating. Without loss of generality, we can assume that $\mathcal{T}(\mathcal{F}) \neq \emptyset$, since by lemma 4.2 and theorem 2.3, adding a constant to $\mathcal{F}$ does not change the total termination of both R and $R \cup \mathcal{E}mb_{\mathcal{F}}$. By theorem 2.4, there is a strict partial order $>$ over $\mathcal{T}(\mathcal{F})$, total and well-founded, and such that:

- $l\sigma > r\sigma$, for any rule $l \to r \in R \cup \mathcal{E}mb_{\mathcal{F}}$, and any ground substitution σ.
- $>$ is closed under ground contexts.

We will prove that $\triangleright \subseteq >$.Then well-foundedness of the later relation will yield well-foundedness of the former relation. Suppose then that $s \triangleright t$, with $s, t \in \mathcal{T}(\mathcal{F})$ and $s \neq t$. Since $>$ is total on $\mathcal{T}(\mathcal{F})$, we have either $s > t$ or $t > s$. If $t > s$, we will see that $R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{t \to s\}$ is terminating (in fact that it is totally terminating), contradicting $s \triangleright t$. We remark that $>$ has the property $t\sigma > s\sigma$, for any (ground) substitution σ, since being s and t ground terms implies that $t\sigma = t$ and $s\sigma = s$. Consequently we can apply theorem 2.4 on the opposite direction to conclude that $R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{t \to s\}$ is totally terminating. □

The relation $\triangleright$ can be used to prove that a system is not totally terminating, as the next example shows. Consider the TRS

$$\begin{array}{lcl} f(g(x)) & \to & f(f(x)) \\ g(f(x)) & \to & g(g(x)) \end{array}$$

The first rule combined with $f(c) \to g(c)$, where c is an arbitrary constant, gives a non-terminating system, hence $g(c) \triangleright f(c)$. Similarly the second rule combined with $g(c) \to f(c)$ results in a non-terminating system, hence $f(c) \triangleright g(c)$. Consequently $\triangleright$ is not well-founded and the system cannot be totally terminating.

The converse of theorem 4.3 does not hold, even if only constant and unary function symbols are allowed. Let R be:

$$\begin{array}{lcl} f(a) & \to & f(b) \\ g(g(b)) & \to & g(c) \\ f(c) & \to & f(g(a)) \end{array}$$

Suppose R is totally terminating and let $(A, >)$ be a total well-founded monotone algebra compatible with R. The first rule tells us that $[a] > [b]$. Then monotonicity of the algebra operations and compatibility with the rules give us $[g(b)] > [c] > [g(a)] > [g(b)]$, which is a contradiction.

We now give a sketch of the proof of well-foundedness of $\triangleright$. Define the following weight function $\rho : \mathcal{T}(\mathcal{F}) \to \mathbb{N}$ by

- $\rho(a) = \rho(b) = 1$; $\rho(c) = 2$
- $\rho(p(t)) = 1 + \rho(t)$, for any $t \in \mathcal{T}(\mathcal{F})$, $p \in \{f, g\}$.

It is easy to see that for any ground substitution σ and any rule $l \to r$, we have

- $\rho(l\sigma) = \rho(r\sigma)$, if $l \to r \in R$.
- $\rho(l\sigma) > \rho(r\sigma)$, if $l \to r \in \mathcal{E}mb_{\mathcal{F}}$.

Furthermore ρ is closed under ground contexts.

The following fact is also not difficult to prove:

$$s \rhd t \Rightarrow \rho(s) \geq \rho(t)$$

As a consequence $\rhd \setminus =_\rho$ is well-founded, where $=_\rho$ is the equivalence relation generated by ρ, i. e. for any $t, s \in \mathcal{T}(\mathcal{F})$, $t =_\rho s \iff \rho(t) = \rho(s)$.

