Treewidth and Small Separators for Graphs with Small Chordality*

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Abstract

A graph G k-chordal, if it does not contain chordless cycles of length larger than k. The chordality cl of a graph G is the minimum k for which G is k-chordal. The degeneracy or the width of a graph is the maximum min-degree of any of its subgraphs. Our results are the following:

1. The problem of treewidth remains NP-complete when restricted on graphs with small maximum degree.
2. An upper bound is given for the treewidth of a graph as a function of its maximum degree and chordality. A consequence of this result is that when maximum degree and chordality are fixed constants, then there is a linear algorithm for treewidth and a polynomial algorithm for pathwidth.
3. For any constant $s \geq 1$, it is shown that any $(s+2)$-chordal graph with degeneracy d contains a $\frac{1}{2}$-separator of size $O((dn)^{\frac{s-1}{s}})$, computable in linear time. Our results extent the many applications of the separator theorems in [1, 33, 34] to the class of k-chordal graphs.

Several natural classes of graphs have small chordality. Weakly chordal graphs and cocomparability graphs are 4-chordal. We investigate the complexity of treewidth and pathwidth on these classes when an additional degree restriction is used. We present an application of our separator theorem on approximating the maximum independent set on k-chordal graphs with small degeneracy.

1 Introduction

In this paper, we study a relatively new graph parameter: the chordality of a graph. (All graphs are assumed to be undirected, finite, and simple.) We call

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a graph k-*chordal*, if it does not contain a chordless cycle of length larger than k. The *chordality* of a graph G is defined as the minimum k for which G is k-chordal. In this paper we investigate the complexity of treewidth and pathwidth in relation to this parameter, the maximum degree, and the degeneracity of G. We also present a separator theorem for graphs with bounded chordality.

The class of k-chordal graphs contains as subclasses many known natural classes of graphs, even for small values of k. Clearly 3-chordal graphs are exactly the chordal graphs. Also the classes of the weakly chordal graphs and, as we prove in Section 6, the cocomparability graphs are subclasses of the class of 4-chordal graphs.

The notions of treewidth and pathwidth appear to play an important role in the analysis of the complexity of several problems in graph theory. They were introduced by Robertson and Seymour in their series of fundamental papers on graphs minors (see [39]). Roughly spoken, the *treewidth* of a graph is the minimum k such that G can be decomposed into a "tree structure" of pieces with at most $k+1$ vertices. (For the precise definition, see Section 2.) A series of recent results show that many NP-complete problems become polynomial or even linear time solvable, or belong to NC, when restricted to graphs with small treewidth (see [5, 7, 9]). Much research has been done on the problem of determining the treewidth and the pathwidth of a graph, and finding optimal tree or path decompositions with optimal treewidth or pathwidth. These problems are NP-complete even if we restrict the input graph on cocomparability graphs [23], bipartite graphs [27] or the complements of bipartite graphs [6]. Moreover, pathwidth remains NP-complete on chordal graphs [22], planar graphs [38] and graphs with bounded maximum degree [38]. In Section 3, we prove that treewidth is also NP-complete on graphs with bounded maximum degree.

Treewidth can be computed in polynomial time on chordal graphs, cographs [15], circular arc graphs [42], chordal bipartite graphs [29], permutation graphs [14], circle graphs [26] and distance hereditary graphs [3]. Bodlaender presented in [10] a linear time algorithm that finds an optimal tree decomposition for a graph with bounded treewidth. Also Bodlaender and Hagerup in [12] provide (near) optimal parallel algorithms for constructing minimum-width tree decompositions of graphs with bounded treewidth. In Section 4, we prove that if a k-chordal graph has maximum degree bounded by a value Δ, then there is a function $f(k, \Delta)$ that is an upper bound for treewidth. A consequence of our result is that for k-chordal graphs with bounded maximum degree, there is a linear time algorithm for computing treewidth and a polynomial time algorithm for computing pathwidth.

In Section 5, we present a connection between the parameters of treewidth and degeneracy for k-chordal graphs. The *degeneracy* of a graph $G = (V, E)$ is defined to be the maximum min-degree of any of the subgraphs of G (see also [16, 20, 31, 35, 36, 41]). In [37], it is proved that the degeneracy of a graph is equal to its width, a graph parameter that is also known as linkage (see also [20, 25, 36]). A layout L of a graph $G = (V, E)$ is a bijective function, mapping its vertices to numbers $\{1, 2, \ldots, |V|\}$. The *width* of a layout L of G is the maximum back-degree of any vertex in L (the *back-degree* of a vertex $v \in L$ is defined to be the number of vertices that are adjacent to v and are preceding it in L). The width of G is the minimum width over all possible layouts of G. Width has been studied in the context of Constraint Satisfaction, as it is known that for constraint graphs of bounded width, it is possible to apply backtrack free search, the classical method to solve the Constraint Satisfaction Problem (see [20]). Also, width appears in many combinatorial results. For instance, in [2], improved time bounds are presented for algorithms that count and find cycles, when the input graph is considered to have small width.

