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Abstract

An l-ruler is a chain of n links, each of length l. The links, which are allowed to cross, are modelled by line segments whose endpoints act as joints. A given configuration of an l-ruler is said to fold if it can be moved to a configuration in which all its links coincide. We show that l-rulers confined inside an equilateral triangle of side 1 exhibit the following surprising alternation property: There exist three values $x_1 \approx 0.483$, $x_2 = 0.5$ and $x_3 \approx 0.866$ such that all configurations of n-link l-rulers fold if $l \in [0, x_1]$ or $l \in (x_2, x_3]$, but for any $l \in (x_1, x_2]$ and any $l \in (x_3, 1]$, there exist configurations of l-rulers that cannot fold. In the folding cases, linear-time algorithms are given that achieve the folding. Also, a general proof technique is given that can show that certain configurations—in the non-folding cases—cannot fold.

1 Introduction

A linkage is a collection of rigid rods or links that are fastened together at their endpoints, about which they may rotate freely. Links may cross over one another. A ruler is a chain of links, i.e., any endpoint is fastened to at most one other endpoint, and two links have an endpoint that is not fastened to any other endpoint.

Several papers have been written on reconfiguration problems for linkages or rulers from a geometric point of view, including a survey [9]. Hopcroft, Joseph and Whitesides [1] proved that reconfiguration of a linkage so that a designated joint reaches a given position is PSPACE-hard. Joseph and Plantinga [3] proved a similar result for moving rulers amidst obstacles. Hopcroft *et al.* [2] proved that folding a ruler to a segment with at most a specified length is an NP-complete problem, but gave a polynomial-time algorithm for reconfiguring a ruler—of which one point is pinned down to the plane—inside a circle. The running time was improved by Kantabutra and Kosaraju [5]. Kantabutra [4] studied rulers inside

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a square, with one end fixed and all links of length at most half the side length of the square. He gave a linear-time reconfiguration algorithm. Lenhart and Whitesides [6, 7, 8] studied the reconfiguration of simple closed chains of links in d dimensions and gave a linear-time reconfiguration algorithm.

We consider a reconfiguration problem for rulers that have all links of equal length and that are confined to an equilateral triangle with unit edge length. The objective is to fold the ruler onto a single link so that all links coincide. This problem is of interest because a confining region having acute angles presents difficulties that have not been studied previously. Also, our results give an additional example of a motion planning problem that can be solved in linear time despite $n+2$ degrees of freedom.

We call a ruler whose links all have equal length l an l-ruler, and we scale the side of the confining triangle to have length 1. Of course there are l-rulers for l close to 1 that cannot be folded onto a single link, and it is not surprising that for sufficiently small values of l, all l-rulers fold. However, we have discovered the following surprising phenomenon. For any n and any link length l in the range $[0, x_1]$ with $x_1 \approx 0.483$, any configuration of an n-link l-ruler folds. For $n \geq 3$ and l in the range $(x_1, x_2]$, where $x_2 = 0.5$, there are configurations of n-link l-rulers that do not fold. For any n and l in the range $(x_2, x_3]$, where $x_3 = \sqrt{3}/2 \approx 0.866$, any configuration of an n-link l-ruler folds. For $n \geq 2$ and l in the range $(x_3, 1]$, there are configurations of n-link l-rulers that do not fold. In the cases where the ruler can always be folded, we give linear-time algorithms that accomplish this. In the cases where not every ruler can be folded, we give a configuration that cannot be folded and prove this.

The remainder of this paper is organized as follows. In Section 2 some notation is introduced, and also simple motions of the ruler. In Section 3, we give a linear-time algorithm to fold l-rulers for $l \in (x_2, x_3]$. Section 4 presents a linear-time algorithm to fold l-rulers for $l \in [0, \frac{1}{3}]$. (The appendix contains the long and highly technical linear-time algorithm for $l \in (\frac{1}{3}, x_1]$.) Non-foldability of rulers is studied in Section 5. The conclusions are given in Section 6.

2 Preliminaries

We denote the links of an n-link l-ruler by $\ell_1, \ldots, \ell_n$, where link ℓ_i has endpoints j_{i-1} and j_i. The angle at j_i is the angle between links ℓ_{i-1} and ℓ_i; the angle at j_0 is the angle ℓ_1 makes with the positive x-axis. A joint j_i is open if the angle is π radians; a joint is closed if the angle is 0 radians.

We denote the unit-side triangle in which l-rulers are confined by Δ, which we visualize as having a horizontal base vw and a top vertex u. Links and joints may lie on the boundary of Δ.

For a joint j_i, we denote with C_i the circle with radius l centered at j_i. This circle may have one, two or three connected components inside Δ, depending on the position of j_i and the value of l.

Algorithms for the reconfiguration of a ruler usually break up the motions for the whole reconfiguration into simple motions, in which only a few joints are used simultaneously. A minimal requirement for a simple motion is that it can be described in constant time [7]. We allow the following type of simple motions for rulers:

- Some joint j_i of the ruler does not change its position, and at most a constant number of angles at joints between a pair of adjacent links change simultaneously.
- No angles at joints change, but the ruler may translate and rotate as a rigid object.

Note that the joints that change angle can be far apart in the ruler. A *dragging motion at joint* j_i is a motion in which the positions of joints j_{i+2} through j_n remain fixed, links ℓ_{i+1} and ℓ_{i+2} act as an elbow to move j_i along some specified line, and j_i drags the first i links so that they translate in the same direction as j_i.

3 Folding rulers with moderately long links

We will show that any configuration of an n-link l-ruler with $l \in (x_2, x_3]$ can be folded, where $x_2 = 0.5$ and $x_3 = \sqrt{3}/2 \approx 0.866$. The bounds are tight, that is, Section 5 shows that there exist configurations of a ruler with $l = 0.5$ that cannot be folded, and the same holds for any $l > \sqrt{3}/2$.

The algorithm to fold an l-ruler with $l \in (x_2, x_3]$ has three phases. The first phase labels all joints in some appropriate way. The second phase brings an arbitrary configuration into one where the joints lie at the vertices of an equilateral triangle inside Δ. The positions correspond to the labels given to the joints. The third phase turns the triangle into a segment.

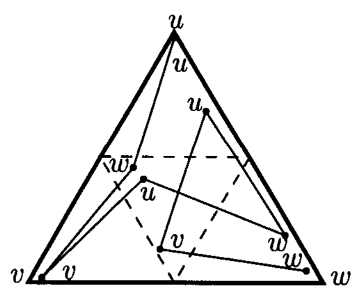

Figure 1: Labeling the joints of moderately long links.

Divide Δ into 4 equal-sized equilateral triangles by connecting the midpoints of the sides of Δ (see Figure 1). Let every joint in the triangle adjacent to u be labeled u, and similarly with v and w. It remains to label the joints in the middle triangle. For any such joint j_i we choose a label that is different from the labels of j_{i-1} and j_{i+1}. If j_{i-1} and j_{i+1} have the same label, say, u, then we assign j_i a label depending on the direction of the link $j_i j_{i+1}$. If its angle with uv is at most $\pi/2$, then j_i is labeled v, otherwise j_i is labeled w.

