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Abstract

Well-foundedness is the essential property of orderings for proving termination. We introduce a simple criterion on term orderings such that any term ordering possessing the subterm property and satisfying this criterion is well-founded. The usual path orders fulfil this criterion, yielding a much simpler proof of well-foundedness than the classical proof depending on Kruskal's theorem. Even more, our approach covers non-simplification orders like *spo* and *gpo* which can not be dealt with by Kruskal's theorem.

For finite alphabets we present completeness results, i. e., a term rewriting system terminates if and only if it is compatible with an order satisfying the criterion. For infinite alphabets the same completeness results hold for a slightly different criterion.

1 Introduction

The usual way of proving termination of a term rewriting system (TRS) is by finding a well-founded order such that every rewrite step causes a decrease according to this ordering. Proving well-foundedness is often difficult, in particular for recursively defined syntactic orderings. It is therefore desirable to have criteria that help decide whether a particular order is well-founded. A standard criterion of this type is implied by Kruskal's theorem: if a monotonic term ordering over a finite signature satisfies the subterm property then it is well-founded. However, this theorem does not apply for all terminating TRS's: there are terminating TRS's like $f(f(x)) \rightarrow f(g(f(x)))$ that are not compatible with any monotonic term ordering satisfying the subterm property. Even *recursive path order (rpo)* with lexicographic status over a varyadic alphabet, is not covered directly by Kruskal's theorem ([5]). This motivated us to look for other conditions ensuring well-foundedness. In this paper we remove the monotonicity condition and replace it by some decomposability condition. For orderings satisfying the subterm property and this decomposability condition we prove well-foundedness in a way that is inspired by Nash-Williams' proof of Kruskal's theorem ([10]; as it appears in [6]), but which is much simpler. A similar technique, for a particular order, has already been used by Kamin and Lévy ([9]). Standard orderings like *recursive path order* ([1, 12]) and *semantic path order (spo)* ([9, 2]) trivially satisfy our conditions, yielding a simple proof of well-foundedness for these orders. Moreover, our conditions cover all terminating TRS's: a TRS terminates if and only if it is compatible with an order satisfying our conditions.

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We are concerned essentially with term rewrite systems over finite signatures. In the case of an infinite signature the same conditions yield well-foundedness if the signature is provided with a partial well-order satisfying some natural compatibility with the given term ordering.

The rest of the paper is organized as follows. In section 2 we give some well-known notions on term rewriting and partial orders. On section 3, we introduce the notion of lifting of an order, which plays an essential role in the theory presented. On section 4 we present our well-foundedness criterion for orders on terms built over a finite signature and give some surprising completeness results involving orders closed under substitutions and orders that are total.

In section 5, we present a well-foundedness criterion for orders on terms built over infinite signatures. First we follow an approach similar to the one used in section 4. For that we need the existence of *well-quasi-orders* on the set of function symbols. This requirement is quite strong and to overcome it we introduce a different notion of lifting of orders on terms. Using this new notion we can present a very general and simple result on well-foundedness and show that in this case the completeness results of section 4 also hold. The criteria presented are used on section 6 to derive well-foundedness of *semantic path order* and *general path order*.

Finally we make some concluding remarks, including some comparison between our results and Kruskal's theorem.

2 Preliminaries

For the sake of self-containment we give some notions over term rewriting systems and orders. For more information the reader is referred to [4].

Let $\mathcal{F}$ be a signature (a set of function symbols) and $\mathcal{X}$ a set of variables with $\mathcal{F} \cap \mathcal{X} = \emptyset$. To each function symbol of $\mathcal{F}$ we associate a set of possible arities given by the function *arity*: $\mathcal{F} \to \mathcal{P}(\mathbb{N}) \setminus \emptyset$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. In the case that *arity*(f) contains only one element for all $f \in \mathcal{F}$, we speak of a fixed-arity signature, otherwise we speak of a varyadic signature.

The set of all terms over $\mathcal{F}$ and $\mathcal{X}$ is defined inductively as usual and denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X})$, the set of ground terms is denoted by $\mathcal{T}(\mathcal{F})$. In the sequel we will consider terms over different kinds of signature, for example finite or infinite signatures and finite or infinite sets of variables. We will make clear which restrictions apply at any point.

Given any term t, s is a *subterm* of t if we can write $t = C[s]$ for some context C. If $C[s] = f(\ldots, s, \ldots)$ and C is not the empty context, we say that s is a *principal subterm* of t. We define $|t|$ to be the depth of a term t. Recall that depth strictly decreases by taking (principal) subterms.

A term rewriting system (TRS) is a tuple $(\mathcal{F}, \mathcal{X}, R)$, where R is a subset of $\mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$. The elements of R are the so called rules of the TRS and are usually denoted by $l \to r$, with l a non-variable term and such that all the variables occurring in r also occur in l.

A TRS R induces a *rewrite relation* over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, denoted by $\to_R$, as follows: $s \to_R t$ iff $s = C[l\sigma]$ and $t = C[r\sigma]$, for some context C, substitution σ and rule $l \to r \in R$. The transitive closure of $\to_R$ is denoted by $\to_R^+$ and its reflexive-transitive closure by $\to_R^*$. A TRS is called *terminating* (strongly normalizing or noetherian) if there exists no infinite sequence of the form $t_0 \to_R t_1 \to_R \cdots$.

We use the terminology *partial order* on a set S meaning an irreflexive and transitive relation on S, that we usually denote by $>$. By *quasi-order* we mean a reflexive and transitive relation, usually denoted by $\geq$. Any quasi-order contains a strict partial order, namely $\geq \setminus \leq$, and an equivalence relation $\geq \cap \leq$, that we usually denote by $\sim$.

A partial order or quasi-order over a set S is said to be *well-founded* if it doesn't admit infinite descending chains of the form

$$x_0 > x_1 > x_2 > \dots$$

We extend the terminology well-founded to the elements of S: we say that $x \in S$ is well-founded if x does not occur in an infinite descending chain as above. Obviously an order $>$ on a set S is well-founded if and only if all elements $s \in S$ are well-founded.

We are interested on orders on the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$. An order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to be *monotonic* if $s > t$ implies $C[s] > C[t]$, for any context C. Given a TRS R and a order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, we say that $>$ is *compatible* with R if $s > t$ whenever $s \to_R t$.

An order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to have the *subterm property* if $f(t_1, \dots, t_n) > t_i$, for any $f \in \mathcal{F}$ and terms $t_1, \dots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, where $n \in \mathit{arity}(f)$.