Several parameters characterizing the sparsity of a graph are related to width. The *arboricity* $\alpha(G)$ of a graph $\alpha(G)$ is defined as the minimal number of edge-disjoint spanning forests into which G can be decomposed. It is known that $\alpha(G) = \Theta(\text{width}(G))$. A graph is called *uniformly β-sparse*, if every subgraph has average degree at most β. It is also known that $\text{width}(G) \leq \beta$ (see [36]). Finally it can be easily proved that if γ is the genus of a graph, then $\text{width}(G) = O(\gamma)$.

In this paper we also use the parameter width_s. When $s = n-1$ or $s = 1$, width_s is equivalent with treewidth and width, respectively. In Section 2, we prove that for all s, width_s is equivalent to the s-dimension of a graph, a parameter defined in [19]. In [19], Dendris, Kirousis, and Thilikos examine various versions of fugitive search games on graphs and present their connections with the parameters of treewidth, pathwidth, and width. They show that the width_s of a graph equals the number of searchers required to catch an inert fugitive that resides on the vertices of G and moves with speed s (an inert fugitive can only move just before a searcher is placed on the vertex it occupies). Using the notion of width_s, we prove a connection between width and treewidth that leads to a separator theorem for $(s+2)$-chordal graphs with small width.

Given a graph $G = (V, E)$, a set $S \subseteq V$ is a separator of G iff the $G[V - S]$, the subgraph of G induced by the vertices in $V - S$, contains at least two connected components. Separator theorems have appeared to play an important role for algorithmic graph theory as these (in combination with a divide and conquer strategy) lead to a series of important complexity results. In [33], Lipton and Tarjan proved that any planar graph with weights assigned to its vertices contains a separator S of size $O(\sqrt{n})$, such that all the connected components of $G[V - S]$ have total vertex weight at most $\frac{2}{3}|V|$. Several applications of this separator theorem are presented in [32], including approximation algorithms for NP-complete problems, nonserial dynamic programming, time-space trade offs' study, lower bounds on boolean circuit size, embedding of data structures, and maximum matching (see also [32] for an application on sparse Gaussian elimination and [30] for results relating small separators with layouts of graphs in a model of VLSI). In [1], Alon, Seymour, and Thomas provide a considerably more general result and extent the previous applications. They prove a separator theorem for any graph not containing a specific graph as a minor.

Our separator theorem guarantees the existence of a separator of size $O(kn^{\frac{s-1}{s}})$ in a $(s+2)$-chordal graph with $\text{width} \leq k$. Moreover our results lead to a linear time algorithm that computes such a separator. As any graph not containing a specific graph as a minor has *bounded* width, our separator theorem gives an extension of the results in [1, 33, 34] for graphs with bounded chordality, and improves the time bounds in [1] for such classes of graphs. We present an application of our separator theorem for the problem of approximating the independent set problem on $(s+2)$-chordal graphs with small width.

As there are sparse graphs with small width that do not contain small separators (see Lemma 3) we feel that the requirements of small width and small chordality help to approach a characterization of the concept of "usefully sparse", a question posed from Lipton and Tarjan in [34] about the existence of separator theorems for non planar sparse graphs.

2 Definitions and preliminary results

In this section some definitions and results will be presented, which are useful in later sections.

Let $G = (V, E)$ be a finite undirected graph without multiple edges or loops. For a set of vertices $V' \subseteq V$, the subgraph of G, induced by V' is denoted by $G[V']$. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

The notions of treewidth and pathwidth were introduced by Robertson and Seymour in [40] and [39].

Definition 1 *A* tree decomposition *of $G = (V, E)$ is defined to be a pair $(\{X_i : i \in I\}, T)$, where $\{X_i : i \in I\}$ is a collection of subsets of V (we call these subsets the* nodes *of the decomposition) and $T = (I, F)$ is a tree having the index set I as set of vertices, such that the following conditions are satisfied:*

1. $\bigcup_{i \in I} X_i = V$.
2. $\forall \{u, w\} \in E, \exists i \in I : u, w \in X_i$.
3. $\forall i, j, k \in I$: *if j is on a path in T from i to k then* $X_i \cap X_k \subseteq X_j$.

The treewidth *of a tree decomposition $(\{X_i : i \in I\}, T)$ is defined to be equal to* $\max_{i \in I} |X_i| - 1$.
The treewidth *of G is defined to be the minimum treewidth over all tree decompositions of G.*

Definition 2 *A* path decomposition *of $G = (V, E)$ is defined to be a sequence $\{X_i : i = 1, \ldots, r\}$ of subsets of V (we call these subsets the* nodes *of the decomposition) such that the following conditions are satisfied:*

1. $\bigcup_{i=1}^{r} X_i = V$.
2. $\forall \{u, w\} \in E, \exists i : u, w \in X_i$.
3. $\forall i, j, k$, *if* $1 \leq i \leq j \leq k \leq r$, *then* $X_i \cap X_k \subseteq X_j$.

The pathwidth *of a path decomposition $\{X_i : i = 1, \ldots, r\}$ is defined to be equal to* $\max_{1 \leq i \leq r} |X_i| - 1$.
The pathwidth *of G is defined to be the minimum pathwidth over all path decompositions of G.*

The problem of computing the treewidth of a given graph has been proved to be NP-complete by Arnborg, Corneil, and Proskurowski in [6]. More precisely, they proved that treewidth is NP-complete even when restricted to the class of cobipartite graphs (i.e. the complements of bipartite graphs).

