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Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

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Karen Aardal
Department of Computer Science, Utrecht University
Utrecht, The Netherlands, aardal@cs.ruu.nl

Martine Labbé
Service de Mathématiques de la Gestion, Université Libre de Bruxelles
Bruxelles, Belgium, mlabbe@ulb.ac.be

Janny Leung
Management Information Systems Department, University of Arizona
Tucson, Arizona, USA, jleung@ventana.bpa.arizona.edu

Maurice Queyranne
Faculty of Commerce and Business Administration, University of British Columbia
Vancouver, British Columbia, Canada, queyranne@mtsg.ubc.ca

Abstract

We study the two-level uncapacitated facility location (TUFL) problem. Given two types of facilities, which we call y-facilities and z-facilities, the problem is to decide which facilities of both types to open, and to which pair of y- and z-facilities each client should be assigned, in order to satisfy the demand at maximum profit.

We first present two multi-commodity flow formulations of TUFL and investigate the relationship between these formulations and similar formulations of the one-level uncapacitated facility location (UFL) problem. In particular, we show that all nontrivial facets for UFL define facets for the two-level problem, and derive conditions when facets of TUFL are also facets for UFL. For both formulations of TUFL, we introduce new families of facets and valid inequalities and discuss the associated separation problems. We also characterize the extreme points of the LP-relaxation of the first formulation.

While the LP-relaxation of a multi-commodity formulation provides good bounds in general, the number of variables and constraints grows rapidly with the size of the problem instance. An alternative model of TUFL is a single-commodity fixed-charge network flow problem. Rardin and Wolsey showed that by projecting a so-called multi-commodity extended formulation of fixed-charge network flow problems onto the space of flow variables used in the weaker flow formulation, a broad class of valid inequalities can be obtained. We discuss a subclass of these inequalities for TUFL that seems particularly useful for computational purposes.

Subject classification: Facility Location: Discrete; Integer Programming: Cutting planes/Facets; Transportation: Location Models.

The two-level uncapacitated facility location (TUFL) problem involves two types of facilities, which we call y-facilities and z-facilities. Decisions have to be made simultaneously on which facilities of both types to open, and to which pair of y- and z-facility each client should be assigned. The objective is to maximize profit under the constraint that the demand of all clients has to be met.

The two-level problem is a natural extension of the well-known uncapacitated facility location (UFL) problem. This two-level problem can be used to model a hierarchical structure with a set of major facilities connected to minor facilities, which in turn are connected to clients. Many practical location situations have a clear two-level structure. For example, the distribution networks of many companies often involve major (central) as well as minor (regional) depots. The central facilities supply the regional ones and such shipment quantities are typically large, whereas each client is served from a regional depot where the transport is usually carried out using smaller vehicles. Other applications involve two types of non-hierarchical facilities. For example, in garbage collection, a truck travels from a depot to the client and then to a disposal plant, so each client is assigned to a truck depot and a disposal plant. In situations where spent products are recycled, each client have to be assigned to a supply facility and a recycling facility. Numerous other applications of TUFL exist within areas such as telecommunication and computer network design. Some applications are described in Barros and Labbé (1992).

Very little is known about the structural properties of TUFL, except that it is an NP-hard problem, since it generalizes UFL which is NP-hard (see Cornuéjols, Fisher and Nemhauser, 1977, and Cornuéjols, Nemhauser and Wolsey, 1990). Only a few algorithms for TUFL have been developed. Kaufman et al. (1977) developed a branch-and-bound algorithm generalizing an algorithm for UFL developed by Efroymson and Ray (1966). Tcha and Lee (1984) also developed a branch-and-bound method where a dual ascent procedure, similar to the one developed by Erlenkotter (1978), is used together with a primal descent method to get good lower and upper bounds. Barros and Labbé (1992) studied various formulations of a problem related to TUFL, and suggested a Lagrangean relaxation and a primal heuristic to derive bounds in a branch-and-bound algorithm. For the capacitated two-level problem, Aardal (1992) studied the cutting plane approach. Balakrishnan and Graves (1989) consider a generalization of TUFL and develop algorithms for finding good lower and upper bounds on the optimal solution value.