It is well known that given a ground TRS, if the system is not terminating then it contains a rule $l \to r$ such that r admits an infinite reduction. Using this fact we can derive that (a, b) is the only pair in $\rhd$ of size one and that $(g(b), c), (c, g(a))$ are the only pairs in $\rhd$ of size two involving c. Also $g(a) \not\rhd g(b)$.

We see now that $g(u) \not\rhd f(v)$, for any ground terms u, v such that $\rho(u) = \rho(v)$. Suppose that is not so, i. e. there are terms $u, v \in \mathcal{T}(\mathcal{F})$ with $\rho(u) = \rho(v)$ and $g(u) \rhd f(v)$. This means that the TRS $R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{f(v) \to g(u)\}$ is not terminating. Since for any rule in this TRS and any ground substitution σ we have $\rho(l\sigma) \geq \rho(r\sigma)$, ρ is closed under contexts and $\mathbb{N}$ is well-founded, we can conclude that if this TRS admits an infinite reduction then so does $R_1 = R \cup \{f(v) \to g(u)\}$, and since R_1 is a ground system, at least one rhs of a rewriting rule admits an infinite reduction. With a bit of case analysis it is possible to see that no reduction rule has a rhs leading to an infinite reduction, giving a contradiction.

Suppose then that $\rhd \cap =_\rho$ is not well-founded and take an infinite chain $t_0 \rhd t_1 \rhd \ldots$, such that the size of the chain, given by $n = \rho(t_i) = \rho(t_j)$, for any i, j, is minimal. Since (a, b) is the only pair in $\rhd$ of size one, it must be $n \geq 2$. If $n = 2$ and c occurs in the chain, its occurrence has to follow the pattern $g(b) \rhd c \rhd g(a)$ or $c \rhd g(a)$. But from what we have seen $g(a) \not\rhd t$, for any $t \in \{c, g(b), f(a), f(b)\}$, which are all the possible terms of size two. Therefore the chain stops at $g(a)$ and cannot be infinite. Consequently any infinite chain of size $n \geq 2$ cannot contain c. So the head symbol of t_0 has to be either f or g. If the head symbol never changes then the chain is of the form

$$p(t'_0) \rhd p(t'_1) \rhd \ldots \rhd p(t'_i) \rhd \ldots$$

where $p \in \{f, g\}$. By eliminating the head symbol, we get an infinite chain $(t'_i)_{i \in \mathbb{N}}$ with a strictly smaller size, contradicting the minimality of $(t_i)_{i \in \mathbb{N}}$. So the head symbol has to change infinitely many often and that contradicts the fact that $g(u) \not\rhd f(v)$, for any terms $u, v \in \mathcal{T}(\mathcal{F})$ with the same weight. As a result $\rhd \cap =_\rho$ is well-founded and so is $\rhd$.

Furthermore $\rhd$ is not complete even for string rewriting systems. If we modify slightly the TRS above we can get a string rewriting system R not totally terminating and such that $R \cup \mathcal{E}mb_{\mathcal{F}}$ terminates and is $\rhd$ is well-founded. In fact the following system

$$\begin{array}{lcl} f(h(x)) & \to & f(k(x)) \\ g(g(k(x))) & \to & g(i(x)) \\ f(i(x)) & \to & f(g(h(x))) \end{array}$$

is a string rewriting system in those conditions. For proving termination of $R \cup \mathcal{E}mb_{\mathcal{F}}$ we choose as monotone algebra $A = \mathbb{N} \times (\{0, 1\} \times \mathbb{N})$ with the order $\succ$ defined by

$$(a, (x, n)) \succ (b, (y, m)) \iff (a > b \text{ or } (a = b \text{ and } x = y \text{ and } n > m))$$