3 Liftings and Status

As mentioned before, we replace monotonicity by another condition. This condition relates the comparison between $f(s_1, \dots, s_m)$ and $f(t_1, \dots, t_n)$ to the comparison of the sequences $\langle s_1, \dots, s_m \rangle$ and $\langle t_1, \dots, t_n \rangle$. Here we need to describe how an ordering on terms is lifted to an ordering on sequences of terms. To be able to conclude well-foundedness it is essential that this lifting preserves well-foundedness.

Definition 3.0.1 *Let $(S, >)$ be a partial ordered set and $S^* = \cup_{n \in \mathbb{N}} S^n$. We define a* lifting *to be a partial order $>^\lambda$ on S^* for which the following holds: for every $A \subseteq S$, if $>$ restricted to A is well-founded, then $>^\lambda$ restricted to A^* is also well-founded. We use the notation $\lambda(S)$ to denote all possible liftings of $>$ on S^*.*

A typical example of a lifting is the *multiset extension* of an order. The usual *lexicographic extension* on unbounded sequences is not a lifting. Just take $S = \{0, 1\}$ with $1 > 0$, then

$$\langle 1 \rangle >^\lambda \langle 01 \rangle >^\lambda \langle 001 \rangle >^\lambda \langle 0001 \rangle >^\lambda \dots$$

If the lexicographic comparison is restricted to sequences whose size is bounded by some fixed natural N, then this is indeed a lifting.

Another type of lifting is a constant lifting, i. e., any fixed well-founded partial order on S^*. Clearly other liftings can be defined, for example as combinations of the ones mentioned. In particular, combinations of multiset and lexicographic order can be very useful. In a partial order $(S, >)$ where $a > b$ and c is incomparable with a and b, one cannot conclude $\langle a, c, c \rangle >^\lambda \langle c, b, a \rangle$, for the multiset lifting nor for any lexicographic lifting. If we define $>^\lambda$ by

$$\langle s_1, \dots, s_m \rangle >^\lambda \langle t_1, \dots, t_n \rangle \iff \begin{cases} (m = n = 3) & \text{and} \\ \langle s_1, s_2 \rangle >^{mul} \langle t_1, t_2 \rangle & \text{or} \\ (\langle s_1, s_2 \rangle =^{mul} \langle t_1, t_2 \rangle) & \text{and } s_3 > t_3 \end{cases}$$

it is not difficult to see that $>^\lambda$ satisfies the definition of lifting and also satisfies $\langle a, c, c \rangle >^\lambda \langle c, b, a \rangle$. This lifting will be used to obtain

$$f(s(x), y, y) >_{rpo} f(y, x, s(x))$$

Classical $>_{rpo}$ cannot be used to compare these two terms.

Definition 3.0.1 is intended to be applied to terms over varyadic function symbols. If we consider signatures with fixed arity function symbols we can simplify the notion of lifting: instead of taking liftings of any order we need only take liftings of fixed order, i. e., the lifting is going to be a partial order over S^n, for a fixed natural number n. This is a special case of a lifting to S^*: $>^\lambda$ is defined on S^* to be the order one has in mind for S^n on sequences of length n, while all other pairs of sequences are defined to be incomparable with respect to $>^\lambda$.

Again typical examples of liftings are the *lexicographic extension* of $>$ on sequences and the *multiset extension* of $>$ restricted to multisets of a fixed size.

We are interested in orders on terms so from now on we choose $S = \mathcal{T}(\mathcal{F}, \mathcal{X})$, with $\mathcal{F}$ containing varyadic function symbols, and we fix a partial order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Definition 3.0.2 *Given $(\mathcal{T}(\mathcal{F}, \mathcal{X}), >)$, a* status function *(with respect to $>$) is a function $\tau : \mathcal{F} \to \lambda(\mathcal{T}(\mathcal{F}, \mathcal{X}))$, mapping every $f \in \mathcal{F}$ to a lifting $>^{\tau(f)}$.*

Again for the case of fixed-arity signatures, a status function will associate to each function symbol $f \in \mathcal{F}$ a order n lifting $>^\lambda$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})^n$, where n is the arity of f.

The following status will be used later in connection with the semantic path order. Let $>$ be a partial order and $\succeq$ a well-founded quasi-order, both defined on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Write $\succ$ for the strict part of $\succeq$ (i. e., $\succ = \succeq \setminus \preceq$) and $\sim$ for the equivalence relation induced by $\succeq$ (i. e., $\sim = \succeq \cap \preceq$). For each $f \in \mathcal{F}$ the lifting $\tau(f)$ is given by

$$\langle s_1, \ldots, s_k \rangle >^{\tau(f)} \langle t_1, \ldots, t_m \rangle \iff \begin{cases} s \succ t, \text{ or} \\ s \sim t \text{ and } \langle s_1, \ldots, s_k \rangle >^{mul} \langle t_1, \ldots, t_m \rangle \end{cases}$$

for any $k, m \in arity(f)$ and where $>^{mul}$ is the multiset extension of $>$, $s = f(s_1, \ldots, s_k)$ and $t = f(t_1, \ldots, t_m)$. It is not difficult to see that $>^{\tau(f)}$ is indeed a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})^*$ and that $>^{\tau(f)}$ respects well-foundedness, being therefore a lifting.

4 Finite signatures

In this section we present one of the main results of this paper. For the sake of simplicity we restrict ourselves to finite signatures. Surprisingly we do not need to fix the arities of the function symbols. Infinite signatures will be treated separately.

4.1 Main result

In the following we consider the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$, over the set of varyadic function symbols $\mathcal{F}$ and such that $\mathcal{F} \cup \mathcal{X}$ is finite.

Recall that a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is *well-founded* (with respect to a certain order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$) if there are no infinite descending chains starting with t.

We introduce some notation.

Definition 4.1.1 *Let $>$ be a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and τ a status function with respect to $>$. We say that $>$ is* decomposable *with respect to τ if $>$ satisfies*

- *if $f(s_1, \ldots, s_k) > f(t_1, \ldots, t_m)$ then either*

- $\exists 1 \le i \le k : s_i \ge f(t_1, \ldots, t_m)$, *or*
- $\langle s_1, \ldots, s_k \rangle >^{\tau(f)} \langle t_1, \ldots, t_m \rangle$.

for all function symbols $f \in \mathcal{F}$, $k, m \in \text{arity}(f)$ *and terms* $s_1, \ldots, s_k, t_1, \ldots, t_m \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.

We can now present the main result of this section.

Theorem 4.1.2 *Let $>$ be a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and τ a status function with respect to $>$. Suppose $>$ has the subterm property and is decomposable with respect to τ, then $>$ is well-founded.*

Proof Suppose that $>$ is not well-founded and take an infinite descending chain $t_0 > t_1 > \cdots > t_n > \cdots$, minimal in the following sense

- $|t_0| \le |s|$, for all non-well-founded terms s;
- $|t_{i+1}| \le |s|$, for all non-well-founded terms s such that $t_i > s$.