Lemma 1 *The labeling defined above has the property that joints incident to the same link have different labels.*

Proof: Since $l > 0.5$, no two joints incident to the same link can be in the same one of the four smaller triangles. By choice, the joints in the middle triangle have a label different from the adjacent ones. ∎

To start up the second phase of the algorithm, we define a triangle Δ' with vertices u', v' and w' and side length l. The sides $u'v'$, $u'w'$ and $v'w'$ are parallel to uv, uw and vw, respectively, and remain that way.

Assume without loss of generality that j_0 is labeled u and j_1 is labeled v. Rotate j_0 counterclockwise around j_1 until it hits uv. We claim that if Δ' is positioned such that u' and j_0 coincide, then Δ' lies inside Δ. This is easy to see, because the link $j_1 j_0$ makes an angle between $\pi/3$ and $2\pi/3$ with $u'v'$ (and uv). We say that j_0 can support Δ' (at u'). More generally, for any link $j_i j_{i+1}$ labeled uv, either j_i can support Δ' (at u'), or j_{i+1} can support Δ' (at v'), or both.

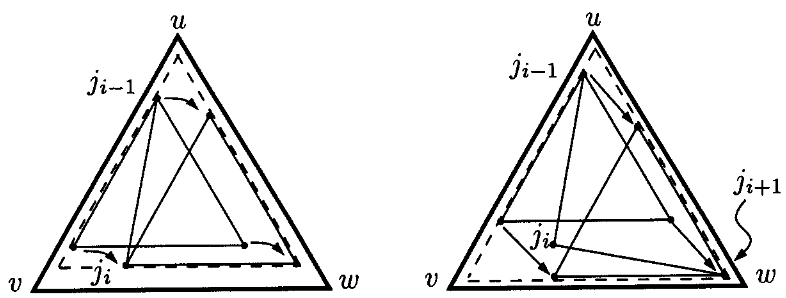

Figure 2: The motion of Δ' stays inside the dashed triangle and thus inside Δ.

By translating Δ' inside Δ, we wrap the ruler onto Δ', such that any joint with label u will be at u'. Assume that we have placed all joints up to j_{i-1} on the vertices of Δ'. Assume without loss of generality that j_{i-1} coincides with u' and j_i has label v. We maintain the invariant that joints $j_i, \ldots, j_n$ have not changed position yet.

First, assume that j_i can support Δ' (see Figure 2, left). Then, by changing the angles at joints j_{i-1} and j_i we let j_i support Δ' at v'. Since the initial and final positions of Δ' lie inside Δ, the circular motions described by the vertices of Δ' are inside Δ. In the figure, Δ' stays inside the dashed triangle.

On the other hand, assume that j_i cannot support Δ'. Then, by the above observations, j_{i+1} can support Δ' (see Figure 2, right). If j_{i+1} has label w, then Δ' can simply be dragged to its new position where j_{i+1} and w' coincide. The motion causes j_i and v' to coincide as well. Next, assume that j_{i+1} is labeled u.

Recall that since j_i is labeled v, the angle which $j_i j_{i+1}$ makes with uv is at most $\pi/2$. Rotate j_{i-1} around j_i until j_{i-1} and j_{i+1} coincide. Then rotate j_i around $j_{i-1} = j_{i+1}$ until it coincides with v'.

Theorem 2 *Any configuration of an n-link l-ruler with $l \in (x_2, x_3]$ can be folded in linear time, changing at most three joints simultaneously.*

4 Folding rulers with short links

The folding of short n-link l-rulers is split into two algorithms—one deals with $l \in [0, \frac{1}{3}]$ and the other with $l \in (\frac{1}{3}, x_1]$. The latter algorithm is long and technical; it can be found in the appendix. We advise the reader not to start with that algorithm before finishing the rest of the paper. This section proves only that l-rulers with $l \in [0, \frac{1}{3}]$ can be folded using a linear number of simple motions. The algorithm attempts to fold the first two links, and then solve the remaining problem on an $(n-1)$-link ruler inductively. Alternatively, it can try to fold links ℓ_2, ℓ_3 and ℓ_4, which leaves a folding problem for an $(n-2)$-link ruler. We show that one of these attempts succeeds without moving $j_5, \ldots, j_n$ from their positions.

We begin with a simple observation, and then put j_2 on the boundary of Δ.

Lemma 3 *If C_1 has j_0 and j_2 on the same component inside Δ, then ℓ_1 and ℓ_2 can be folded without changing the position of j_1.*

Proof: Simply rotate j_0 around j_1 onto j_2. ∎

Lemma 4 *Without changing the position of j_3, links ℓ_1 and ℓ_2 can be folded, or joints j_1 and j_2 can be put against a side of Δ.*

Proof: Translate j_0 toward j_2. If j_0 reaches j_2, then ℓ_1 and ℓ_2 are folded, otherwise, j_1 has hit a side of Δ. Assume without loss of generality that j_1 has hit vw, and that j_1 is closer to v. Move j_1 leftward along vw towards the middle, with $j_3 j_2 j_1$ acting as an elbow; note that j_0 cannot hit any side of Δ during this motion. If j_1 reaches the middle, then j_0 can be rotated onto j_2 because C_1 has only one component inside Δ. Otherwise, j_2 has hit the side of Δ, or j_2 is open and the angle $v j_1 j_3$ is at most $\pi/2$ radians. But then j_1 is at least at distance $2l/\sqrt{3}$ from v, and C_1 has only one component inside Δ. ∎

Define the *u-triangle* as the equilateral triangle inside Δ with a vertex at u and with side length $l/\sqrt{3}$. Define the v-triangle and the w-triangle similarly. We continue in one of two ways, depending on whether j_2 is in a u-, v- or w-triangle, or outside all of them.

Lemma 5 *If j_1 and j_2 are on sides of Δ, and j_2 is outside the u-, v- and w-triangle, then ℓ_1 and ℓ_2 can be folded without changing the position of j_2.*

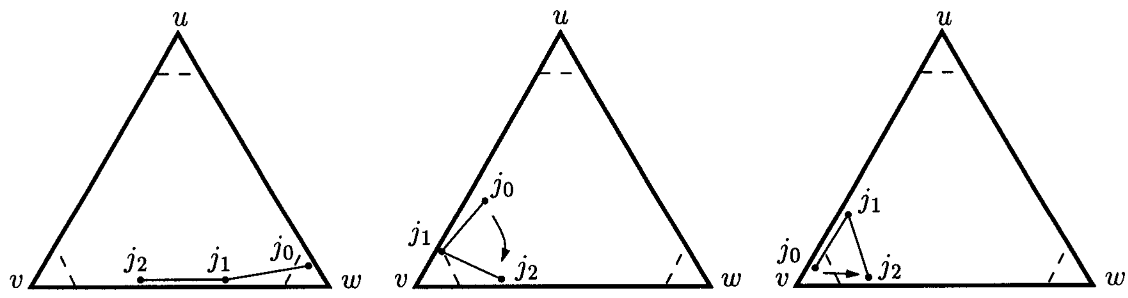

Figure 3: Three cases of folding ℓ_1 and ℓ_2 when j_2 is on vw, closer to v, and outside the v-triangle. In the leftmost case, j_0 is rotated clockwise onto vw and then dragged to j_2.

Proof: Assume without loss of generality that j_2 is on the side vw, and closer to v than to w (see Figure 3). If j_1 is on vw, then either j_0 can be rotated onto j_2 directly, or j_0 can be rotated against vw and then dragged towards j_2.