where $>$ is the usual order in the natural numbers, and the interpretations

$$\begin{array}{lcll} k_A((a,(x,n))) & = & (a+1,(1,n)) & \text{for } x \in \{0,1\} \\ h_A((a,(x,n))) & = & (a+1,(0,n)) & \text{for } x \in \{0,1\} \\ i_A((a,(x,n))) & = & (a+2,(0,n)) & \text{for } x \in \{0,1\} \\ f_A((a,(x,n))) & = & \begin{cases} (a+1,(0,3n+1)) & \text{if } x=0 \\ (a+1,(0,n)) & \text{otherwise} \end{cases} & \\ g_A((a,(x,n))) & = & \begin{cases} (a+1,(1,n)) & \text{if } x=0 \\ (a+1,(1,2n+1)) & \text{otherwise} \end{cases} & \end{array}$$

It is not difficult to see that these functions are strictly monotone and that for every $\alpha : \mathcal{X} \to A$ and every rule $l \to r \in R \cup \mathcal{E}mb_{\mathcal{F}}$, $[l,\alpha] > [r,\alpha]$. The system cannot be totally terminating since for any possible total interpretation we would have $i_A(a) > g_A(h_A(a)) > g_A(k_A(a)) > i_A(a)$, for any algebra element a. For the well-foundedness of $\rhd$ we proceed as in the previous example (with substantially more case analysis).

The next step was based on the following observation: if $C_0[t] \to_R^+ C_1[u]$ and $C_1[t] \to_R^+ C_0[u]$ then adding $t \to u$ to R still does not affect total termination. These ideas were combined in the following definition.

Definition 4.4 *Given a TRS R we define the relation $\gg \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ as follows: $s \gg t$ if*

- *$s \to_R^+ t$ or $s \to_{\mathcal{E}mb_{\mathcal{F}}}^+ t$*
- *$s = C[a]$ and $t = C[b]$ and $a \gg b$*
- *for some $n > 0$, there are contexts $C_0[\;], \ldots, C_n[\;]$ such that $C_0[\;] = C_n[\;]$ and $C_i[s] \gg C_{i+1}[t]$, for each $0 \le i < n$,*
- *$\exists u \in \mathcal{T}(\mathcal{F}) :\; s \gg u$ and $u \gg t$*

The relation $\gg$ is a bit more elaborate than $\rhd$ but a similar result as theorem 4.3 holds for $\gg$. Again we need some auxiliary results.

Lemma 4.5 *Suppose R is totally terminating and let $(A,>)$ be a total well-founded monotone algebra compatible with R. If $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and $[s,\sigma] \ge [t,\sigma]$, for some $\sigma \in A^{\mathcal{X}}$, then $[C[s],\sigma] \ge [C[t],\sigma]$, for any context $C[\;]$.*

Proof We proceed by induction. The assertion holds for the trivial context $\Box$ by hypothesis. Suppose it also holds for a context $C'[\;]$. Then

$$\begin{array}{lll} [f(\ldots,C'[s],\ldots),\sigma] & = & \text{(by definition of } [\;]) \\ f_A(\ldots,[C'[s],\sigma],\ldots) & \ge & \text{(by IH and monotonicity of } f_A) \\ f_A(\ldots,[C'[t],\sigma],\ldots) & \ge & \text{(by definition of } [\;]) \\ [f(\ldots,C'[t],\ldots),\sigma] & & \end{array}$$

$\Box$

Lemma 4.6 *Let $(A,>)$ be any total well-founded monotone algebra compatible with R. Then $C[s] >_A C[t] \Rightarrow s >_A t$, for any terms s,t and context $C[\;]$, where $>_A$ is the order over terms induced by $(A,>)$. Furthermore if $(A,>)$ is also compatible with $\mathcal{E}mb_{\mathcal{F}}$, then $C[s] >_A s$, for any non-trivial context $C[\;]$ and term s.*

Proof Let then $C[s] >_A C[t]$. We have to see $\forall\sigma \in A^{\mathcal{X}} : [\![s,\sigma]\!] > [\![t,\sigma]\!]$. Suppose $\exists\tau \in A^{\mathcal{X}} : [\![s,\tau]\!] \not> [\![t,\tau]\!]$. Due to the totality of $>$, this means that $[\![s,\tau]\!] \leq [\![t,\tau]\!]$. By lemma 4.5 we have $[\![C[s],\tau]\!] \leq [\![C[t],\tau]\!]$, contradicting $C[s] >_A C[t]$. So $s >_A t$.