Note that from the first minimality condition follows that any principal subterm of t_0 is well-founded. Assume that $t_{i+1} = f(u_1, \ldots, u_k)$ and some u_j, with $1 \le j \le k$, is not well-founded. From the subterm property and transitivity of $>$, we obtain $t_i > t_{i+1} > u_j$, hence the second minimality condition yields $|t_{i+1}| \le |u_j|$ which is a contradiction. We conclude that all principal subterms of any term t_i, $i \ge 0$, are well-founded.

Since $\mathcal{F} \cup \mathcal{X}$ is finite, the (infinite) sequence $(t_i)_{i \ge 0}$ must contain a subsequence $(t_{\phi(i)})_{i \ge 0}$ with $t_{\phi(i)} = f(u_{i,1}, \ldots u_{i,n_i})$, for a fixed $f \in \mathcal{F}$. By hypothesis, for each $i \ge 0$, either

- $\exists 1 \le j \le n_i : \ u_{i,j} \ge t_{\phi(i+1)}$; or
- $\langle u_{i,1}, \ldots, u_{i,n_i} \rangle >^{\tau(f)} \langle u_{i+1,1}, \ldots, u_{i+1,n_{i+1}} \rangle$.

Since all terms $u_{i,j}$ are well-founded, the first case never occurs. Consequently we have an infinite descending chain

$$\langle u_{0,1}, \ldots, u_{0,n_0} \rangle >^{\tau(f)} \langle u_{1,1}, \ldots, u_{1,n_1} \rangle >^{\tau(f)} \langle u_{2,1}, \ldots, u_{2,n_2} \rangle >^{\tau(f)} \ldots$$

Since $>$ is well-founded over the set $\bigcup_{i \ge 0} (\bigcup_{j=1}^{n_i} \{u_{i,j}\})$, this contradicts the assumption that $\tau(f)$ preserves well-foundedness. □

Theorem 4.1.2 provides a way of proving well-foundedness of orders on terms, including orders which are not closed under contexts nor closed under substitutions.

Consider the *recursive path order* with status ([1, 12]) whose definition we present below.

Definition 4.1.3 (RPO with status) *Let $\rhd$ be a partial order on $\mathcal{F}$ and τ a status function with respect to $>_{rpo}$. Given two terms s, t we say that $s >_{rpo} t$ iff $s = f(s_1, \ldots, s_m)$ and either*

1. $t = g(t_1, \ldots, t_n)$ *and*
   (a) $f \rhd g$ *and* $s >_{rpo} t_i$, *for all* $1 \le i \le n$, *or*
   (b) $f = g$, $\langle s_1, \ldots, s_m \rangle >_{rpo}^{\tau(f)} \langle t_1, \ldots, t_n \rangle$ *and* $s >_{rpo} t_i$, *for all* $1 \le i \le n$; *or*

2. $\exists 1 \leq i \leq m:\ s_i >_{rpo} t$ *or* $s_i = t$.

Irreflexivity and transitivity of $>_{rpo}$ are cumbersome but not difficult to check. Well-foundedness of $>_{rpo}$, as defined in definition 4.1.3, follows from theorem 4.1.2. If we take the definition of $>_{rpo}$ over a precedence that is a quasi-order with the additional condition that each equivalence class of function symbols has one status associated, well-foundedness is still a direct consequence of theorem 4.1.2. We remark that by using our definition of lifting and status, definition 4.1.2 is a generalization of $>_{rpo}$ orders as found in the literature. With this definition we are able to prove termination of the following TRS (originally from [7]):

$$f(s(x), y, y) \;\rightarrow\; f(y, x, s(x))$$

For that we use a lifting given earlier, namely

$$\langle s_1, \ldots, s_m \rangle >^{\lambda} \langle t_1, \ldots, t_n \rangle \iff \begin{cases} (m = n = 3) & \text{and} \\ \langle s_1, s_2 \rangle >^{mul} \langle t_1, t_2 \rangle & \text{or} \\ (\langle s_1, s_2 \rangle =^{mul} \langle t_1, t_2 \rangle) & \text{and } s_3 > t_3 \end{cases}$$

and then take $>_{rpo}^{\tau(f)} = >_{rpo}^{\lambda}$. Termination of this system cannot be handled by earlier versions of $>_{rpo}$.

In section 6 we shall see that well-foundedness of both *semantic path order* and *general path order* also follow from theorem 4.1.2.

4.2 Completeness results

The next result states that the type of term orderings described in theorem 4.1.2 covers all terminating TRS's.

Theorem 4.2.1 *A TRS R is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a status function τ satisfying the following conditions:*

- *$>$ has the subterm property*
- *$>$ is decomposable with respect to τ*
- *if $s \rightarrow_R t$ then $s > t$.*

Proof The "if" part follows from theorem 4.1.2: the order $>$ is well-founded and the assumption $\rightarrow_R \subseteq >$ implies that R is terminating.

For the "only-if" part we define the relation $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ by:

$$s > t \iff s \neq t \text{ and } \exists C[\,] : s \rightarrow_R^* C[t]$$

By definition, the relation $>$ is irreflexive and has the subterm property. Transitivity is checked straightforwardly using termination of R.

We check that $>$ is well-founded. Suppose it is not and let $s_0 > s_1 > \cdots$ be an infinite descending chain. By definition of $>$, for each $i \geq 0$, we have $s_i \rightarrow_R^* C_i[s_{i+1}]$, for some context $C_i[\,]$, so we obtain the infinite chain

$$s_0 \rightarrow_R^* C_0[s_1] \rightarrow_R^* C_0[C_1[s_2]] \rightarrow_R^* \cdots$$

From termination of R, we conclude that there is an index $j \geq 0$ such that

$$s_j = C_j[s_{j+1}] = C_j[C_{j+1}[s_{j+2}]] = \ldots$$

Since the sequence is infinite and $C_k[\ ] \neq \Box$ (since $s_k \neq s_{k+1}$), for all $k \geq j$, this is a contradiction.

For each function symbol $f \in \mathcal{F}$ we define $>^{\tau(f)}$ by:

$$\langle u_1, \ldots u_k \rangle >^{\tau(f)} \langle v_1, \ldots, v_m \rangle \iff f(u_1, \ldots, u_k) > f(v_1, \ldots, v_m)$$

for any $k, m \in arity(f)$. Since $>$ is well-founded, we see that $>^{\tau(f)}$ is indeed a lifting.