Bodlaender proved the following result about the fixed parameter complexity of treewidth (see [10]).

Theorem 1 *For any fixed integer k, there is a linear time algorithm, that tests whether a given graph $G = (V, E)$ has treewidth at most k, and if so, outputs a tree decomposition of G with treewidth at most k.*

Treewidth can be characterized in terms of elimination orderings. Arnborg introduced in [5] the notion of the elimination dimension of a graph and proved that it is equivalent with treewidth. We give the definition of the s-elimination dimension of a graph introduced in [19]. For the case where $s = n - 1$, this parameter is equivalent to treewidth.

An elimination ordering of a graph $G = (V, E)$ is an ordering $\pi = (v_1, \ldots, v_n)$ of its vertices ($n = |V|$).

Given an elimination ordering π and an integer s ($1 \leq s \leq n - 1$), the graphs G_i, $i = 1, \ldots, n$ generated during an *s-elimination of the vertices of G* according to π are defined to be: $G_1 = G$; $V_{i+1} = V_i - \{v_i\}$ and E_{i+1} is the set of pairs $\{u, v\}$ such that $u, v \in V_{i+1}$ and there is a path in G that from u to v with length at most s and all its vertices except u and v (we call these vertices *internal* vertices of a path) belong to the set $\{v_1, \ldots, v_i\}$.

Definition 3 *The* s-dimension of v_i with respect to π *is defined as the degree of v_i in G_i.*
The s-dimension of π *is defined as the maximum s-dimension of any vertex v_i with respect to π.*
The s-elimination dimension of G *is the minimum s-dimension over all elimination orderings of G.*

A cycle $C = (v_1, \ldots, v_l, v_1)$ in a graph $G = (V, E)$ is *chordless* if it does not contain any chords (i.e. $\forall v_i, v_j,\ l - 1 > |i - j| > 1\ \ \{v_i, v_j\} \notin E$). We denote as $\mathrm{lc}(G)$ the length of the longest chordless cycle in G and call this parameter the *chordality* of a graph (in the case that G is a tree we take $\mathrm{lc}(G) = 2$). A graph G is *k-chordal* if $\mathrm{cl}(G) \leq k$. It is easy to see that $\mathrm{lc}(G)$ can take a wide range of values as is shown in the following easy extremal result.

Lemma 1 *If G is a graph with n vertices and $e \geq n$ edges and $\rho = \binom{n}{2} - e$ then the following holds:*

$$3 \leq \mathrm{lc}(G) \leq \frac{1}{2}(3 + \sqrt{9 + 8\rho})$$

moreover the above inequality is tight.

Proof It is sufficient to observe that if G contains a chordless cycle of length s, then it cannot have more that $\binom{n}{2} - \binom{s}{2} + s$ edges. □

Using the notation above we mention the following result proved in [19].

Theorem 2 *Let G be a graph such that $lc(G) \leq s+2$. The treewidth of G equals its s-elimination dimension.*

Theorem 3 *The problem of computing the s-dimension of a graph is NP-complete when $s > 1$.*

Proof As mentioned above, treewidth is NP-complete also when restricted on the class of cobipartite graphs. As for any graph G in this class $\mathrm{lc}(G) = 4$, the result follows from Theorem 2 and the NP-completeness of treewidth. □

The parameter $\mathrm{width}_s(G)$ introduced below characterizes treewidth in terms of layouts when $s = n - 1$.

Let $L = (v_1, \ldots, v_n)$ be a layout of the vertices in G.

Definition 4 *The* width_s *of a vertex $v \in L$ is the number of vertices preceding v in L that are connected with v via a path of vertices not preceding v which has length at most s.*
The width_s *of a layout of G is the maximum $width_s$ over all vertices of G.*
The width_s *of a graph G is the minimum $width_s$ over all possible layouts of G.*

Theorem 4 *For any graph G and $s \geq 1$, s-dimension(G) =width$_s(G)$.*

Proof Suppose that s-dimension$(G) = k$ and let $\pi = (v_1, \ldots, v_n)$ be an elimination ordering such that s-dimension$(\pi) = k$. Consider the layout $L = (v_n, \ldots, v_1)$ obtained by reversing π. We claim that for any vertex $v_i \in L$ the width_s of v_i is equal to the s-dimension of v_i with respect to the ordering π and hence $\leq k$: note that the number of vertices that precede v_i and are connected with v_i via paths of length at most s with internal vertices in the set $\{v_1, \ldots, v_{i-1}\}$ is equal, by definition, to the degree of v_i in G_i with respect to π.

Suppose now that $\mathrm{width}_s = k$. Using a similar argument we can prove that given a minimal layout L with $\mathrm{width}_s(L) = k$, for any vertex v_i in the ordering obtained by he inversion of L, the s-dimension of v_i is equal to the $\mathrm{width}_s(v_i)$ and thus $\leq k$. □

A graph H is a minor of a graph G if H can be obtained from a subgraph of G by a number of edge contractions. (A contraction of an edge $\{u, v\}$ replaces the vertices u and v by a new vertex that is adjacent to all vertices that were adjacent to v or u.)