Suppose now that A is compatible with $\mathcal{E}mb_{\mathcal{F}}$. Let $C[s] = f(t_1,\ldots,s,\ldots,t_n)$, with s occurring in position i, $1 \leq i \leq n$. Since $f(\ldots,x_i,\ldots) \to x_i$ is a rule in $\mathcal{E}mb_{\mathcal{F}}$, compatibility ensures that $f(\ldots,x_i,\ldots) >_A x_i$. We define a substitution $\tau : \mathcal{X} \to \mathcal{T}(\mathcal{F},\mathcal{X})$ by

$$\tau(x) = \begin{cases} t_j & \text{if } x = x_j, \text{ for some } j \neq i \\ s & \text{if } x = x_j \\ x & \text{otherwise} \end{cases}$$

Since $>_A$ is closed under substitutions, we have $C[s] = f(x_1,\ldots,x_n)\tau >_A \tau(x_i) = s$.

Suppose $C'[s] >_A s$ for some context $C'[\ ]$. Since $>_A$ is closed under contexts, we get $f(t_1,\ldots,C'[s],\ldots,t_n) >_A f(t_1,\ldots,s,\ldots,t_n)$. But $f(t_1,\ldots,s,\ldots,t_n) >_A s$, so transitivity of $>_A$ yields the result. □

Theorem 4.7 *If R is totally terminating then $\gg$ is well-founded.*

Proof Due to lemma 4.2 we can assume without loss of generality that $\mathcal{F}$ contains at least one constant, so $\mathcal{T}(\mathcal{F})$ is not empty. Since R is totally terminating, from theorem 2.3 we know that $R \cup \mathcal{E}mb_{\mathcal{F}}$ is also totally terminating. By theorem 2.4 we know there is a total well-founded order $>$ over $\mathcal{T}(\mathcal{F})$ such that:

- $l\sigma > r\sigma$, for any rule $l \to r \in R \cup \mathcal{E}mb_{\mathcal{F}}$ and any ground substitution σ.
- $>$ is closed under ground contexts.

We will see, by induction on the definition of $\gg$, that $s \gg t \Rightarrow s > t$. Well-foundedness of $>$ will then yield the result. Suppose that $s \gg t$, for some terms s,t.

- If $s \to_R^+ t$ or $s \to_{\mathcal{E}mb_{\mathcal{F}}}^+ t$, since $>$ is compatible with $R \cup \mathcal{E}mb_{\mathcal{F}}$ we have $\to_{R\cup\mathcal{E}mb_{\mathcal{F}}}^+ \subseteq >$ and therefore $s > t$.
- If $s = C[a]$, $t = C[b]$ with $a \gg b$ and $a > b$ (by induction hypothesis) then $s > t$, since $>$ is closed under ground contexts.
- If $s \gg t$ because for some $n > 0$, there are contexts $C_0[\ ],\ldots,C_n[\ ]$ such that $C_0[\ ] = C_n[\ ]$ and for each $0 \leq i < n$, $C_i[s] \gg C_{i+1}[t]$, then by induction hypothesis we have $C_0[s] > C_1[t], C_1[s] > C_2[t]$, etc. . Since $>$ is total either $s > t$ or $t \geq s$. Suppose that $t \geq s$. Using the induction hypothesis, the fact that $>$ is closed under ground contexts and its transitivity, we get

  $$C_0[s] > C_1[t] \geq C_1[s] > C_2[t] > \ldots > C_n[t] \geq C_n[s] = C_0[s]$$

  contradicting well-foundedness of $>$; therefore $s > t$.
- Finally if $\exists u \in \mathcal{T}(\mathcal{F}) : s \gg u$ and $u \gg t$, then also by induction hypothesis $s > u$ and $u > t$ and transitivity of $>$ gives the result.