From the above reasoning follows that all the conditions of theorem 4.1.2 are satisfied. Finally if $s \rightarrow_R t$, we obviously have $s \rightarrow_R^* C[t]$, with C the empty context. Since R is terminating we must have $s \neq t$ and consequently $s > t$. $\Box$

An alternative proof of theorem 4.2.1 can be given using the fact that a TRS R is terminating if and only if it is compatible with a semantic path order; in the proof of this fact the same order as above is used. Since *spo* fulfils the conditions of theorem 4.1.2, as we shall see in section 6, this provides an alternative proof for theorem 4.2.1.

The order defined in the proof of theorem 4.2.1 has the additional property of being closed under substitutions (but not under contexts). Consequently we also have the following stronger result.

Theorem 4.2.2 *A TRS R is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a status function τ satisfying the following conditions:*

- *$>$ has the subterm property*
- *$>$ is decomposable with respect to τ*
- *$>$ is closed under substitutions*
- *if $s \rightarrow_R t$ then $s > t$.*

An interesting question raised by J.-P. Jouannaud is what can be said about totality of orders satisfying the conditions of theorem 4.1.2. It turns out that totality can very easily be achieved as we now show. However totality is not compatible with closedness under substitutions. First we present a well-known lemma.

Lemma 4.2.3 *Any partial well-founded order $>$ on a set A can be extended to a total well-founded order on A.*

Proof We give a sketch of one possible proof. Consider K the set of partial orders $(S, >_S)$ satisfying the following conditions:

1. $S \subseteq A$.
2. if $s, t \in S$ and $s > t$ then $s >_S t$.
3. $>_S$ is total and well-founded in S.
4. if $s > t$ and $t \in S$ then $s \in S$ and $s >_S t$.

Note that since $>$ is well-founded on A, minimal elements do exist and any set containing a minimal element and ordered by the empty order is an element of K, so K is not empty.

We now turn K to a partially ordered set itself by defining the order $\sqsubset$ as follows:

$$(S, >_S) \sqsubset (T, >_T) \iff \begin{cases} S \subset T \text{ (as sets) and } >_S \subset >_T \\ \text{if } s >_T t \text{ and } s \in S \text{ then } t \in S \text{ and } s >_S t \end{cases}$$

It is easy to check that $\sqsubset$ is a partial order on K. The next step is to verify that $(K, \sqsubset)$ satisfies the conditions of Zorn's lemma. We establish then the existence of a maximal element in K and finally see that that maximal element is a total well-founded order extending the original one. □

Note that Zorn's lemma cannot be applied with the usual subset ordering since well-foundedness is not preserved under infinite unions.

Theorem 4.2.4 *A TRS R is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and a status function τ satisfying the following conditions:*

- *$>$ has the subterm property*
- *$>$ is decomposable with respect to τ*
- *$>$ is total*
- *if $s \rightarrow_R t$ then $s > t$.*

Proof Again the "if" part follows from theorem 4.1.2: the order $>$ is well-founded and the assumption $\rightarrow_R \subseteq >$ implies that R is terminating.

For the "only-if" part we use theorems 4.1.2 and 4.2.1. Since R is terminating, by theorem 4.2.1 there is an order $\gg$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a status function τ satisfying the conditions of theorem 4.1.2 and such that $s \rightarrow_R t \Rightarrow s \gg t$. By theorem 4.1.2 the order $\gg$ is well-founded, but not necessarily total. By lemma 4.2.3, let $>$ be a total well-founded order extending $\gg$. Since $\gg$ has the subterm property, so does $>$. Furthermore $>$ is also compatible with $\rightarrow_R$, for if $s \rightarrow_R t$ then $s \gg t$ and so $s > t$. In order to apply theorem 4.1.2 we still have to define a status function τ for which $>$ is decomposable. For each function symbol $f \in \mathcal{F}$ we define: $\langle u_1, \dots u_k \rangle >^{\tau(f)} \langle v_1, \dots, v_m \rangle \iff f(u_1, \dots, u_k) > f(v_1, \dots, v_m)$, for any $k, m \in arity(f)$. Since $>$ is well-founded, $>^{\tau(f)}$ is indeed a lifting. Theorem 4.1.2 now gives the result. □

The previous result may seem a bit strange since it tells us that we can achieve totality on all terms and not only ground terms. This is so because we do not impose any closure conditions on the order. Note that a total order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is never closed under substitutions as long as $\mathcal{X}$ contains more than one element. As for closure under contexts, this property is usually not maintained by naive extensions of the order, it may even make the existence of certain extensions impossible. In our case the conditions imposed are subterm property and compatibility with the reduction relation and so any extension will comply with those conditions whenever the original order does.

4.3 Infinitely many variables

If we allow $\mathcal{X}$ to be an infinite set, the conditions imposed on the order on theorem 4.1.2 are not enough to guarantee that the order is well-founded. Just consider a set of variables $\mathcal{X} = \{x_i\}_{i \geq 0}$ and $>$ satisfying $x_0 > x_1 > x_2 > \ldots$. Even if the conditions of theorem 4.1.2 are satisfied, the order is obviously not well-founded. However, in the presence of an infinite set of variables, well-foundedness of $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is equivalent to well-foundedness of $>$ on $\mathcal{X}$, i. e., theorem 4.1.2 can be rewritten as:

Theorem 4.3.1 *Let $>$ be a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and τ a status function with respect to $>$. Suppose $>$ has the subterm property and is decomposable with respect to τ. Then $>$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ if and only if $>$ is well-founded on $\mathcal{X}$.*

One direction is trivial, the other is almost identical to the proof of theorem 4.1.2.

As for theorems 4.2.1, 4.2.2 and 4.2.4, they all hold under the additional assumption that the order considered is well-founded when restricted to $\mathcal{X}$.

5 Infinite Signatures

In the previous section we presented some results which are applicable to orders and TRS's over finite signatures. Here we turn to the infinite case, i. e., we consider the set of terms over an infinite alphabet $\mathcal{F}$, with varyadic function symbols, and an infinite set of variables $\mathcal{X}$. As usual we require that $\mathcal{F} \cap \mathcal{X} = \emptyset$.

We first discuss orders which are based on a precedence on the set of function symbols. Afterwards we will present another simplified approach in which we can dispense with the precedence. This approach is based on a generalization of the notion of lifting.

5.1 Precedence-based orders

It turns out that theorem 4.1.2 can also be extended to infinite signatures. We do however need to impose some extra conditions.

We introduce some more notation. Let $\trianglerighteq$ be a quasi-order over $\mathcal{F}$, called a *precedence*. We denote the strict partial order $\trianglerighteq \setminus \trianglelefteq$ by $\triangleright$ and the equivalence relation $\trianglerighteq \cap \trianglelefteq$ by $\sim$.