We investigate some structural properties of TUFL. In Section 1, we introduce the necessary notation and a multi-commodity flow formulation of TUFL, and characterize the extreme points of its LP-relaxation. We investigate the relationship between TUFL and UFL in Section 2. Since UFL is a relaxation of TUFL, any inequality valid for UFL is also valid for TUFL. We show that certain inequalities that define facets for TUFL induce facets for UFL, and that basically all facet-defining inequalities for UFL can be extended to facet-defining inequalities for TUFL. Consequently, we obtain several classes of facets for TUFL from previously-known facet-defining inequalities for UFL (see Cho et al., 1983a,b,

Cornuéjols and Thizy, 1982, Guignard, 1980, and Cornuéjols et al., 1977).

In Section 3, we introduce two sets of constraints to the multi-commodity flow formulation which ensure that a facility is not open unless a client is assigned to it. We present new families of facets and valid inequalities for both formulations, and discuss the corresponding separation problems. If we consider UFL with the extra constraints that a facility cannot be opened unless a client is assigned to it, we show in Section 4 that almost all facets of this variant of UFL can be extended to facets for the corresponding version of TUFL.

In Section 5, we present a single-commodity flow formulation of TUFL. This formulation provides a weaker LP-bound than the multi-commodity flow formulation, but may for large instances be computationally more tractable, as it contains substantially fewer variables and constraints. In order to strengthen the flow formulation, we consider a subclass of a general family of inequalities called dicut-collection inequalities, developed by Rardin and Wolsey (1993). The general class is obtained by projecting a so-called multi-commodity extended formulation onto the space of the variables of the flow formulation.

1 A Multi-Commodity Formulation of TUFL

Here, we introduce a formulation of the two-level uncapacitated facility location problem that is a natural extension of the one-level facility location problem. This formulation can be viewed as a multi-commodity fixed-charge network flow problem.

Let I, J, and K denote the sets of z-facilities, y-facilities and clients respectively. Let $m = |I|$, $n = |J|$ and $q = |K|$. Furthermore, let

$$z_i = \begin{cases} 1 & \text{if z-facility } i \text{ is open,} \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if y-facility } j \text{ is open,} \\ 0 & \text{otherwise, and} \end{cases}$$

x_{ijk} = fraction of demand of client k that is served by z-facility i and y-facility j.

We want to determine which z- and y-facilities should be opened and to which pair of facilities each client should be assigned so as to maximize profit under the constraint that all demand has to be satisfied. To avoid trivial cases, we henceforth assume that m, n and q are all at least equal to 2.

The two-level facility location problem can be modeled as:

$$\text{(MC):} \qquad \max \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ijk}x_{ijk} - \sum_{j\in J} f_j y_j - \sum_{i\in I} g_i z_i$$

subject to

$$\sum_{i\in I}\sum_{j\in J} x_{ijk} = 1 \quad \text{for all } k \in K, \tag{1}$$

$$\sum_{j\in J} x_{ijk} \leq z_i \quad \text{for all } i \in I,\ k \in K, \tag{2}$$

$$\sum_{i\in I} x_{ijk} \leq y_j \quad \text{for all } j \in J,\ k \in K, \tag{3}$$

$$0 \leq z_i \leq 1 \quad \text{for all } i \in I, \tag{4}$$

$$0 \leq y_j \leq 1 \quad \text{for all } j \in J, \tag{5}$$

$$x_{ijk} \geq 0 \quad \text{for all } i \in I,\ j \in J,\ k \in K, \tag{6}$$

$$z_i, y_j \text{ integer valued} \quad \text{for all } i \in I,\ j \in J, \tag{7}$$

where c_{ijk} is the profit of supplying the demand of client k from z-facility i and y-facility j, f_j and g_i are the fixed costs of opening y-facility j and z-facility i respectively.

Model MC is sufficient even for the case when the demand of each client is not allowed to be split. Because there are no capacity constraints on the depots and facilities, the problem will always have an optimal solution where the x_{ijk}'s are integer-valued. Let

$$X_{\text{MC}} = \{(x,y,z) \in R^{mnq+m+n} : (x,y,z) \text{ satisfies } (1)-(7)\},$$

$$\bar{X}_{\text{MC}} = \{(x,y,z) \in R^{mnq+m+n} : (x,y,z) \text{ satisfies } (1)-(6)\},$$

and let $P_{\text{MC}} = \text{conv}(X_{\text{MC}})$. An alternative model of TUFL is obtained by considering each pair (i,j) of z- and y-facilities as a single uncapacitated facility represented by the 0-1-variable w_{ij}, thus obtaining an UFL-type model. The fixed charges are correctly accounted for by introducing variables z_i, $i \in I$, and y_j, $j \in J$, and the forcing constraints

$$w_{ij} \leq z_i, \text{ and } w_{ij} \leq y_j, \text{ for all } i \in I,\ j \in J.$$

This model allows the results for UFL to be applied directly, at the expense of introducing an additional $m \times n$ 0-1-variables.