□

The previous result can be used to show that a TRS is not totally terminating and in particular that it cannot be proven terminating by any $>_{rpo}$ (or $>_{kbo}$). For example let R be:

$$p(f(f(x))) \to q(f(g(x))) \qquad p(g(g(x))) \to q(g(f(x)))$$
$$q(f(f(x))) \to p(f(g(x))) \qquad q(g(g(x))) \to p(g(f(x)))$$

This system (actually $R \cup \mathcal{E}mb_{\mathcal{F}}$) is terminating (in each step the number of redexes decreases) but not totally terminating. Let c be a constant, then from the leftmost rules we get $p(f(f(c))) > q(f(g(c)))$ and $q(f(f(c))) > p(f(g(c)))$ and consequently $f(c) > g(c)$ (with $C_0 = p(f(\Box)) = C_2$ and $C_1 = q(f(\Box))$). Similarly using the rightmost rules we get $g(c) > f(c)$; therefore $>$ is not well-founded and so R cannot be totally terminating.

One can wonder whether the reverse of theorem 4.7 holds. This is not the case. For example one can prove that the system[6]

$$f(0,a) \to f(1,b) \qquad h(1,a) \to h(0,b)$$
$$g(0,b) \to g(1,a) \qquad k(1,b) \to k(0,a)$$

is not totally terminating while $>$ is well-founded. To see this note that $>$ coincides with $\to_{R\cup\mathcal{E}mb_{\mathcal{F}}}$ and $R \cup \mathcal{E}mb_{\mathcal{F}}$ is terminating since in each R-rewriting step the number of redexes decreases and R is length-preserving (for every rule the length of the lhs equals the length of the rhs). It is easy to see that the interpretations of a and b (or 0 and 1) have to be incomparable and so the system is not totally terminating.

It is also interesting to remark that we can prove that the TRS's presented in connection with the relation $\rhd$ can be proven not totally terminating using $>$. For example for the TRS

$$\begin{array}{lcl} f(h(x)) & \to & f(k(x)) \\ g(g(k(x))) & \to & g(i(x)) \\ f(i(x)) & \to & f(g(h(x))) \end{array}$$

given an arbitrary constant c, from the definition and properties of $>$ we can derive $g(k(c)) > i(c) > g(h(c)) \Rightarrow k(c) > h(c)$. From the first rule we get $h(c) > k(c)$, so $>$ is not well-founded.

It is not clear whether the reverse of theorem 4.7 holds for string rewriting systems.

5 A complete characterization

The results presented so far apply to TRS's over finite or infinite signatures. In this section we assume that $\mathcal{F}$ is finite.

As we saw the characterization of section 4 is not complete. One can wonder whether completeness can be obtained by adding purely syntactical rules to definition 4.4. We did not succeed, but closely related we arrived at the following result. As in theorem 2.4 we assume that $\mathcal{T}(\mathcal{F})$ is non-empty (again lemma 4.2 justifies that assumption).

Theorem 5.1 *A TRS R is totally terminating if and only if there exists a strict partial order $\gg$ on $\mathcal{T}(\mathcal{F})$ satisfying*

1. $\to^{+}_{R\cup\mathcal{E}mb_{\mathcal{F}}} \subseteq \gg$.

2. *$\gg$ is closed under ground contexts.*

[6]Due to U. Waldmann.