Definition 5.1.1 *Given an order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a precedence $\trianglerighteq$ on $\mathcal{F}$, we say that $>$ is* compatible *with $\trianglerighteq$ if whenever $f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n)$ and $g \triangleright f$ then $s_i \geq g(t_1, \ldots, t_n)$, for some $1 \leq i \leq m$.*

In theorem 4.1.2 we only needed to take into account comparisons between terms with the same head function symbol, but now we also need to consider the comparisons between terms whose head function symbols are equivalent under the precedence considered. As a consequence we need to impose some constraint on the status associated with a function symbol.

Definition 5.1.2 *Given a precedence $\trianglerighteq$ on $\mathcal{F}$, an order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a status function τ, with respect to $>$, we say that τ and $\trianglerighteq$ are* compatible *if whenever $f \sim g$ then $\tau(f) = \tau(g)$.*

As usual a *well quasi-order*, abbreviated to *wqo*, is a quasi-order $\succeq$ such that any extension of it is well-founded. We can now formulate theorem 4.1.2 for infinite signatures:

Theorem 5.1.3 *Let $\trianglerighteq$ be a precedence on $\mathcal{F}$, $>$ a partial order over $\mathcal{T}(\mathcal{F},\mathcal{X})$, and τ a status function with respect to $>$, such that that both $>$ and $\trianglerighteq$ and τ and $\trianglerighteq$ are compatible. Suppose $>$ has the subterm property and satisfies the following condition:*

- $\forall f, g \in \mathcal{F},\ m \in \text{arity}(f), n \in \text{arity}(g), s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$:
  if $f(s_1,\ldots,s_m) > g(t_1,\ldots,t_n)$ *with* $f \sim g$, *then either*
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1,\ldots,t_n)$, *or*
  - $\langle s_1,\ldots,s_m\rangle >^{\tau(f)} \langle t_1,\ldots,t_n\rangle$.

Suppose additionally that $\trianglerighteq$ is a wqo on $\mathcal{F}\setminus\mathcal{F}_0$ and $>$ is well-founded on $\mathcal{X}\cup\mathcal{F}_0$, where $\mathcal{F}_0 = \{f \in \mathcal{F} : \text{arity}(f) = \{0\}\}$. Then $>$ is well-founded on $\mathcal{T}(\mathcal{F},\mathcal{X})$.

Proof We proceed, as in proof of theorem 4.1.2, by contradiction. First we remark that any infinite descending sequence $(t_i)_{i\geq 0}$ contains an infinite subsequence $(t_{\phi(i)})_{i\geq 0}$ such that $arity(t_{\phi(i)}) \neq \{0\}$, for if that would not be the case, the sequence would contain infinitely many variables or constants, contradicting the fact that $>$ is well-founded on $\mathcal{X}\cup\mathcal{F}_0$.

We take a minimal infinite descending sequence $(t_i)_{i\geq 0}$, in the same sense as in theorem 4.1.2. Again, as remarked in the proof of theorem 4.1.2, from the minimality of $(t_i)_{i\geq 0}$, the subterm property and transitivity of $>$, it follows that all (principal) subterms of any term t_i, $i \geq 0$, are well-founded.

Let $root(t)$ be the head symbol of the term t. Consider the infinite sequence $(root(t_i))_{i\geq 0}$. From the first observation above it follows that this sequence contains infinitely many terms such that the root function symbol of those terms has arities greater than 0. Consequently and since $\trianglerighteq$ is a *wqo* on $\mathcal{F}\setminus\mathcal{F}_0$, we can conclude that this sequence contains an infinite subsequence $(root(t_{\phi(i)}))_{i\geq 0}$ such that $root(t_{\phi(i+1)}) \trianglerighteq root(t_{\phi(i)})$ and $arity(root(t_{\phi(i)})) \neq \{0\}$, for all i.

Also the infinite sequence $(root(t_i))_{i\geq 0}$ contains no infinite subsequence $(root(t_{\psi(i)}))_{i\geq 0}$ such that $root(t_{\psi(i+1)}) \sim root(t_{\psi(i)})$, for all i. Suppose it is not so and let $(root(t_{\psi(i)}))_{i\geq 0}$ be such a sequence. Since $t_{\psi(i)} > t_{\psi(i+1)}$, by hypothesis we must have

1. $s_{i,k} \geq t_{\psi(i+1)}$, with $s_{i,k}$ a principal subterm of $t_{\psi(i)}$, or
2. $\langle s_{i,1},\ldots,s_{i,k_{\psi(i)}}\rangle >^{\lambda} \langle s_{i+1,1},\ldots,s_{i+1,k_{\psi(i+1)}}\rangle$, where $>^{\lambda}$ is the lifting given by the status of $root(t_{\psi(0)})$[1], and $s_{i,1},\ldots,s_{i,k_{\psi(i)}}$ and $s_{i+1,1},\ldots,s_{i+1,k_{\psi(i+1)}}$ are the principal subterms of respectively $t_{\psi(i)}$ and $t_{\psi(i+1)}$, for all i.

Due to the minimality of $(t_i)_{i\geq 0}$ and the subterm property, case 1 above can never occur. Therefore we have an infinite descending sequence

$$\langle s_{0,1},\ldots,s_{0,k_{\psi(0)}}\rangle >^{\lambda} \langle s_{1,1},\ldots,s_{1,k_{\psi(1)}}\rangle >^{\lambda} \langle s_{2,1},\ldots,s_{2,k_{\psi(2)}}\rangle >^{\lambda} \ldots$$

Since $>$ is well-founded on $\bigcup_{i\geq 0}\bigcup_{j=1}^{k_{\psi(i)}} \{s_{i,j}\}$, this contradicts the definition of lifting.

[1] Recall that for equivalent function symbols, their status coincides.

Therefore, and without loss of generality, we can state that the infinite subsequence $(root(t_{\phi(i)}))_{i\geq 0}$ has the additional property $root(t_{\phi(i+1)}) \rhd root(t_{\phi(i)})$, for all i.[2] Since $t_{\phi(i)} > t_{\phi(i+1)}$ and $>$ is compatible with $\unrhd$, we must have $u \geq t_{\phi(i+1)}$, for some principal subterm u of $t_{\phi(i)}$, contradicting the minimality of $(t_i)_{i\geq 0}$. □

Some remarks are in order. Since there are no substitutions involved, there is no essential difference between elements of $\mathcal{X}$ and $\mathcal{F}_0$. The condition stating that $>$ is well-founded on $\mathcal{X}$ is imposed to disallow the bizarre case where we can have an infinite descending sequence constituted solely by variables. Usually (e. g. in Kruskal's theorem) it is required that the precedence $\unrhd$ be a *wqo* over $\mathcal{F}$, we can however relax that condition to $\unrhd$ being a *wqo* over $\mathcal{F} \setminus \mathcal{F}_0$ provided $>$ is also well-founded on $\mathcal{F}_0$. This is weaker than requiring that $\unrhd$ be a *wqo* on $\mathcal{F}$. The *wqo* requirement cannot be weakened to well-foundedness as the following example shows. Consider $\mathcal{F} = \{f_i | i \geq 0\}$ with $arity(f_i) = \{1\}$, for all $i \geq 0$. Let $>$ be an order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ with the subterm property and such that

$$f_0(x) > f_1(x) > f_2(x) > \dots$$

Take $\unrhd$ to be the empty precedence. Obviously $\unrhd$ is well-founded and all the other conditions of theorem 5.1.3 are satisfied, however the order $>$ is not well-founded.