We use the following well known result (for a proof see e.g., [8]).

Lemma 2 *Let H be a minor of G. Then treewidth$(H) \leq$ treewidth(G).*

For $s = 1$, Definition 4 gives the width of a graph. An already known corollary of Theorems 4 and 2 is that for chordal graphs, treewidth is equal to width, which is polynomially computable and has an NC approximation algorithm for constant approximation factors $< \frac{1}{2}$ (see [4]). It can be easily proved that width is bounded for classes of graphs with an excluded minor, i.e. graphs with no minor isomorphic to a given graph H (see [17]). However the class of graphs with bounded width is larger: there are graphs with small width containing arbitrary large minors, as is shown in the following lemma.

Lemma 3 *For any $k \geq 3$ and any graph H, there is a graph G such that width$(G) \leq k$ and H is a minor of G.*

Proof Suppose H has h vertices. It is enough to construct a graph G of width 3, containing as a minor a clique with h vertices. The construction is given by the following procedure:

1. Let i be the smaller integer such that $h - 1 \leq 2^i + 2$.
2. Initialize G as a clique with $2^i + 3$ vertices where each vertex has degree $d = 2^i + 2$.
3. Repeat the following step as long as there are vertices with degree d: Let v be a vertex with neighborhood $\{v_1, \ldots, v_d\}$. Replace v by two vertices u, w and add the edges $\{u, w\}, \{u, v_1\}, \ldots, \{u, v_{\frac{d}{2}}\}, \{w, v_{\frac{d}{2}+1}\}, \ldots, \{w, v_d\}$.
4. $d = \frac{d}{2} + 1$.
5. if $d > 3$ goto *step* 3.

From the procedure above we see that the obtained graph G has all the vertices of degree $3 \leq k$ and contains a clique with $2^{i-1} + 3 \geq h$ vertices as a minor. □

Lick and White in [31] proved the following extremal result about width (see also [25]).

Theorem 5 *Let G be a graph with n vertices, e edges and width$(G) \leq k$. Then $e \leq \binom{k}{2} + k(n-k)$.*

Clearly width$_{s_1}(G) \leq$width$_{s_2}(G)$ when $s_1 \leq s_2$. Using this fact, we can see that the above extremal result holds also for width$_s$ $\forall s \geq 1$.

Definition 5 *Let $G = (V, E)$ be a graph and $w : V \to R^+$ a function assigning a positive real weight to each vertex in V. We call the sum of the weights over all the vertices of a set $V' \subseteq V$ the* total weight *of V' and denote it as $w(V')$.*

A set $S \subseteq V$ is a $\frac{1}{2}$-separator of the function w in G iff the sum of the weights of the vertices in each of the connected components of $G[V - S]$ is no more than $\frac{1}{2}w(V)$.

It seems to be useful to have results that tell how to find separators of small cardinality in graphs, as these have several applications in combination with a divide-and-conquer strategy. For such theorems and applications see e.g. [34, 33, 1]. A well known separator result is the following.

Theorem 6 *Let $G = (V, E)$ be a graph and $w : V \to R^+$ a function assigning a positive real weight to each vertex in V. Then any tree decomposition $(\{X_i : i \in I\}, T)$ of G with treewidth$\leq k$, contains a node that is an $\frac{1}{2}$-separator of w in G.*

This theorem is a straightforward generalization of Theorem 3.5 in [11], and can be proved in the same way.

3 Treewidth is NP-complete for graphs with bounded max-degree

In this section we prove that treewidth is also NP-complete when restricted on graphs with maximum degree at most 9.

The decision version of the treewidth problem is the following:

TREEWIDTH
Instance: Graph $G = (V, E)$, integer $k \leq |V| - 1$.
Question: Is the treewidth of G at most k?

Definition 6 *We call a graph (n_1, m_1, n_2, m_2)-*bigrid *if it can be constructed in the following way:*

Take two grids G_1 and G_2 with sizes n_1, m_1 and n_2, m_2 respectively. Extend each grid G_i $i = 1, 2$ in the following way:

Let $S_i = \{v_1^i, \ldots, v_{n_i}^i\} \subseteq V(G_i)$ be the vertices of a side of G_i containing n_i vertices. Add a vertex set $S_i' = \{u_1^i, \ldots, u_{n_i}^i\}$ and connect v_j^i with u_j^i for $j = 1, \ldots, n_i$. We call the two graphs obtained, the pruned grids *of the construction and we denote them as G_1' and G_2'.*

The construction is completed by adding an arbitrary number of edges, each between a vertex in S_1' and a vertex in S_2'.

We call the transformation below *a q-clique-grid transformation* from a cobipartite graph G to a bigrid graph G'.

Let $G = (V, E)$ be a cobipartite graph where V_1, V_2 induce disjoint cliques and $|V_1| + |V_2| = |V|$. Let $n_1 = |V_1|$ and $n_2 = |V_2|$. Now we take a (n_1, q, n_2, q)-bigrid $G' = (V', E')$ in the following way: each vertex in S_i' represents a vertex in V_i and each edge $e = \{v_k^1, v_l^2\}$, $v_k^1 \in S_1', v_l^2 \in S_2'$ represents the edge in E which has as endpoints the corresponding to u_k^1 and u_l^2 vertices of V.