If j_1 is against uv and below the perpendicular to uv through j_2, then j_0 can be rotated around j_1 onto j_2 because C_1 has only one component inside Δ. If j_1 is against uv and above the perpendicular to uv, then the link $j_1 j_2$ divides Δ into two parts. If j_0 is in the triangle $j_1 j_2 v$, then j_0 can be translated onto j_2. If j_0 is in the quadrilateral part, then j_0 can be rotated onto j_2. ■

If the above method fails to fold ℓ_1 and ℓ_2, then we will drag j_2 and possibly also j_3 and j_4. First, we wish to not worry about the first two links hitting sides as long as j_2 is in the v-triangle. To this end, we make the links ℓ_1 and ℓ_2 parallel to vw with joint j_1 open, and we keep these links this way until specified otherwise. Note that j_1 and j_0 cannot hit any side (in particular, uw) unless j_2 leaves the v-triangle.

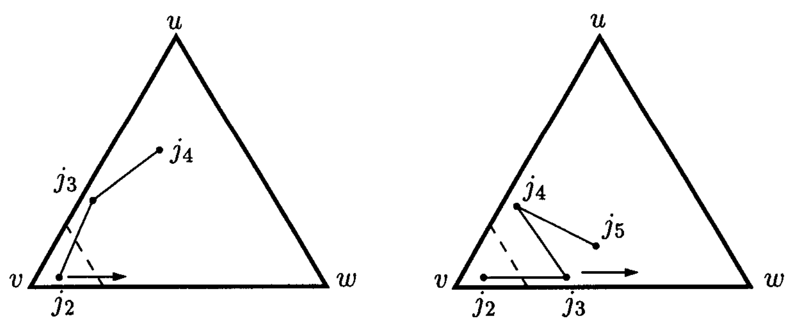

Figure 4: Left: putting j_3 on a side, or getting j_2 outside the v-triangle. Right: getting j_2 outside the v-triangle by dragging j_3.

Lemma 6 *If j_2 is in the v-triangle and on vw, then j_2 can be put outside the v-triangle, or j_2 and j_3 can be put on the same side of Δ, without changing the* *position of* j_4.

Proof: Drag j_2 along vw toward w, keeping j_4's position fixed (see Figure 4). If j_2 does not get out of the v-triangle, then j_3 has hit uv or vw. If j_3 is on uv, then rotate j_2 around j_3 to that side as well. ∎

If j_2 is put outside the v-triangle, then ℓ_1 and ℓ_2 can be folded according to Lemma 5. Otherwise, assume without loss of generality that j_2 and j_3 are both on vw.

Lemma 7 *If* j_2 *is in the* v*-triangle on side* vw *and* j_3 *is on side* vw*, then* j_2 *can be moved outside the* v*-triangle, or* ℓ_2*,* ℓ_3 *and* ℓ_4 *can be folded, without changing the position of* j_5.

Proof: If j_2 can be rotated against uv outside the v-triangle, then we are done. Otherwise, if j_4 is on the same component of C_3 as j_2, then rotate j_2 onto j_4, and then rotate j_3 around $j_2 = j_4$ onto j_1. Otherwise, drag j_3 along vw toward w, with $j_5 j_4 j_3$ acting as an elbow (see Figure 4). If j_2 does not leave the v-triangle, then j_4 must have hit a side of Δ. This side cannot be uw, since the distance from the v-triangle to the side uw is greater than $2l$. If the side is vw and joint j_3 is open, then j_2 can be dragged toward j_4 (and w), with j_3 leaving vw. This will bring j_2 outside the v-triangle. If the side is vw and joint j_3 is closed, then ℓ_3 and ℓ_4 coincide, and we can make ℓ_2 to coincide with these links as well by rotating j_3 around $j_2 = j_4$. If the side hit by j_4 is uv, then drag j_2 toward w with j_3 leaving the side vw, and j_2 will leave the v-triangle. This is possible since the angle $\angle j_2 j_3 j_4$ is between $\pi/6$ and $\pi/3$ radians in this case. ∎

Theorem 8 *Any configuration of an* n*-link* l*-ruler with* $l \in [0, \frac{1}{3}]$ *can be folded in linear time, changing at most three joints simultaneously.*

Proof: The lemmas above show that with only a constant number of simple motions, either ℓ_1 and ℓ_2 can be folded, or ℓ_2, ℓ_3 and ℓ_4 can be folded. Thus the problem reduces to an $(n-1)$-link or $(n-2)$-link l-ruler. The theorem follows by induction. The base cases are easy (observe for instance that imaginary links can be added to one end to reduce the number of cases). ∎

5 Non-foldable rulers

It will be shown that not every configuration of an l-ruler is foldable if $l \in (x_3, 1]$ where $x_3 = \sqrt{3}/2 \approx 0.866$, or if $l \in (x_1, x_2]$ where $x_1 \approx 0.48348$ and $x_2 = 0.5$. One can make a distinction between two types of non-foldability. It may be that the ruler is rigidly stuck, or it may be that small motions are possible, but not enough to fold it. Besides giving examples of stuck rulers, we also provide a proof technique to show that a ruler is stuck.

The first example of a rigidly stuck ruler consists of two links of length 1, one coinciding with the edge uv of Δ, and the other coinciding with vw. It is easy to see that this configuration cannot be folded, and that it is rigidly stuck. Next, assume that the link length is less than 1, joint j_1 coincides with v, link ℓ_1 lies on the edge uv and link ℓ_2 lies on the edge vw. This configuration is not rigidly stuck. However, if $l > x_3 = \sqrt{3}/2$, then joint j_0 cannot rotate past the bisector of v to reach the joint j_2. Nor can j_2 reach j_0. The illustrated configuration is non-foldable.

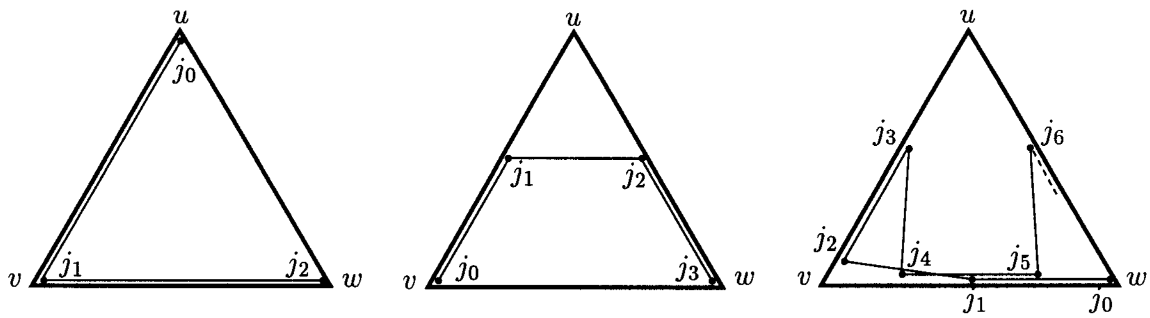

Figure 5: Three rulers that are rigidly stuck.

The second example of a rigidly stuck ruler consists of three links of length 0.5. Joint j_0 coincides with v, joint j_1 coincides with the midpoint of uv, joint j_2 coincides with the midpoint of uw and j_3 coincides with w. As in the previous example, one can decrease the link length slightly and start with roughly the same configuration, and obtain a non-foldable ruler that is not rigidly stuck. We prove that this example provides a non-foldable ruler when $l \in (x_1, x_2]$ where $x_1 \approx 0.48348$ and $x_2 = 0.5$, by using a proof technique which we explain after the third example.