3. *if for some $n \geq 1$, contexts $D_0[\ ], \dots, D_n[\ ]$ and terms $s_0, \dots, s_{n-1}, t_0, \dots, t_n \in \mathcal{T}(\mathcal{F})$ exist such that $D_0[\ ] = D_n[\ ]$, $t_0 = t_n$ and for each $0 \leq i < n$, $D_i[s_i] \gg D_{i+1}[t_{i+1}]$, then $s_i \gg t_i$, for some $i \in \{0, \dots, n-1\}$.*

Proof For the only if part since R is totally terminating so is $R \cup \mathcal{E}mb_{\mathcal{F}}$ (theorem 2.3). By theorem 2.4, there is a total well-founded order $>$ over $\mathcal{T}(\mathcal{F})$, closed under ground contexts and verifying $l\sigma > r\sigma$, for any rule $l \to r \in R \cup \mathcal{E}mb_{\mathcal{F}}$ and ground substitution σ. Consequently $>$ satisfies conditions (1) and (2). We check that $>$ also satisfies (3).

Suppose that for some $n \geq 1$, there are terms $s_0, \dots, s_{n-1}, t_0, \dots, t_n \in \mathcal{T}(\mathcal{F})$ and contexts $C_0[\ ], \dots, C_n[\ ]$ such that $C_0[\ ] = C_n[\ ]$, $t_0 = t_n$ and for each $0 \leq i < n$, $C_i[s_i] > C_{i+1}[t_{i+1}]$. We have to see that there is an index $i \in \{0, \dots, n-1\}$, such that $s_i > t_i$. Suppose no such index exists, then $\forall j \in \{0, \dots n-1\}$ we have $t_j \geq s_j$, since $>$ is total. From the hypothesis and the fact that $>$ is closed under ground contexts, we get

$$C_0[s_0] > C_1[t_1] \geq C_1[s_1] > C_2[t_2] \geq \dots > C_n[t_n] = C_0[t_0] \geq C_0[s_0]$$

which is a contradiction. Since $>$ fulfils all the conditions of the theorem, the result holds.

For the if part, suppose there is an order $>$ fulfilling conditions (1) – (3). Let Z_R denote the set of all partial orders over $\mathcal{T}(\mathcal{F})$ satisfying those conditions, and which is non empty by hypothesis. We order Z_R by $\subset$, the strict set inclusion and will see that in this poset every chain has an upper bound. Then by Zorn's lemma, Z_R has a maximal element.

Let then $\theta_0 \subset \theta_1 \subset \dots \subset \theta_n \subset \dots$, be a chain in Z_R and let $\Theta = \bigsqcup_{i \in \mathbb{N}} \theta_i$. We shall prove that $\Theta \in Z_R$. Irreflexivity and transitivity of Θ are not difficult to check. It is also easy to check that Θ fulfils conditions (1) and (2).

For condition (3), suppose that for some $n \geq 1$, there exist contexts $C_0[\ ], \dots, C_n[\ ]$ and terms $s_0, \dots, s_{n-1}, t_0, \dots, t_n \in \mathcal{T}(\mathcal{F})$ such that $C_0[\ ] = C_n[\ ]$, $t_0 = t_n$ and for each $0 \leq i < n$, $C_i[s_i] \ \Theta \ C_{i+1}[t_{i+1}]$. We have to see that $s_i \Theta t_i$, for some index $i \in \{0, \dots, n-1\}$. For each pair $(C_i[s_i], C_{i+1}[t_{i+1}]) \in \Theta$ there is an index $k_i \in \mathbb{N}$ such that $(C_i[s_i], C_{i+1}[t_{i+1}]) \in \theta_{k_i}$. Take $k = max\{k_0, \dots, k_{n-1}\}$, then $(C_i[s_i], C_{i+1}[t_{i+1}]) \in \theta_k$, for all $0 \leq i < n$. Since θ_k satisfies condition (3), we conclude that $\exists i \in \{0, \dots, n-1\}$ such that $s_i \theta_k t_i$ and therefore $s_i \Theta t_i$.