If we remove the condition "$>$ is well-founded on $\mathcal{X} \cup \mathcal{F}_0$", and strengthen the condition on $\unrhd$ to "$\unrhd$ is a *wqo* on $\mathcal{F} \cup \mathcal{X}$", then the same statement as above can be proved (and the proof is very similar). In this case and for finite signatures, theorem 4.1.2 is a direct consequence of theorem 5.1.3, since the discrete order is a *wqo* and the compatibility conditions are trivially fulfilled.

Theorem 5.1.3 holds in particular for precedences that are *partial well-orders (pwo's)*. In this case we only need to compare terms with the same root function symbol and the compatibility condition of definition 5.1.2 is trivially verified.

As in the finite case, well-foundedness of orders as *rpo* over infinite signatures, is a consequence of theorem 5.1.3. For that we only need to extend the well-founded precedence to a total well-founded one, maintaining the equivalence part the same, which is then a *wqo*. All the other conditions also hold, so the theorem can be applied.

Another interesting result arises if we relax the requirements on the precedence and strengthen the ones on the order.

Theorem 5.1.4 *Let $\unrhd$ be a well-founded precedence on $\mathcal{F}$, $>$ a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and τ a status function with respect to $>$, such that that τ and $\unrhd$ are compatible. Suppose $>$ has the subterm property and satisfies the following condition:*

- $\forall f, g \in \mathcal{F},\ m \in \text{arity}(f), n \in \text{arity}(g), s_1, \dots, s_m, t_1, \dots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$:
  if $f(s_1, \dots, s_m) > g(t_1, \dots, t_n)$ *then either*
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1, \dots, t_n)$, *or*
  - $f \rhd g$, *or*
  - $f \sim g$ *and* $\langle s_1, \dots, s_m \rangle >^{\tau(f)} \langle t_1, \dots, t_n \rangle$.

Then $>$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

[2] Strictly speaking, an infinite subsequence of this sequence has that property.

The proof is very similar to the proof of theorem 5.1.3, therefore we omit it. Note that well-foundedness of *rpo*, for an arbitrary well-founded precedence, is a direct consequence of this result. In the "classical" approach, first the precedence has to be extended via lemma 4.2.3 to a well-founded total precedence, maintaining the equivalence part, before Kruskal's theorem yields the desired result.

It would also be interesting to have a theorem similar to theorem 4.2.1 for the case of infinite signatures. However for infinite signatures the empty relation is not a *wqo* any longer and it is not clear how to choose an appropriate *wqo*. A possibility is to take $\trianglerighteq$ defined by $f \sim g$ for any $f, g \in \mathcal{F}$, which is trivially a *wqo*, however this choice will not always work as the following example shows. Consider the infinite terminating TRS given by

$$a_i \rightarrow a_j$$

for any $i \geq 0$ and any $0 \leq j < i$ and where each a_i is a constant. Then any order compatible with R will never be compatible with a precedence in which $a_i \sim a_j$, for all $i, j \geq 0$.

Another alternative is to take a total well-founded order on $\mathcal{F}$, again by definition a *wqo*, but then other compatibility problems arise. Just consider the rule

$$a \rightarrow f(0)$$

If we choose the precedence as an arbitrary total well-founded order on $\mathcal{F}$, we may have $f \triangleright a$, and the conditions of theorem 5.1.3 will never hold.

5.2 Generalizing liftings on orders

The decomposability restriction $\langle s_1, \ldots, s_m \rangle >^{\tau(f)} \langle t_1, \ldots, t_n \rangle$ has the inconvenient of forgetting about the root symbols of the terms compared. In the case of finite signatures, that is irrelevant since we only need to compare terms with the same head symbol and the symbol can be encoded in the status τ. For infinite signatures however, that information is essential, since given an infinite sequence of terms we no longer have the guarantee that it contains an infinite subsequence of terms having the same root symbol. As a consequence we need to impose some strong conditions both on the set of function symbols and on the status and order used. A way of relaxing these conditions is by remembering the information lost with the decomposition and this can be achieved by changing the definition of lifting.

In this section we present another condition for well-foundedness on term orderings. Now we do not require the existence of an order or quasi-order on the set of function symbols $\mathcal{F}$. Instead we will use a different definition of lifting for orderings on terms.

Definition 5.2.1 *Let $(\mathcal{T}(\mathcal{F}, \mathcal{X}), >)$ be a partial ordered set of terms. We define a* term lifting *to be a partial order $>^{\Lambda}$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ for which the following holds: for every $A \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, if $>$ restricted to A is well-founded, then $>^{\Lambda}$ restricted to $\bar{A}$ is also well-founded, where*

$$\bar{A} = \{f(t_1, \ldots, t_n) : f \in \mathcal{F},\ n \in \text{arity}(f),\ \text{and } t_i \in A,\ \text{for all } i, 0 \leq i \leq n\}$$

We use the notation $\Lambda(>)$ to denote all possible term liftings of $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

We remark that term liftings can make use of liftings and status functions since the well-foundedness requirement is preserved. Given an order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, every lifting in the sense of definition 3.0.1 induces a term lifting of the same order as follows:

$$f(s_1, \ldots, s_m) >^{\Lambda} g(t_1, \ldots, t_n) \iff \langle s_1, \ldots, s_m \rangle >^{\lambda} \langle t_1, \ldots, t_n \rangle$$

We present a new well-foundedness criterion.

Theorem 5.2.2 *Let $>$ be a partial order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and let $>^{\Lambda}$ be a term lifting of $>$. Suppose $>$ has the subterm property and satisfies the following condition:*

- $\forall f,g \in \mathcal{F},\ m \in \text{arity}(f), n \in \text{arity}(g), s_1,\ldots,s_m,t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$:
  if $s = f(s_1,\ldots,s_m) > g(t_1,\ldots,t_n) = t$ *then either*
  - $\exists 1 \le i \le m : s_i \ge g(t_1,\ldots,t_n)$, *or*
  - $s >^{\Lambda} t$

Then $>$ is well-founded on $\mathcal{T}(\mathcal{F},\mathcal{X})$.