The third example of a rigidly stuck ruler has nine links of length ≈ 0.483576. Joint j_0 coincides with w, joint j_1 lies on the side vw, joint j_2 lies on uv, joint j_3 also lies on uv, joint j_4 lies on vw and of the two possibilities, closest to v. Joints $j_9, \ldots, j_5$ are the mirror images of $j_0, \ldots, j_4$ when reflected in the bisector at u.

To prove that a configuration of a ruler is stuck, we define the *state* of a configuration, which is a discretization of it. We use the states to show that a given configuration cannot change to a different state. We study the possible *state transitions* for any configuration, and show that none can take place first. A state of a configuration consists of the following items (see Figure 6):

1. For any joint j and incident link ℓ, draw from the joint j the perpendiculars to the three edges of the triangle Δ. The link ℓ can be in any of the three sectors centered at j, which define one item of the state of the ruler. We denote the sectors as u-sector, v-sector and w-sector. The boundaries of the sectors are assigned arbitrarily to one of the incident sectors.
2. For three consecutive joints j_{i-1}, j_i and j_{i+1}, the sidedness of the triangle $j_{i-1}j_ij_{i+1}$ (a left turn or a right turn) is an item of the state. If joint j_i is

open or closed, then one of the possible item instances is assigned arbitrarily.

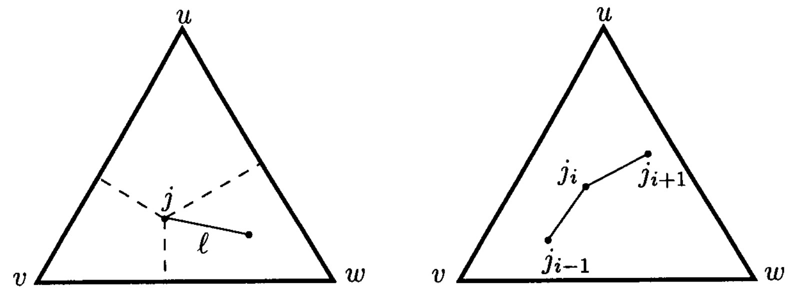

Figure 6: Left: ℓ is in the w-sector of j. Right: $j_{i-1}j_ij_{i+1}$ make a right turn.

It follows that any configuration of an n-link ruler with at least two links has $3n-1$ items in its state. There are two possible state transitions for a configuration of a ruler, for which the following states are critical (in other words, when an item is about to change):

1. A link ℓ makes an angle of $\pi/2$ radians with one of the edges of Δ.
2. Three consecutive joints are colinear (the middle joint is open or closed).

If two consecutive links, both incident to some joint j, are in the same sector, then one need not test whether the three joints incident to these links are colinear with j open. For this to happen, one of the links must leave the sector first. Similarly, if two consecutive links, both incident to some joint j, are in different sectors, then one need not test whether the three joints incident to these links are colinear with j closed.

There are more choices of defining states, which lead to different transitions and different critical states. Whichever choice is made, the following is sufficient for a proof that a configuration of a ruler is non-foldable. It is necessary that the initial and final configurations be in separate connected components. It is sufficient that the initial configuration be in an isolated vertex of the state graph that is different from the final configuration. Following this approach, we show that the configurations of the rulers of the first and second examples are non-foldable for the appropriate link lengths.

Lemma 9 *For each $l \in (x_3, 1]$, there exists a configuration of an l-ruler that cannot be folded (where $x_3 = \sqrt{3}/2$).*

Proof: Consider the configuration of example 1. In a folded configuration of this ruler, links ℓ_1 and ℓ_2 are in the same state with respect to joint j_1. For the initial configuration of example 1, this is not the case. We consider which critical state can occur as the first one (possibly, simultaneously with others). Consider the state of joint j_1 and link ℓ_1. The link ℓ_1 is in the u-sector with respect to j_1. If ℓ_1 were to change its state to be in the w-sector, then ℓ_1 must make an angle of $\pi/2$ radians with the edge uw, but this is impossible, before

Δ cannot contain a link with the given link length perpendicular to any of its sides. The other transitions of the first type cannot occur for the same reasons. A transition of the second type can occur in one of two forms. The joint j_1 can be open, i.e., j_0 and j_2 are distance $2l$ apart, or the joint j_2 can be closed, i.e., j_0 and j_2 coincide. Clearly, Δ cannot contain a configuration of this ruler with j_1 open. Also, j_1 cannot close before another state transition occurs before or simultaneously, because when j_1 closes the links ℓ_1 and ℓ_2 are in the same state with respect to j_1. ∎

Lemma 10 *For each $l \in (x_1, x_2]$, there exists a configuration of an l-ruler that cannot be folded (where $x_1 \approx 0.483$ and $x_2 = 0.5$).*

Proof: Consider the configuration of example 2. In a folded configuration of this ruler, links ℓ_1 and ℓ_2 are in the same state with respect to joint j_1. For the initial configuration of example 2, this is not the case. We consider which critical state can occur as the first one (possibly, simultaneously with others). Consider link ℓ_3, which is in the w-sector with respect to joint j_2. Assume that the first state transition brings ℓ_3 in the v-sector. Then j_2 must lie at least a distance l above the edge vw in the critical state. Since link ℓ_2 is in the v-sector with respect to j_2, link ℓ_1 is in the v-sector with respect to j_1, and j_0, j_1, j_2 make a right turn, the ruler in this critical configuration only fits inside Δ if $l \leq x_1$. Next, assume that the first state transition brings ℓ_3 in the u-sector with respect to j_2. This state transition can never occur as the first, since the state of ℓ_3 with respect to j_3 will always change before. The other possible state transitions of this type can be handled similarly.

Consider joints j_0, j_1, j_2, which make a right turn, and assume that the first state transition brings this into a left turn. Since ℓ_1 and ℓ_2 are in different sectors with respect to j_1, joint j_1 cannot close without having another state transition before or simultaneously. Furthermore, ℓ_1 is in the u-sector of j_0 and ℓ_1 is in the w-sector of j_1. If joint j_1 is open, these sectors must be the same. Therefore, another state transition must occur before or simultaneously. Hence, we need not consider state changes for three consecutive joints as the first state change. ∎

6 Conclusions

We have studied folding an n-link ruler with equal link lengths inside an equilateral triangle. This paper gives one of the first results on the reconfiguration of rulers when there are acute angles that constrain the motion of the ruler. Even in the simple setting of this paper, a surprising result shows up: rulers with short links can always be folded, rulers with midsize links cannot always be folded, rulers with fairly long links can always be folded, and rulers with long links cannot always be folded. We showed these results using techniques that can be used in other ruler-folding situations as well.

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A Folding l-rulers for $1/3 < l \leq x_1 \approx 0.4834$

In Section 3 we folded moderately long l-rulers onto an equilateral triangle with side length l and then folded this triangle. In this appendix we fold l-rulers for $0 < l \leq x_1 \approx 0.4834$ onto a *trellis*. Then we fold the trellis to a triangle and fold the triangle. We prove in the reverse order that these three foldings are possible.