We now need the following, rather technical lemmas.

Lemma 4 *Consider a tree decomposition $(\{X_j, j \in J\}, T)$ of a graph $G = (V.E)$. Then for any clique K of G, $\exists j \in J : V(K) \subseteq X_j$.*

For a proof of this lemma see e.g., [15, 21].

Lemma 5 *Let G be a grid with sizes n, q and $(\{X_j : j \in J\}, T)$ be a tree decomposition of G with width $\leq k$. Then if $q \geq 2k + 3$, there is a node X_j in the decomposition that contains at least one vertex of each of the q rows of G.*

Proof Let $w : V \to R^+$ be a function such that $\forall v \in V$, $w(v) = 1$. As treewidth$(G) \leq k$, from Theorem 6 it follows that there must exist a node X_j in the decomposition tree that is a $\frac{1}{2}$-separator of G. Clearly X_j has common vertices with at most $k+1$ columns in G. So there are at least $q-(k+1)$ columns in G not meeting X_j. Suppose that all of the vertices of these columns belong in the same component of $G[V - X_j]$. As X_j is a $\frac{1}{2}$-separator of w in G, this component must contain $\leq \frac{1}{2}(nq)$ vertices. Therefore $n(q - (k + 1)) \leq \frac{1}{2}(nq)$ which gives $q \geq 2k + 2$, a contradiction. Hence, there are two columns in different components of $G[V - X_j]$. Now any row contains a vertex of each of the two components, and hence X_j must contain a vertex of each row, by definition of tree decomposition. □

The following lemma asserts that treewidth is an invariant of the q-clique-grid transformation when q is sufficiently large.

Lemma 6 *Let G be a cobipartite graph. Let G' be the graph we obtain from G if we apply a q-clique-grid transformation on G with $q \geq 2k + 3$. Then treewidth(G) $\leq k$, if and only if treewidth(G') $\leq k$.*

Proof Suppose that treewidth(G) $\leq k$. We will prove that treewidth(G') $\leq k$. Notice first that G contains two cliques C_i of n_i vertices each ($i = 1, 2$). By Lemma 4, G must have a tree decomposition that has a node X_j^i containing C_i. An easy construction shows that any (n_i, m_i)-grid has a tree decomposition of treewidth $\leq n_i$, that contains all the vertices of some side of n_i vertices in one of its nodes. Using this fact, we can build a tree decomposition of each pruned grid G'_i, $i = 1, 2$ in G' that has width $\leq n_i$ and has a node containing S'_i. Now if we identify the vertices of each clique C_i in G with the corresponding set S'_i in each pruned grid, we can see that, composing the tree decompositions of G, G'_1 and G'_2, the graph G_{merge} thus obtained has a tree decomposition of width $\leq \max\{n_1, n_2, \text{treewidth}(G)\} \leq k$. As G' is a subgraph of G_{merge} we have the required result.

Suppose now that treewidth(G') $\leq k$. Fix a tree-decomposition $(\{X_i \mid i \in I\}, T)$ of G' of treewidth at most k. Let G'' be obtained from G' by adding edges between all pairs of vertices v, w for which there is at least one node $i \in I$ with $v, w \in X_i$. Clearly, $(\{X_i \mid i \in I\}, T)$ is also a tree-decomposition of G''. Let G''' be the graph, obtained by contracting all rows in both grids with the corresponding vertex u_i. Note that G is a subgraph of G''': edges between vertices in the different cliques clearly are present. We must verify all edges in the cliques are present in G''': when v, w belong to the same clique in G, then two rows in one of the grids in G'' have been contracted to v and w, respectively. As there is a node that contains a vertex of each row of this grid (Lemma 5), it follows there is an edge between a vertex of v's row and a vertex of w's row in G'', hence v and w are adjacent in G'''. So, G is a minor of G'''. The result now follows by Lemma 2. □

Theorem 7 *Treewidth remains NP-complete when restricted to graphs with maximum degree 9.*

Proof In the NP-completeness proof of Arnborg et al [6], a transformation from the cutwidth problem to cobipartite graphs is given. Cutwidth is NP-complete when restricted to graphs with maximum degree at most three [38]. Applying the transformation from [6] to graphs of maximum degree three yields cobipartite graphs where any vertex is connected to at most eight vertices in the clique to which it does not belong. Hence, treewidth is NP-complete for the latter type of cobipartite graphs. When we apply a q-clique-grid transformation on such a cobipartite graph, we obtain a graph of degree at most 9. Applying such a transformation with a properly chosen value of q (e.g., take $q = 2k + 3$, where k is the desired treewidth), yields the result. □

4 Graphs with bounded $\Delta(G)$ and $\mathbf{lc}(G)$

In the previous section we proved that treewidth is NP-complete for graphs with maximum degree ≥ 9. Similarly, pathwidth is NP-complete for graphs of maximum degree 3 [38]. In this section we show hat if both max-degree and the length of the chordless cycles are bounded, then the treewidth is bounded by a constant, and hence computable in linear time. It also follows that there is a polynomial time algorithm for pathwidth in this case.