Proof Suppose that $>$ is not well-founded and take an infinite descending chain $t_0 > t_1 > \cdots > t_n > \cdots$, minimal in the same sense as in the proof of theorem 4.1.2, i. e.,

- $|t_0| \le |s|$, for all non-well-founded terms s;
- $|t_{i+1}| \le |s|$, for all non-well-founded terms s such that $t_i > s$.

As remarked in the proof of theorem 4.1.2, from the minimality of $(t_i)_{i\ge 0}$, the subterm property and transitivity of $>$, it follows that all principal subterms of any term t_i, $i \ge 0$, are well-founded.

Since $t_i > t_{i+1}$, for all $i \ge 0$, we must have

1. $u_i \ge t_{i+1}$, for some principal subterm u_i of t_i, or
2. $t_i >^{\Lambda} t_{i+1}$

Due to the minimality of the sequence, the first case above can never occur. Therefore we have an infinite descending chain

$$t_0 >^{\Lambda} t_1 >^{\Lambda} t_2 >^{\Lambda} \ldots$$

But due also to minimality, the order $>$ is well-founded over the set of terms

$$A = \{u : u \text{ is a principal subterm of } t_i, \text{ for some } i \ge 0\}$$

By definition of term lifting we have that $>^{\Lambda}$ is well-founded over

$$\bar{A} = \{f(u_1,\ldots,u_k) : f \in \mathcal{F}, k \in \mathit{arity}(f) \text{ and } u_i \in A, \text{ for all } 1 \le i \le k\}$$

and since $\{t_i : i \ge 0\} \subseteq \bar{A}$, we get a contradiction. □

It is interesting to remark that theorem 4.1.2 is a consequence of theorem 5.2.2. To see that we define the following order $\gg$:

$$s \gg t \iff (\mathit{root}(s) = \mathit{root}(t)) \text{ and } (s > t)$$

Now we define the following term lifting

$$f(s_1,\ldots,s_m) \gg^{\Lambda} g(t_1,\ldots,t_n) \iff (f = g) \text{ and } \langle s_1,\ldots,s_m\rangle >^{\tau(f)} \langle t_1,\ldots,t_n\rangle$$

where $>^{\tau(f)}$ is the lifting associated by the status function τ to the function symbol f. It is not difficult to see that since the lifting $>^{\tau(f)}$ respects well-foundedness of $>$, $\gg^{\Lambda}$ is a well-defined term lifting. Now theorem 5.2.2 gives well-foundedness of $\gg$. But since non-well-foundedness of $>$ would imply non-well-foundedness of $\gg$ (by an argument similar to the proof of theorem 4.1.2), we are done.

Furthermore when $\mathcal{F}$ is finite, theorem 5.2.2 is also a consequence of theorem 4.1.2 (i. e., they are equivalent). For that we define the status

$$\langle s_1, \ldots, s_m \rangle >^{\tau(f)} \langle t_1, \ldots, t_n \rangle \iff f(s_1, \ldots, s_m) >^{\Lambda} f(t_1, \ldots, t_n)$$

It is now not difficult to check that the other implication holds.

Due to the required existence of a partial order on the set of function symbols, the relation of this theorem with theorems 5.1.3 and 5.1.4 is not yet clear.

An important consequence of the use of term liftings is that we manage to recover the completeness results stated on section 4.2 and that we could not state for precedence-based orders.

Theorem 5.2.3 *Let R be a TRS over an infinite varyadic signature. Then R is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a term lifting $>^{\Lambda}$ satisfying the following conditions:*

- *$>$ has the subterm property (and $>$ is closed under substitutions)*
- $\forall f, g \in \mathcal{F},\ m \in \text{arity}(f), n \in \text{arity}(g), s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$:
  if $s = f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n) = t$ then either
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1, \ldots, t_n)$, *or*
  - $s >^{\Lambda} t$
- *if $s \to_R t$ then $s > t$.*

Proof Sketch. The "if" part follows from theorem 5.2.2: the order $>$ is well-founded and the assumption $\to_R \subseteq >$ implies that R is terminating.

For the "only-if" part the proof is similar to the proof of theorem 4.2.1. We define again the relation $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$: $s > t \iff s \neq t$ and $\exists C[\,] : s \to_R^* C[t]$. The only different part is the definition of term lifting. Since the order $>$ is well-founded we can use it as the term lifting itself. □

As for the finite case the completeness result concerning totality also holds and the proof is very similar, so we omit it.

Theorem 5.2.4 *Let R be a TRS over an infinite varyadic signature. Then R is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a term lifting $>^{\Lambda}$ satisfying the following conditions:*

- *$>$ has the subterm property*
- $\forall f, g \in \mathcal{F},\ m \in \text{arity}(f), n \in \text{arity}(g), s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$:
  if $s = f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n) = t$ then either

- $\exists 1 \leq i \leq m : s_i \geq g(t_1, \ldots, t_n)$, *or*
- $s >^{\Lambda} t$

- $>$ *is total*
- *if* $s \rightarrow_R t$ *then* $s > t$.

6 Semantic Path Order and General Path Order

In this section we show how well-foundedness of *semantic path order* [9] and *general path order* [3] can be derived using either theorem 4.1.2 or theorem 5.2.2.

Definition 6.0.5 Semantic Path Order.
Let $\geq$ *be a well-founded quasi-order on* $\mathcal{T}(\mathcal{F})$. *The* semantic path order $\succeq_{spo}$ *is defined on* $\mathcal{T}(\mathcal{F})$ *as follows:* $s = f(s_1, \ldots, s_m) \succeq_{spo} g(t_1, \ldots, t_n) = t$ *if either*

1. $s > t$ *and* $s \succ_{spo} t_i$, *for all* $1 \leq i \leq n$, *or*
2. $s \sim t$ *and* $s \succ_{spo} t_i$, *for all* $1 \leq i \leq n$ *and* $\langle s_1, \ldots, s_m \rangle \succeq_{spo}^{mul} \langle t_1, \ldots, t_n \rangle$, *where* $\succeq_{spo}^{mul}$ *is the multiset extension of* $\succeq_{spo}$, *or*
3. $\exists i \in \{1, \ldots, m\} : s_i \succeq_{spo} t$.

It can be seen that the $\succ_{spo}$ has the subterm property and is in general not monotonic.