A *trellis* is composed of four equilateral triangles of side length l—three *corner* triangles homothetic to Δ and one upside-down *center* triangle, as in Figure 7. If we translate a trellis in Δ, keeping sides parallel, then the six vertices of the trellis sweep out six equilateral *frame triangles*, also shown in Figure 7. These are called the u, v and w frame triangles for the corners, and the uv, uw and vw frame triangles for the others.

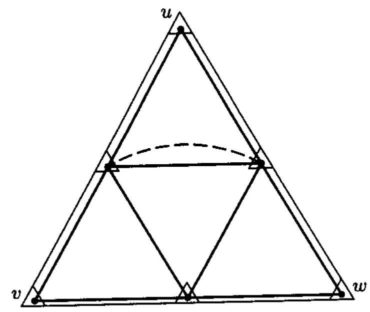

Figure 7: The trellis and frame.

Recall that C_i denotes the circle with radius l that is centered at joint j_i, and A_i denotes the set of circular arcs that are the connected components of $C_i \cap \Delta$. The *vw-fence* is the line segment that is the intersection of Δ with a line parallel to $\overline{vw}$ at distance l. We say that a joint j_i is *above the vw-fence* if the disk inside circle C_i does not intersect the line $\overline{vw}$. Define the uv-fence and uw-fence similarly.

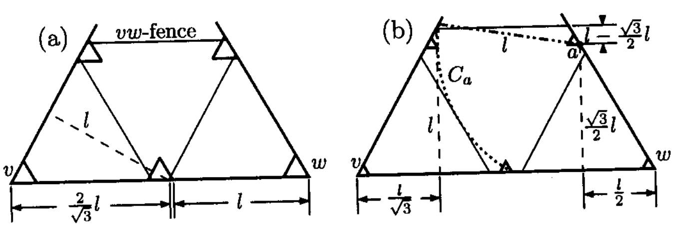

Figure 8: The relationship between fences and frame depends on l.

There are critical values for l that determine the relationship between middle frame triangles and fences. We assume throughout this appendix that $1/3 < l \leq x_1$, which is the larger critical value.

Lemma 11 *If* $l \leq \sqrt{3}/(2+\sqrt{3}) \approx 0.4641$ *then any point of* $\overline{vw}$ *is above the uv- or uw-fence or is in the vw frame triangle.*

Let a be the corner of the uw frame triangle nearest w. If $l \leq 0.4834$ *then the circle* C_a *intersects* $\overline{uv}$ *above the vw-fence.*

Proof: Figure 8(a) illustrates the first part of the lemma: the fences touch the middle frame triangles when $l \leq 1 - 2l/\sqrt{3}$.

Figure 8(b) illustrates the second part: the lemma is satisfied if l is at most the distance between the lower right corner of the uw frame triangle and the left end of the vw-fence. That is, if

$$l^2 \leq (l - (\sqrt{3}/2)l)^2 + (1 - l/\sqrt{3} - l/2)^2.$$

■

A.1 Folding a triangle onto a link, folding a trellis onto a triangle

To begin, we prove that any prefix of links that lie on edges of the center triangle in a trellis can be folded in place to a single link.

Lemma 12 *Let $\ell_1, \ell_2, \ldots, \ell_k$ be a ruler on the center triangle τ of a trellis. Then the ruler can be folded onto ℓ_k inside the trellis.*

Proof: The circular sector formed by pivoting link ℓ_1 about joint j_1 onto link ℓ_2 is entirely within the trellis. By induction, we can therefore fold all links onto ℓ_k. ∎

We can now fold a trellis. To reduce the number of cases, we always fold the ruler onto a triangle τ that has one vertex in the corner of the trellis—if we ever put τ in the center of the trellis, then Lemma 12 says that we can fold the links on τ to a single link and take a new triangle τ that is incident to this link and a corner of the trellis.

Lemma 13 *A ruler on a trellis can be folded to a single segment if $l \leq x_1 \approx 0.4834$*

Proof: As an induction hypothesis, suppose that all links from ℓ_1 to ℓ_i, for some $i \geq 1$, lie on τ, which is a corner triangle of the trellis. This is easy to obtain in the base case: link ℓ_1, being on the trellis, is an edge of a unique corner triangle that can be chosen as τ. If the next link ℓ_{i+1} is already on τ then nothing needs to be done. Otherwise, we have three cases depicted in Figure 9 for folding ℓ_{i+1} onto τ, which depend on the locations of joints j_{i+1} and j_{i-1}.

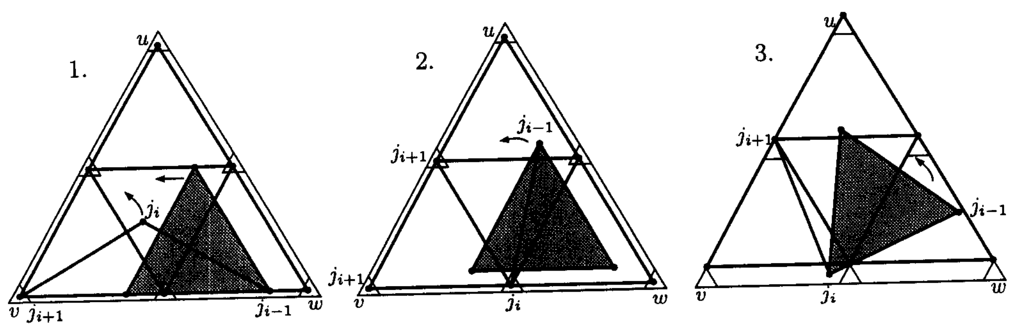

Figure 9: Cases for folding the trellis.

Case 1: Joints j_{i+1} and j_{i-1} are in corners of the trellis. Then j_i is at the side between j_{i-1} and j_{i+1}. Translate τ, moving j_{i-1} towards j_{i+1} and j_i away from the side of the trellis until τ is again a corner triangle of the trellis and has vertices j_{i-1}, j_i, and j_{i+1}.

Case 2: Joints j_i and j_{i-1} lie at the sides of the trellis; joint j_{i+1} lies at the side or corner. Rotate j_i to bring j_{i-1} to the side near j_{i+1} while rotating j_{i-1} to keep τ homothetic to the corner triangles. This also makes τ a corner triangle of the trellis having vertices j_{i-1}, j_i, and j_{i+1}.

Case 3: Joint j_{i+1} is on a side. The triangle τ must touch the opposite corner of the trellis or else joint j_{i+1} and link ℓ_{i+1} are already on τ. This is the most complicated case—it cannot be folded inside the trellis, but can be folded inside a unit equilateral triangle if $l \le x_1$. To prove this, let us be more specific about the locations of the trellis and the joints.

Let joint j_{i+1} be at the side $\overline{uv}$ of the triangle Δ, and j_i at the side $\overline{vw}$, and τ near w. We translate the trellis so that one of its vertices coincides with u. Next, we pivot $j_i$about j_{i+1}, keeping τ homothetic to Δ, until j_i moves above the uw-fence. Since $l \le x_1$, the triangle τ can then swing freely on C_i to hit $\overline{uv}$ at j_{i+1}. Next, rotate about j_{i+1} to bring j_i back onto the trellis, making τ the center triangle. Finally, fold τ to a segment according to Lemma 12 and choose a new τ incident to this segment and a corner of the trellis. This completes case 3.