For graphs G, let $D(G)$ denote $\sum_{u \in V(G)}(\Delta - \deg(u))$.

Lemma 7 *Let k, Δ, s be fixed constants. Let $\mathcal{D}$ be the class of connected graphs such that for any $G \in \mathcal{D}$ holds that:*

1. $2 \leq \Delta(G) \leq \Delta$
2. *there exists a vertex $v \in V(G)$ such that $deg(v) \leq k \leq \Delta$ and for any vertex $u \in V(G)$ there is a path between u and v of length at most s.*

Then $D(G) \leq k(\Delta - 1)^s$.

Proof First observe that if $|V(G)| = n$ and $|E(G)| = e$, $D(G) = n\Delta - 2e$. Consider fixed values of k, Δ and s. If $\Delta = 2$, then $D(G) = \max\{2, k\}$ and if $s = 0$, then $D(G) = k$. We now examine the case that $\Delta \geq k > 2$ and $s > 0$.

Note that $\mathcal{D}$ is a finite set. Consider a graph G in $\mathcal{D}$ such that $D(G)$ is maximal. We will prove first that G is a tree. Assume that G contains a cycle. Let T be a breath first spanning tree of G with root v, and let e be the edge of the cycle not in $E(T)$. Now let G' be the graph obtained by deleting e from G. Clearly $G' \in \mathcal{D}$ and $D(G') = D(G) + 2$, a contradiction.

We claim now that each vertex in $V(G) - \{v\}$ that is not a leaf has degree Δ and that $\deg(v) = k$. Suppose w is not a leaf, and either $v = w$ and $\deg(v) < k$, or $v \neq w$ and $\deg(v) < \Delta$. We construct a graph G' by adding a new vertex and connecting it with w. Now $G' \in \mathcal{D}$ and $D(G') = D(G) + \Delta - 2$, a contradiction. Finally, for each leaf u, the unique path between u and v in G must have length s because otherwise we can add a vertex w to G connected with u and the thus obtained graph G' also belongs to $\mathcal{D}$ and $D(G') = D(G) + \Delta - 2$, a contradiction.

Now observe that there is only one possibility left for G. It is easy to see that $e = k + k(\Delta - 1) + \ldots + k(\Delta - 1)^{s-1} = \frac{k}{\Delta - 2}((\Delta - 1)^s - 1)$ and, as G is a tree, we have that $D(G) = (e + 1)\Delta - 2e = e(\Delta - 2) + \Delta = k(\Delta - 1)^s$. □

Definition 7 *Let $G = (V, E)$ be a graph and let $A, B \subset V, A \cap B = \emptyset$. We define the* degree of A in B, *denoted by* $\deg(A, B)$, *as the number of vertices in B that are connected with vertices in A.*

Theorem 8 *Let G be a graph with $\Delta(G) = \Delta \geq 2$ and width$(G) \leq k$. Let $s \geq 1$. Then width$_s(G) \leq k(\Delta - 1)^{s-1}$.*

Proof We examine the nontrivial case when $s > 1$. As width$(G) \leq k$, there is a layout L such that width$(L) \leq k$. Consider L' as the layout of V obtained by reversing L. Let v be any vertex in L'. It is sufficient to prove that the

s-width of v cannot be more than $k(\Delta-1)^{s-1}$. Let A be the set of vertices not preceding v in L' that are connected with v via paths of vertices not preceding v_i and of length $\leq s-1$. Also let B be the set of vertices preceding v. Clearly $\mathrm{width}_s(v) = \deg(A,B)$. Using the notation $G_A = G[A]$, we can see that $\deg(A,B) \leq D(G_A) = \sum_{u \in V(G_A)}(\Delta - \deg(u))$ (the degree of u is taken with respect to G_A). Also notice that G_A is a connected graph with $\Delta(G) \leq \Delta$, vertex $v \in G_A$ has degree$\leq k$ and is connected with any vertex in $V(G_A)$ with a path of length $\leq s-1$. Now by Lemma 7 we have the required result. □

As $\mathrm{width}(G) \leq \Delta(G)$ we have the following corollary.

Theorem 9 *Let G be a graph where $2 \leq \Delta(G) \leq \Delta$, Let $s \geq 1$. Then $width_s(G) \leq \Delta(\Delta-1)^{s-1}$.*

The following result can easily be derived from Theorems 2, 4, and 9.

Theorem 10 *Let G be a graph such that $lc(G) \leq s+2$ and $\Delta(G) \leq \Delta$. Let $s \geq 1$. Then $treewidth(G) \leq \Delta(\Delta-1)^{s-1}$.*

Corollary 1 *Let $s \geq 1$, and Δ be fixed constants. Let $\mathcal{G}$ be the class of graphs with $lc(G) \leq s+2$, and $\Delta(G) \leq \Delta$. Then there exist:*

1. *A linear time algorithm that computes the treewidth of graphs in $\mathcal{G}$.*
2. *A polynomial time algorithm that computes the pathwidth of graphs in $\mathcal{G}$.*
3. *A $O(log^2 n)$ time parallel algorithm that computes the treewidth of graphs in $\mathcal{G}$ and that uses $O(n/log^2 n)$ processors on an EREW PRAM.*

Each of the above algorithms outputs the corresponding tree or path decomposition of minimum treewidth or pathwidth.