We have seen that every chain in Z_R is majorated in Z_R. Since Z_R is not empty, we can apply Zorn's lemma to conclude that Z_R has a maximal element that we denote by Θ_m. The last main step of our proof is to show that Θ_m is a total order over $\mathcal{T}(\mathcal{F})$. We proceed by contradiction. Suppose there are two ground terms $p \neq q$ such that $(p, q), (q, p) \notin \Theta_m$. Consider the relation

$$\Upsilon = (\Theta_m \cup \{(C[p], C[q]) : C[\ ] \text{ is any ground context } \})^+$$

By definition Υ is transitive. For irreflexivity, suppose that there is a ground term a such that $a \Upsilon a$. Then one of the following three cases must hold:

1. $a \Theta_m a$
2. $C[p] = a = C[q]$

3. for some $n \geq 0$, contexts $C_0[\,], \ldots, C_n[\,]$ exist such that $a\Theta_m C_0[p]$, $C_i[q]\Theta_m C_{i+1}[p]$, for $0 \leq i < n$, and $C_n[q]\Theta_m a$

In the first two cases we immediately get a contradiction since Θ_m is irreflexive and $p \neq q$. The last case is an instance of condition (3) with $D_0 = \square$, $s_0 = a = t_0 = t_{n+1}$, $s_i = q$ and $t_i = p$, for all $1 \leq i \leq n$. Since Θ_m satisfies the aforementioned condition, we have that either $a\Theta_m a$ or $q\Theta_m p$, contradicting either irreflexivity of Θ_m or the choice of p and q.

We check that Υ is closed under ground contexts. Suppose that $s\Upsilon t$ for some ground terms s, t, and let $C[\,]$ be any ground context. As for irreflexivity we have to distinguish three cases, namely

1. $s\Theta_m t$
2. $s = D[p]$ and $t = D[q]$, for some ground context $D[\,]$
3. for some $n \geq 0$, contexts $C_0[\,], \ldots, C_n[\,]$ exist such that $s\Theta_m C_0[p]$, $C_i[q]\Theta_m C_{i+1}[p]$, for $0 \leq i < n$, and $C_n[q]\Theta_m t$

In the first case we can conclude that $C[s]\Theta_m C[t]$. In the second case we have $C[s] = C[D[p]]$ and $C[t] = C[D[q]]$. In both cases we conclude that $C[s]\Upsilon C[t]$. For the last case, since Θ_m is closed under ground contexts, we derive

$$C[s]\ \Theta_m\ C[C_0[p]],\ C[C_i[q]]\ \Theta_m\ C[C_{i+1}[p]],\ \text{for } 0 \leq i < n, \text{ and } C[C_n[q]]\ \Theta_m\ C[t]$$

Again we have an instance of condition (3), with $D_0 = \square$, $s_0 = C[s]$, $t_0 = C[t] = t_{n+1}$, and $s_i = q$ and $t_i = p$, for all $1 \leq i \leq n$. Since Θ_m satisfies the condition, either $C[s]\Theta_m C[t]$ or $q\Theta_m p$. In the first case we get the desired result and in the second we have a contradiction.

We finally check that Υ satisfies condition (3). Suppose then that for some $n \geq 1$ there are contexts $F_0[\,], \ldots, F_n[\,]$ and terms $u_0, \ldots, u_{n-1}, v_0, \ldots, v_n$ such that $F_0 = F_n$, $t_0 = t_n$ and $F_i[u_i]\Upsilon F_{i+1}[v_{i+1}]$, for $0 \leq i < n$. We want to conclude that $u_i\Upsilon v_i$, for some $0 \leq i < n$. Fix any $0 \leq i < n$. Then since $F_i[u_i]\ \Upsilon\ F_{i+1}[v_{i+1}]$, there is $k_i \geq 0$ and there are ground contexts $C_1[\,], \ldots, C_{k_i}[\,]$ such that