At the completion of these cases, we have all the links folded onto a corner triangle. We can move this triangle to the center and fold it according to Lemma 12. ■

A.2 An analysis of two-link rulers

In this section we study the motion of a two link ruler when one end is dragged along the side of the triangle Δ. This dragging motion will be the primary tool in the next and final section, which folds a ruler onto the trellis.

These sections become increasingly complex. It may help to consider why this is so. A k link ruler has $k+2$ degrees of freedom—it can be described by the position of the first joint and angles for each successive link. Thus, a ruler placed in the triangle is a point in a $(k+2)$-dimensional *configuration space* consisting of all possible placements. Showing that every ruler can be folded to a particular segment implies that this configuration space has a single connected component.

To prove directly that a $(k+2)$-dimensional set is connected can be daunting, even when $k = 2$. Thus, we look at configurations where joints are on the sides of Δ. With a two link ruler abc, for example, we place c on a side and drag it, pivotting on a, until b hits a side (or joints go onto a trellis). This reduces the problem to three and then to two degrees of freedom—the placement of a (which we draw in Figure 11). Thus, by proving lemmas about these contact configurations, we avoid having to look at the entire configuration space.

Consider a ruler consisting of two segments $\overline{ab}$ and $\overline{bc}$, where c is along the $\overline{vw}$ side of Δ. Let $\overline{vw}$ be horizontal with w on the right. Let's say that a *wall* is any portion of an edge of Δ that is not contained in a frame triangle. In the next lemmas, we investigate how b can hit a wall when we drag c along $\overline{vw}$. Figure 10 illustrates these different cases.

Lemma 14 *Given a ruler abc with c on $\overline{vw}$. If we fix the location of a and drag c toward w, then one of the following occurs.*

1. *Joint c or b reaches a frame triangle.*

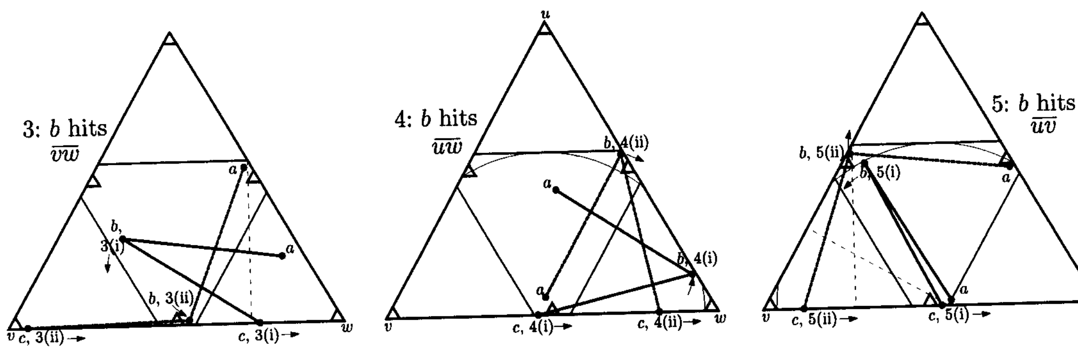

Figure 10: Illustrations of the cases of Lemma 14, in which b hits a wall as c is dragged along $\overline{vw}$ toward w.

2. *Joints a, b and c become colinear.*
3. *Joint b hits a wall on* $\overline{vw}$
   - *(i). between the v and vw frame triangles, or*
   - *(ii). between the vw frame triangle and the vertical line through the right endpoint of the vw-fence.*
4. *With* $l > \sqrt{3}/(2+\sqrt{3}) \approx 0.4641$*, joint b hits a wall on* $\overline{uw}$
   - *(i). inside the circle C centered at the left corner of the vw frame triangle, or*
   - *(ii). between the vw-fence and the uw frame triangle.*
5. *With* $l > \sqrt{3}/(2+\sqrt{3}) \approx 0.4641$*, joint b hits a wall on* $\overline{uv}$
   - *(i). inside the circle C centered at the right corner of the vw frame triangle, or*
   - *(ii). between the vw-fence and the uv frame triangle.*

Proof: The only events that can prevent c from reaching the frame triangle at w are joint b hitting a side of Δ or the ruler abc straightening. We can look at the cases in which b hits sides of Δ without b or c being in a frame triangle. Note that b is below the vw-fence since c is on $\overline{vw}$.

In case 3(ii), joint a must be right of the vertical line through b, or else dragging c right would move b away from the wall. But a must then be left of the vertical line through the right endpoint of the vw-fence.

In 4(ii), there is room for b between the vw-fence and the uw frame triangle only if link length $l > 0.4641$, by Lemma 11. In 4(i), c must be between the uw-fence and the vw frame triangle for b to hit the wall between the uw and w frame triangles. The fact that c is left of the vw frame triangle means that b hits inside the circle centered at the left corner of the vw frame triangle. This case occurs only if $l > 0.4641$. Furthermore, b is above the uv-fence if $l \le 0.4834$ by Lemma 11.

The cases for 5 (i) and (ii) are similar to those for 4 (i) and (ii). ■

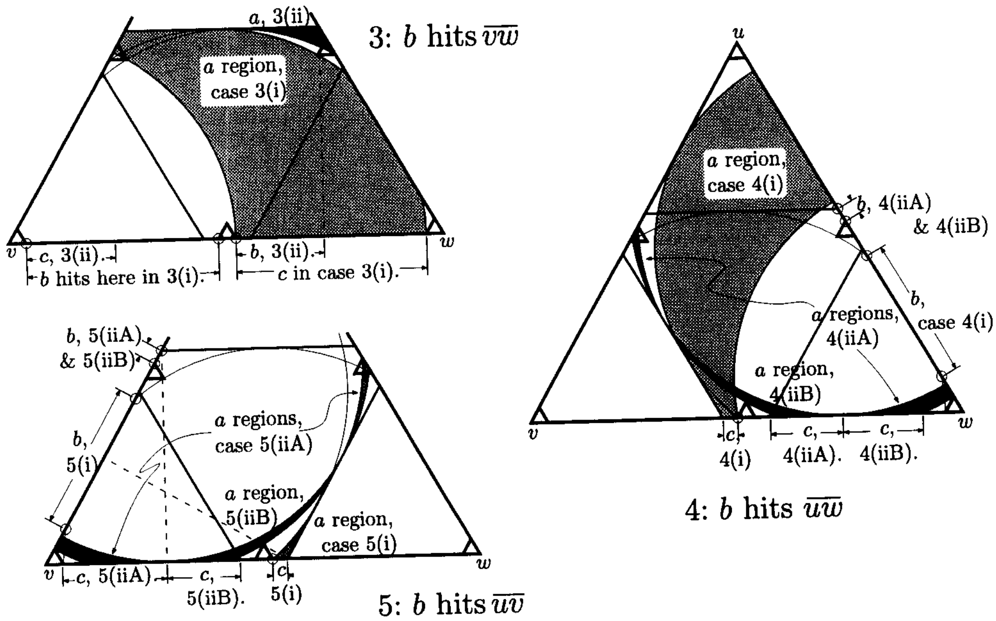

Figure 11: Locations for a that make b hit a wall in the cases of Lemma 14 as c is dragged along $\overline{vw}$ toward w. (Small circles mark the centers of relevant arcs.)