Proof Theorems 1 and 10 imply the first result. The second result follows from the result in [13] stating that for graphs with bounded treewidth, there is a polynomial time algorithm for pathwidth. The third result is obtained by combining the parallel algorithm given in [12] with Theorem 10. □

5 A separator theorem for k-chordal graphs with small width

In this chapter we will prove a separator theorem for s-chordal graphs with small width. We first give the following lemma about the high degree vertices of a graph with small width.

Lemma 8 *Let G be a graph where $width(G) \leq k$ and let V_d be the set of vertices that have degree $\geq d \geq k$. Then $|V_d| \leq \frac{2kn}{d}$.*

Proof Each vertex in V_d has degree at least d. Therefore $d|V_d| \leq \sum_{v \in V_d} \deg(v) \leq \sum_{v \in V} \deg(v) \leq 2|E|$. By Theorem 5 we have that $d|V_d| \leq k^2 - k + 2kn - 2k^2 \leq 2kn$ which completes the proof of the lemma. □

When the width of a graph is given, the following theorem provides an upper bound to width_s.

Theorem 11 *Let G be a graph G with width$(G) \leq k$. Then width$_s(G) \leq (2k)^{\frac{s-1}{s}} n^{\frac{s-1}{s}}$.*

Proof By Lemma 8 we have that if $d = \alpha n$, then there are at most $\frac{2k}{\alpha}$ vertices with degree $\geq \alpha n$ in G (α is a value to be chosen later). Let L be a layout of width $\leq k$ and V_{rich} be the set of vertices with degree at least αn. Notice that any vertex in L that is not in V_{rich} is connected with at most αn vertices in V_{rich}

We take a layout L' of G such that the vertices in V_{rich} are the $|V_{\text{rich}}|$ first vertices. We arrange the rest of the vertices (we call them *poor vertices*) following the reversed order of their arrangement in L. Clearly the width$_s$ of each of the first $\frac{2k}{\alpha}$ vertices in L is at most $\frac{2k}{\alpha}$.

Notice that any poor vertex v can be adjacent to at most k vertices not preceding it in L'. Following the notation of Theorem 8, we define A as the set of vertices not preceding v in L' that are connected with v via paths of vertices not preceding v_i of length $\leq s-1$ and denote $G_A = G[A]$. Also, let B be the set of vertices preceding v. Clearly width$_s(v) = \deg(A, B) \leq D(G_A) = \sum_{u \in V(G_A)} (\Delta - \deg(u))$ (the degree of u is taken with respect to G_A). If we observe that $\deg(v) \leq k$ in G_A and all the vertices in $V(G_A)$ have degree $< \alpha n$, then from Lemma 7, it follows that width$_s(v) \leq k(\alpha n)^{s-1}$. So width$_s(L') \leq \max\{\frac{2k}{\alpha}, k(\alpha n)^{s-1}\}$. We obtain the optimum value if we choose $\alpha = 2^{\frac{1}{s}} n^{-\frac{s-1}{s}}$, where we have that width$_s(G) \leq k(2n)^{\frac{s-1}{s}}$. □

Corollary 2 *If lc$(G) \leq s+2$ and width$(G) \leq k$, then treewidth$(G) \leq k(2n)^{\frac{s-1}{s}}$.*

We see that for a graph where lc$(G) \leq s+2$, we can find a layout L as in Theorem 11 that has width$_s(L) \leq (2k)^{\frac{s-1}{s}} n^{\frac{s-1}{s}}$. As treewidth$(G) \leq (2k)^{\frac{s-1}{s}} n^{\frac{s-1}{s}}$, from the proof of Theorem 4 it follows that reversing L, we can obtain an elimination ordering for G. As, when given an elimination ordering, a tree decomposition can be found in linear time (see [5]), this leads to a linear time algorithm that finds a $\frac{1}{2}$-separator in G with the required size. Thus we have the following theorem.

Theorem 12 *Let $G = (V, E)$ be a graph and $w : V \to R^+$ a function assigning a positive real weight to each vertex in V. Then if width$(G) \leq k$ and lc$(G) \leq s+2$, there is a linear time algorithm computing a $\frac{1}{2}$-separator of w in G that has size $\leq k(2n)^{\frac{s-1}{s}}$.*

6 Graphs with small chordality

In this section, we give small upper bounds for the chordality of some well known classes of graphs, thus enabling to apply results of the previous section to such graphs.

Definition 8 *A graph G is a* weakly chordal graph *iff neither G or G^c contain a chordless cycle of length ≥ 5.*

The class of weakly chordal graphs was introduced by Hayward in [24]. An immediate corollary is the following.

Lemma 9 *If G is weakly chordal then G, G^c are 4-chordal.*

We mention that the class of weakly chordal graphs is quite a large one, as it contains the classes of co-chordal graphs, chordal bipartite graphs, permutation graphs, trapezoid graphs, tolerance graphs, 2-threshold graphs and others (see also [18]).