- $F_i[u_i]\ \Theta_m\ C_1[p]$
- $C_j[q]\ \Theta_m\ C_{j+1}[p]$, for $1 \leq j < k_i$
- $C_{k_i}[q]\ \Theta_m\ F_{i+1}[v_{i+1}]$

Again we have an instance of condition (3) with

- $n = k_i + 1$
- $D_0 = D_n = \square$
- $s_0 = F_i[u_i]$, $t_0 = t_n = F_{i+1}[v_{i+1}]$, and $s_j = q$ and $t_j = p$, for $1 \leq j \leq k_i$

So we conclude that $F_i[u_i]\Theta_m F_{i+1}[v_{i+1}]$ or $q\Theta_m p$. Since the last case gives a contradiction, the first must hold. Given the arbitrariety of i and since Θ_m satisfies condition (3), we conclude that $\exists 0 \leq j < n :\ u_j\Theta_m v_j$, implying $u_j\Upsilon v_j$, as we wanted.

We have seen that $\Upsilon \in Z_R$ and since Υ is strictly bigger that Θ_m, this contradicts the maximality of Θ_m. Therefore Θ_m is total on $\mathcal{T}(\mathcal{F})$. Since Θ_m contains the embedding relation ($\rightarrow_{R\cup\mathcal{E}mb_{\mathcal{F}}} \subseteq \Theta_m$) and $\mathcal{F}$ is finite, by Kruskal's theorem Θ_m is well-founded so we can apply theorem 2.4 to conclude that R is totally terminating. □

Although this result yields completeness, it is not easy to apply for proving that a particular TRS is not totally terminating, in contrast to the result of section 4.

The type of orders described in theorem 5.1 are not necessarily total, but combining this result with theorem 2.4, we see that existence of a total well-founded order compatible with a TRS R is equivalent to the existence of a compatible order of the type described in theorem 5.1, so we can say that this results provides another characterization of totality.

6 Conclusions

In this paper the notion of total termination is treated syntactically in two ways. On the one hand we analyzed how total termination covers precedence based orderings like recursive path order. Surprisingly this led to a slight generalization of versions of recursive path order as they appeared in the literature and to a new proof of well-foundedness. Only after this generalization could we prove total termination.

On the other hand we tried to find a syntactical characterization of total termination of the following shape: if a TRS is totally terminating then some syntactically defined relation is well-founded. This led to a method of proving non-total termination: if the constructed relation admits an infinite descending chain then the TRS is not totally terminating. The converse is not true: we constructed TRS's for which the constructed relations are well-founded while the TRS's are not totally terminating. Finally we found an "if and only if"-characterization of total termination covering the previous constructions. However, this characterization is not of practical use to determine whether a given TRS is totally terminating or not.

References

[1] BEN-CHERIFA, A., AND LESCANNE, P. Termination of rewriting systems by polynomial interpretations and its implementation. *Science of Computing Programming 9*, 2 (1987), 137–159.

[2] DERSHOWITZ, N. Termination of rewriting. *Journal of Symbolic Computation 3*, 1 and 2 (1987), 69–116.

[3] DERSHOWITZ, N., AND JOUANNAUD, J.-P. Rewrite systems. In *Handbook of Theoretical Computer Science*, J. van Leeuwen, Ed., vol. B. Elsevier, 1990, ch. 6, pp. 243–320.

[4] DERSHOWITZ, N., AND MANNA, Z. Proving termination with multiset orderings. *Communications ACM 22*, 8 (1979), 465–476.

[5] FERREIRA, M. C. F., AND ZANTEMA, H. Total termination of term rewriting. In *Proceedings of the 5th Conference on Rewriting Techniques and Applications* (1993), C. Kirchner, Ed., vol. 690 of *Lecture Notes in Computer Science*, Springer, pp. 213–227. Full version submitted for publication.

[6] FERREIRA, M. C. F., AND ZANTEMA, H. Well-foundedness of term orderings. To appear at CTRS 94 (Workshop on Conditional and Typed Term Rewriting Systems).