We can characterize the locations for a (and c) in terms of the location that b hits the wall. For example, a lies on C_b when b is on the wall—additional conditions may restrict which portion of C_b. One can determine all locations for a that cause b to hit a certain wall segment by taking the union of the appropriate portions of C_b for all positions where b hits that segment. Figure 11 illustrates the regions for a that are described in the next lemma.

Lemma 15 *When b is on a wall, we have additional restrictions in the following cases of Lemma 14:*

3. Joint a is below the vw-fence and right of the vertical line through b.

4(i). Joint a is above the 30° line through b.

4(ii). Joint c is either (A) left or (B) right of the vertical line through b. Joint a is either (A) right of the vertical or else (B) left of the vertical through b and below the 30° line through b. (Actually, a can be coincident with c in (B), but then the three joints are colinear.)

5(i). Joint a is below the -30° line through b or coincident with c.

5(ii). Joint c is either (A) left or (B) right of the vertical line through b. Joint a is either (A) left of the vertical or above the -30° line or else (B) right of the vertical and below the -30° line through b.

Proof: Since a is fixed, joint b moves along C_a in a direction determined by the motion of c. The conditions on a (and c) ensure that this motion is into the wall. ∎

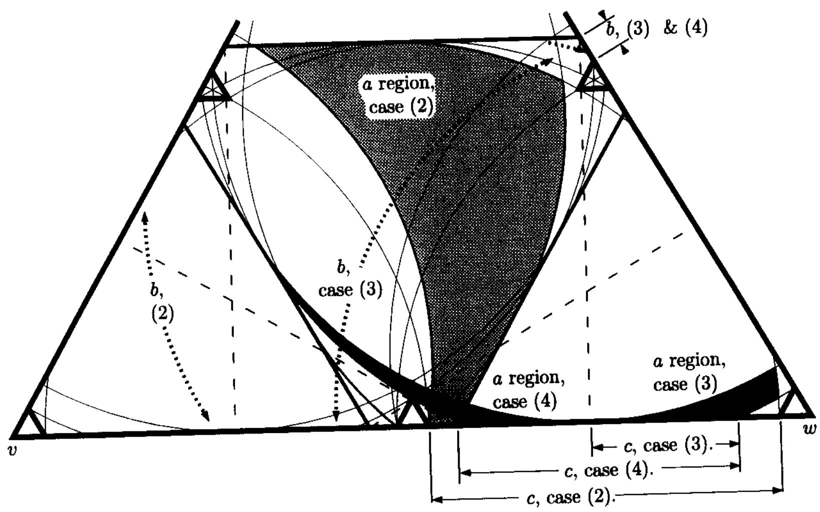

Figure 12: Locations for a and b that prevent motion of c both left and right.

Next we look at what can happen when we try to drag c either right or left. The cases are illustrated in Figure 12.

Corollary 16 *Given a ruler abc with c on* $\overline{vw}$*, by dragging c towards v and w we get b or c into a frame triangle unless*

(1). joints a, b and c become colinear,

(2). case 3(i) of Lemma 14 applies in one direction and 4(i) in the other,

(3). cases 3(ii) and 5(iiA) of Lemma 14 apply, or

(4). cases 4(iiB) and 5(iiB) of Lemma 14 apply.

Proof: If we take the union of the a regions described in Lemma 15 and intersect them with the reflection about a vertical line, then we find the positions in which a can prevent motion in both directions. Figure 12 illustrates the combinations for which the regions for a intersect and the resulting sliding ranges for c on $\overline{vw}$ remain between the vw and w frame triangles. The motions of b are also shown. (Other potential combinations in which regions for a intersect are 3(i) & *refl*(3(ii)), 3(i) & *refl*(5(iiB)), and 4(iiA) & *refl*(5(iiA)). These do not appear because the conditions regarding vertical lines cannot be met by sliding c.) ∎

By way of remark, if $l \leq \sqrt{3}/(2+\sqrt{3})$, then cases 2–4 of Corollary 16 cannot apply.

A.3 Folding a ruler onto the trellis

We are finally ready to fold a ruler with length $l \leq x_1 \approx 0.4834$ onto the trellis. We first put joint j_0 into a frame triangle (and thus onto the trellis), then we look at how the two-link rulers $j_0j_1j_2$ and $j_4j_3j_2$ work together and show that by dragging j_2 either j_1 can be moved onto the trellis or three links can be folded to one. Once we have the first joint on the trellis, frame triangles can be a big help.

Lemma 17 *Suppose that j_0 is on the trellis. If one of the joints j_1, j_2, j_3, or j_4 ever gets into a frame triangle, then we can put j_1 onto the trellis.*

Proof: If j_1 is in a frame triangle, then we can drag the trellis by moving j_0 on C_1 until j_1 is on the trellis. If joint j_2, j_3, or j_4 is in a frame triangle, then we can drag the trellis toward that joint until a lower-numbered joint enters a frame triangle. ∎

Now, consider the ruler $j_0, j_1, \ldots, j_n$.

Lemma 18 *Given a ruler j_0, j_1, ..., one can move j_0 into a frame triangle or fold the first two links.*

Proof: Consider the ruler $j_2j_1j_0$ with the position of j_2 fixed. If j_1 or j_2 are in frame triangles, then we can put j_0 into a frame triangle. Otherwise, rotate j_0 to a wall and apply Corollary 16. The only way for $j_2j_1j_0$ to be colinear in Δ minus the frame triangles is to fold j_0 onto j_2. If, on the other hand, one of the cases (2)–(4) hold, then dragging j_0 along the wall moves j_1 above some fence so that j_0 can rotate on C_1 to j_2. ∎

We make one more useful observation. If we can put two joints together above a fence, then we can fold three links to one.

Lemma 19 *Given a ruler with joints abcd, if a and c are positioned at a common point above some fence, then we can fold all three links onto $\overline{cd}$ without moving d.*

Proof: Joints b and d lie on the single arc $A_c = A_a$. ∎

Lemma 20 *If j_0 is on the trellis, we can put j_1 onto the trellis or fold three links to one by rotating at most seven joints.*

Proof: We apply our analysis of two-link rulers to $j_0j_1j_2$ and $j_4j_3j_2$. First, we make sure that colinearity can never prevent joint j_2 from reaching a frame triangle. Then we rotate j_2 to a wall and drag it until $j_0j_1j_2$ or $j_4j_3j_2$ stop the motion according to Corollary 16. We handle mixed cases—where $j_4j_3j_2$ prevents motion of j_2 in one direction and $j_0j_1j_2$ prevents motion in the other—by reducing them to cases where the ruler $j_0j_1j_2$ does not restrict the motion of j_2. Finally, we show how to solve these cases by folding three links to one or moving a joint into a frame triangle and applying Lemma 17.

If j_0 is in a corner frame triangle, then we move the trellis away from this corner, pivoting on j_2, until j_0 is at the edge of the frame triangle strictly inside Δ. (Notice that if joint j_1 hits an edge of Δ during this process, then j_1 is in a frame triangle.) Now, since Δ minus the corner frame triangles has diameter at most $2l$ and j_0 is in the frame inside this region, any future colinearity of $j_0j_1j_2$ will imply that j_2 has entered a frame triangle.