It is known that for chordal bipartite graphs, treewidth is polynomially computable in time $O(e^3)$ (see [29]) and pathwidth is NP-complete [28].

Definition 9 *A* partial order *P on a set V is an irreflexive and transitive binary relation on V. If $a, b \in V$ and $(a, b) \in P$ then we write $a < b$ and call a, b* comparable*. If two elements of P are not comparable, then we call them* incomparable.

Definition 10 *A graph $G = (V, E)$ is a* comparability *graph if there exist a partial order P on V such that $\forall v, u \in V$, $\{v, u\} \in E$ iff $v < u$ or $u < v$ in P. A graph $G = (V, E)$ is a* cocomparability *graph if it is the complement of a comparability graph.*

Lemma 10 *If G is a cocomparability graph then G is 4-chordal.*

Proof Suppose that G contains a chordless cycle $C = (v_1, \ldots, v_l, v_1)$. of length $l \geq 5$. We denote the complement of G as G^c. Clearly G^c is a comparability graph. Hence, G^c does not contain an odd chordless cycle (see [18], page 43). If $l = 5$, then the complement of $G^c[\{v_1, \ldots, v_5\}]$ is also a cycle with five vertices, contradiction. If $l \geq 6$, then v_1, v_3, v_5 form a triangle in G^c, contradiction. Hence, G is 4-chordal. □

As mentioned above, treewidth and pathwidth are NP-complete when restricted to cocomparability graphs (see [23]). When additionally a degree restriction is put on the graphs, we have the following result, which can be derived directly from Theorem 10 and Lemmas 10 and 9.

Theorem 13 *For any constant Δ, there exist:*

1. *A linear time algorithm for computing the treewidth of cocomparability graphs or weakly chordal graphs, with maximum degree Δ.*

2. *An optimal parallel algorithm for computing the treewidth of cocomparability graphs or weakly chordal graphs, with maximum degree Δ.*

3. *A polynomial algorithm for computing the pathwidth of cocomparability graphs or weakly chordal graphs, with maximum degree Δ.*

Also, the next results follows from Theorem 12.

Corollary 3 *Let $G = (V, E)$ be a graph and let $w : V \to R^+$ be a function assigning a positive real weight to each vertex in V. If G is a 4-chordal graph and width(G) $\leq k$, then treewidth(G) $\leq k\sqrt{2n}$ and there is a linear time algorithm computing a $\frac{1}{2}$-separator of w in G of size at most $k\sqrt{2n}$.*

As an additional example of classes of graphs with a constant upper bound on the chordality, we mention the the graphs that are complements of r-partite graphs (graphs with chromatic number at most r): these do not have a chordless cycle of length more than $2r$.

It is easy to prove that any graph that does not contain a specific graph as a minor has constant bounded width. Therefore separator Theorem 12 straightforwardly extents the results of Lipton and Tarjan in [33, 34] and Alon, Seymour, and Thomas in [1] in the setting of 4-chordal graphs with small width. Moreover, Theorem 12 can give applications for any class of graphs where chordality and width are bounded.

We present below the application of our separator theorem to the problem of approximating the independent set problem on $(s + 2)$-chordal graphs with small width.

We examine the non trivial case where $s + 2 > 3$. Let G be a given $(s + 2)$-chordal graph where width$\leq k$ where k is a fixed constant. By repeatedly finding a separator as in Theorem 12 we can obtain the following immediate generalization of Theorem 3 in [34].

Proposition 1 *Let $s \geq 1$, k be constants. Let $\mathcal{G}$ be the set of $(s + 2)$-chordal graphs G with width(G) $\leq k$, given with a function $w : V \to R^+$ assigning weights to the vertices of G such that $\sum_{v \in V} w(v) = 1$. Then there is an $O(n \log n)$ algorithm, that when given an ϵ $0 < \epsilon \leq 1$, and a graph $G \in \mathcal{G}$, finds a set of at most $O(n^{\frac{s-1}{s}} \epsilon^{-\frac{1}{s}})$ vertices, whose removal leaves G with no connected component of total weight exceeding ϵ.*

Applying now the previous proposition with $\epsilon = \log n/n$ and giving each vertex weight $1/n$, we can find a set of vertices C of size $O(n/\log^{\frac{1}{s}} n)$, whose removal leaves no connected component with more than $\log n$ vertices. If now we apply exhaustive search to each connected component, we can find a collection of independent sets whose union is denoted as I. Let I^* be a maximum independent set in G. Following the same arguments as in [34] and taking in mind that if width(G) $\leq k$ then $|I^*| \geq \frac{n}{2k+1}$, we can conclude to the following:

Proposition 2 *Given an $(s+2)$-chordal graph G that has constant width, there is an $O(n^2)$ algorithm that finds an independent set I in G with relative error $\frac{|I^*|-|I|}{|I|} = O(1/\log^{\frac{1}{s}} n)$, where I^* is a maximum independent set.*

7 Discussion

From Theorem 6 it follows that, in a graph whose treewidth is small comparatively to the number of its vertices (e.g., treewidth(G) $= O(\sqrt{n})$), there exist also an equal size $\frac{1}{2}$-separator. Using this fact, it would be useful to determine classes of graphs where treewidth is small enough to provide a separator theorem.

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