In the case the alphabet we consider is finite, define the following status. Let $\succeq$ be the well-founded quasi-order used in the definition of $\succ_{spo}$. For each $f \in \mathcal{F}$ the lifting $\tau(f)$ is given by

$$\langle s_1, \ldots, s_k \rangle \succ_{spo}^{\tau(f)} \langle t_1, \ldots, t_m \rangle \iff \begin{cases} s \succ t, \text{ or} \\ s \sim t \text{ and } \langle s_1, \ldots, s_k \rangle \succ_{spo}^{mul} \langle t_1, \ldots, t_m \rangle \end{cases}$$

for any $k, m \in arity(f)$ and where $\succ_{spo}^{mul}$ is the multiset extension of $\succ_{spo}$, $s = f(s_1, \ldots, s_k)$ and $t = f(t_1, \ldots, t_m)$. It is not difficult to see that $\succ_{spo}^{\tau(f)}$ is indeed a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})^*$ and that $\succ_{spo}^{\tau(f)}$ respects well-foundedness, being therefore a lifting. Since $\succ_{spo}$ has the subterm property and satisfies the other conditions of theorem 4.1.2, its well-foundedness follows from application of the theorem.

For the case we consider an infinite signature, we define the following term lifting: for $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$

$$s \succ_{spo}^{\Lambda} t \iff \begin{cases} (s \succ t) & \text{or} \\ (s \sim t) \quad \text{and} & \langle s_1, \ldots, s_m \rangle \succ_{spo}^{mul} \langle t_1, \ldots, t_n \rangle \end{cases}$$

where again $\succeq$ is the well-founded quasi-order used in the definition of $\succ_{spo}$. Since $\succ$ is well-founded and the multiset extension respects well-foundedness, $\succ_{spo}^{\Lambda}$ is indeed a term lifting. Using this term lifting, we can apply theorem 5.2.2 to conclude that $\succ_{spo}$ is well-founded.

The *general path order*, that we denote by $\succeq_{gpo}$, was introduced in [3]. We present the definition and show how well-foundedness of this order can be derived from theorem 4.1.2 or theorem 5.2.2.

Definition 6.0.6 *A* termination function θ *is a function defined on the set of terms* $\mathcal{T}(\mathcal{F},\mathcal{X})$ *and is either*

1. *a homomorphism from terms to a set S such that*
$$\theta(f(s_1,\ldots,s_n)) = f_\theta(\theta(s_1),\ldots,\theta(s_n))$$

2. *an extraction function that given a term associates to it a multiset of principal subterms, i. e.,*
$$\theta(f(s_1,\ldots,s_n)) = [s_{i_1},\ldots,s_{i_k}]$$
where $i_1,\ldots,i_k \in \{1,\ldots,n\}$.

Definition 6.0.7 *A* component order $\phi = \langle\theta,\geq\rangle$ *consists of a termination function defined on the set* $\mathcal{T}(\mathcal{F})$ *of ground terms, along with an associated well-founded quasi-order* $\geq$ *(defined on the codomain of* θ*).*

Definition 6.0.8 General Path Order.
Let $\phi_i = \langle\theta_i,\geq_i\rangle$*, with* $0 \leq i \leq k$*, be component orders, such that if* θ_j *is an extraction function then* $\geq_j$ *is the multiset extension of the general path order* $\succeq_{gpo}$ *itself. The induced* general path order $\succeq_{gpo}$ *is defined on* $\mathcal{T}(\mathcal{F})$ *as follows:* $s = f(s_1,\ldots,s_m) \succ_{gpo} g(t_1,\ldots,t_n) = t$ *if either*

1. $\exists i \in \{1,\ldots,m\} : s_i \succeq_{gpo} t$ *or*
2. $s \succ_{gpo} t_j$*, for all* $1 \leq j \leq n$*, and* $\Theta(s) >_{lex} \Theta(t)$*, where* $\Theta = \langle\theta_0,\ldots,\theta_k\rangle$ *and* $>_{lex}$ *is the lexicographic combination of the component orderings* θ_i *with* $0 \leq i \leq k$.

The equivalence part is defined as: $s = f(s_1,\ldots,s_m) \sim_{gpo} g(t_1,\ldots,t_n) = t$ *if* $s \succ_{gpo} t_j$*, for all* $1 \leq j \leq n$*, and* $t \succ_{gpo} s_j$*, for all* $1 \leq j \leq m$*, and* $\theta_i(s) \sim_i \theta_i(t)$*, for all* $0 \leq i \leq k$*, and where* $\sim_i$ *is the equivalence contained in* $\geq_i$.

It is known ([3]) that $\succeq_{gpo}$ is a quasi-order with the subterm property.

Well-foundedness of $\succeq_{gpo}$ is a consequence of the results previously presented. For the case of finite signatures we define the following status

$$\langle s_1,\ldots,s_m\rangle \succ_{gpo}^{\tau(f)} \langle t_1,\ldots,t_n\rangle \iff \Theta(f(s_1,\ldots,s_m)) >_{lex} \Theta(f(t_1,\ldots,t_k))$$

where as in definition 6.0.8, $\Theta(v) = \langle\theta_0(v),\ldots,\theta_k(v)\rangle$ and $>_{lex}$ is the lexicographic combination of the component orderings θ_i with $0 \leq i \leq k$. If θ_i is an homomorphism to a well-founded set, then θ_i is obviously a lifting, and if θ_i is a multiset extracting function, since the multiset construction preserves well-foundedness, we also have that θ_i is a lifting. Finally the finite lexicographic composition of liftings is still a lifting. As a consequence $\succ_{gpo}^{\tau(f)}$ is a well-defined status, and since $\succ_{gpo}$ has the subterm property and satisfies the other conditions of theorem 4.1.2, we can apply this result to conclude $\succ_{gpo}$ is well-founded.

For infinite signatures, well-foundedness of $\succ_{gpo}$ is a consequence of theorem 5.2.2. If we define the term lifting $\succ_{gpo}^{\Lambda}$ as Θ, we see that $\succ_{gpo}^{\Lambda}$ is indeed well-defined. Since the other conditions of theorem 5.2.2 are satisfied, we can apply it to conclude well-foundedness of $\succ_{gpo}$. Finally it is interesting to remark that if we allow the termination function to be not only a multiset extraction function but an arbitrary lifting, we obtain a generalization of $\succ_{gpo}$ whose well-foundedness can still be derived from the results presented.[3]

[3]For other similar generalization of $\succ_{gpo}$ see [8].

7 Conclusions

We presented some criteria for proving well-foundedness of orders on terms. Our approach was inspired by Kruskal's theorem but is simpler. Kruskal's theorem (and extensions as the one in [11]) is a stronger result in the sense that it establishes that a certain order is a *well-quasi-order* (or *partial-well-order*). Our result allows to conclude well-foundedness directly. However the essential difference is the domain of application: Kruskal's theorem implies well-foundedness of orders extending any monotonic order with the subterm property, hence only covers simplification orders and it is well-known that those orders do not cover all terminating TRS's. Our criteria do not require monotonicity and as a consequence, cover all terminating TRS's.

For infinite signatures we managed to present a well-foundedness criterion even simpler and the completeness results still hold.

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