Since j_0 is in a frame triangle, j_1 is on an arc of A_1 that intersects a frame triangle. We can move j_1 into that frame triangle, pivoting on j_3, unless j_2 hits a wall. By rotating and reflecting Δ, we can assume that this wall is $\overline{vw}$.

Suppose, without loss of generality, that the ruler $j_0j_1j_2$ does not allow j_2 to slide freely to the right. We will show how to either satisfy the theorem or else arrange that one joint (j_2 or j_3) can slide without restriction from preceding links. Since j_0 is in a frame triangle, j_2 can be restricted only by cases 3(i), 3(ii), or 5(iiA) of Lemma 14—only these cases have an region for a that intersects a frame triangle. (See Figures 11 and 13.)

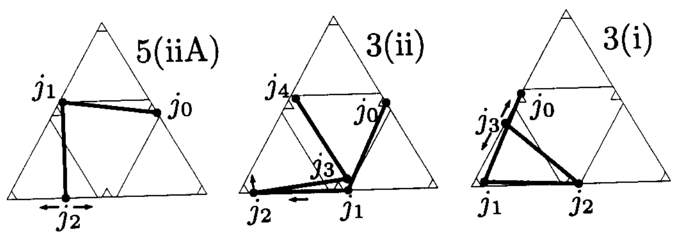

Figure 13: Dealing with cases in which $j_0j_1j_2$ restricts j_2.

Case 5(iiA): This case is the easiest—we move the trellis to have a vertex at w and j_0 moves out of the critical region and no longer restricts the motion of j_2. (This is because the arc A_0 goes above the vw-fence after the move.)

Case 3(ii): Joint j_1 is between the vw frame triangle and the uw-fence, which means that j_2 is near v. If we drag j_2 toward v, Lemma 14 implies that only the ruler $j_4j_3j_2$ can prevent j_2's entry into the v frame triangle.

If joints j_4, j_3, and j_2 become colinear by folding then j_2=j_4 and Lemma 19 implies that we can fold three links. With any other colinearity, j_4 is in the frame. The only case of Lemma 14 that applies to the ruler $j_4j_3j_2$ is 3(i). (Joint j_2 is too close to v for 4(iiB).) In that case, drag j_2 and j_1 along $\overrightarrow{wv}$, pivoting on j_4 and moving the trellis as necessary. Joint j_3 hits the wall at j_1. Next, move j_2 on $C_3 = C_1$ to $\overline{uv}$ and move the trellis to u. Then the ruler $j_0j_1j_2$ does not restrict the motion of j_2 on $\overline{uv}$.

Case 3(i): Joint j_2 is below the uv-fence and can move to the vw frame triangle unless j_3 hits $\overline{uv}$ according to case 4(i). But then the trellis can be moved to u so that j_2 can slide freely between the uv-fence and the vw frame triangle. Thus, j_3 can slide on $\overline{uv}$ without constraint from $j_0j_1j_2j_3$.

We can now slide a joint freely along a wall, with respect to preceding links. We shall call the joint j_2 and assume that the wall is between the vw

and v frame triangles on $\overline{vw}$. According to Corollary 16, we can put j_2 or j_3 onto the frame unless (1) $j_4j_3j_2$ become colinear or $l > \sqrt{3}/(2+\sqrt{3}) \approx 0.4641$ and one of the cases (2), (3), or (4) depicted in Figure 12 (and Figure 14) occurs.

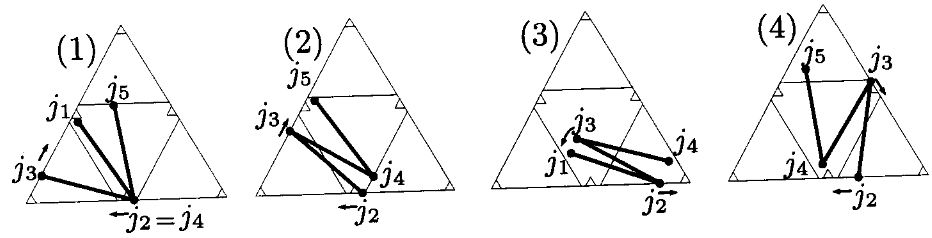

Figure 14: Only the ruler $j_4j_3j_2$ restricts j_2

Case (3): This is the easiest case. Joint j_2 (as c) is always above the uw-fence, so A_2 has one connected component. Joint j_3 sweeps this component, so must hit j_1. Then the positions of j_2 and j_4 place them on the same connected component of $A_3 = A_1$; we can move j_2 to fold $j_1j_2j_3j_4$ to a single link.

Case (4): In this case, joint j_3 stops j_2 from reaching the vw frame triangle by hitting $\overline{uw}$ according to 5(iiB). Move j_2 as close to the vw frame triangle as possible. Apply Lemma 14 to ruler $j_5j_4j_3$ in an attempt to drag j_3 into the uw frame triangle. (Notice that we can slide j_2 toward the vw frame triangle so that j_2 never prevents this motion of j_3.) One of four outcomes occurs. First, if j_3 reaches the frame, then we are done by Lemma 17. Second, if j_4 exits the case (4) region of Figure 12 then we are done because j_2 is no longer restricted in both directions by $j_4j_3j_2$. Third, if j_4 hits a wall in the case (4) region, then it does so at j_2 and above the uv-fence; Lemma 19 says we can fold three links to one. Finally, if j_5, j_4 and j_3 become colinear, then $j_5 = j_3$. Joints j_2 and j_4 are on the same connected component of $A_5 = A_3$, so moving j_3 folds $j_2j_3j_4j_5$ to a single link.

Case (2): In this case, joint j_3 stops j_2 from reaching the vw frame triangle by hitting $\overline{uv}$ above the uw-fence. Attempt to drag j_3 on $\overline{uv}$; notice that we can slide j_2 so that it never prevents the motion of j_3.

Either j_3 reaches the frame triangle at v, and we are done by Lemma 17, or j_3 goes below the uw-fence and j_2 enters the vw frame triangle, or one of the cases of Corollary 16 occur for $j_5j_4j_3$. In case (1), joint j_3 becomes coincident with j_5 above the uw-fence and Lemma 19 says that we can fold $j_2j_3j_4j_5$ to a single link. We need not consider (2), because there j_3 goes below the uw-fence. In cases (3) and (4), we slide j_3 as far toward the uv frame triangle as possible and j_4 hits $\overline{vw}$ at j_2. Now, j_3 and j_5 are on the same connected component of $A_3 = A_5$ and we can again fold $j_2j_3j_4j_5$.

Case (1): In the last case, j_4, j_3 and j_2 become colinear. If one of these joints is in a frame triangle then Lemma 17 applies—this must occur if the ruler $j_4j_3j_2$ straightens. Otherwise, $j_4j_3j_2$ folds so that $j_2 = j_4$. If $j_2 = j_4$ is above a fence, then Lemma 19 applies. Otherwise, we have two components of $A_2 = A_4$. If joint j_3 is on a component that intersects a frame triangle or one of joints j_1 or j_5, then we are done by Lemma 17 or by folding three links to one. In the remaining case, which is illustrated in Figure 14(1), joint j_3 can be moved to $\overline{uv}$ and dragged into the uv frame triangle without interference from the rulers $j_1j_2j_3$ or $j_5j_4j_3$.

This completes the proof that joints can be moved onto the trellis or links folded. Since our motions affect at most three links before and three links after the freely sliding vertex, we move at most seven joints. ■