Below we derive the dimension of P_{MC} and determine which of the defining inequalities are facet defining. The proofs of Propositions 1 and 2 (part c) are straightforward and are omitted. We defer the proofs of the rest of Proposition 2 to Section 2.

Proposition 1 *The dimension of the polytope P_{MC} is $mnq + m + n - q$.*

Proposition 2

(a) *For any* $j \in J$ *and* $k \in K$, $\sum_{i \in I} x_{ijk} \leq y_j$ *defines a facet of* P_{MC}.

(b) *For any* $i \in I$ *and* $k \in K$, $\sum_{j \in J} x_{ijk} \leq z_i$ *defines a facet of* P_{MC}.

(c) *For any* $i \in I$, $j \in J$ *and* $k \in K$, $x_{ijk} \geq 0$ *defines a facet of* P_{MC}.

(d) *For any* $j \in J$, $y_j \leq 1$ *defines a facet of* P_{MC}.

(e) *For any* $i \in I$, $z_i \leq 1$ *defines a facet of* P_{MC}.

1.1 Extreme Points of $\bar{X}_{\mathrm{MC}}$

The next proposition gives a characterization of the extreme points of the LP-relaxation $\bar{X}_{\mathrm{MC}}$. Let (x, y, z) be a point in $\bar{X}_{\mathrm{MC}}$. We define the sets

$$\begin{aligned} I_1 &= \{i : 0 < z_i < 1\}, \\ J_1 &= \{j : 0 < y_j < 1\}, \text{ and} \\ K_1 &= \{k : \text{ there exists an } (i,j) \text{ pair such that } 0 < x_{ijk} < 1\}. \end{aligned}$$

For each $k \in K_1$, we define the sets

$$\begin{aligned} \tilde{I}_k &= \{i : 0 < \sum_{j \in J} x_{ijk} \leq z_i < 1 \text{ or } 0 < \sum_{j \in J} x_{ijk} < z_i = 1\}, \\ \text{and } \tilde{J}_k &= \{j : 0 < \sum_{i \in I} x_{ijk} \leq y_j < 1 \text{ or } 0 < \sum_{i \in I} x_{ijk} < y_j = 1\}. \end{aligned}$$

and the directed graph $G_k = (N_k, A_k)$ where $N_k = \{s, t, d\} \cup \tilde{I}_k \cup \tilde{J}_k$ and

$$\begin{aligned} A_k = \; & \{(s,i) : i \in \tilde{I}_k \cap I_1\} \\ & \cup \{(i,j) : x_{ijk} > 0, i \in \tilde{I}_k, j \in \tilde{J}_k\} \\ & \cup \{(j,t) : j \in \tilde{J}_k \cap J_1\} \\ & \cup \{(d,i) : \sum_{j \in J} x_{ijk} < z_i, i \in \tilde{I}_k\} \\ & \cup \{(j,d) : \sum_{i \in I} x_{ijk} < y_j, j \in \tilde{J}_k\}. \end{aligned}$$

The arcs incident to the dummy node d indicate which of the constraints (2) and (3) are slack for client k. An arc incident to the dummy source node s indicates that the corresponding z-variable is fractional; similarly, the arcs (j,t) indicate that y_j is fractional.

Example 1

Consider the following point in $\bar{X}_{\mathrm{MC}}$ for $I = J = \{1,2,3\}$ and $K = \{1,2\}$:

$$z_1 = z_2 = \frac{1}{3}, z_3 = 1, \qquad y_1 = y_2 = \frac{1}{3}, y_3 = 1,$$
$$x_{131} = x_{211} = x_{321} = \frac{1}{3}, \quad x_{122} = x_{232} = x_{312} = \frac{1}{3}, \qquad \text{all other } x_{ijk} = 0.$$

For this point

$$I_1 = J_1 = \{1,2\}, \quad \tilde{I}_1 = \tilde{J}_1 = \{1,2,3\}, \quad \tilde{I}_2 = \tilde{J}_2 = \{1,2,3\},$$

and the graphs G_1 and G_2 are shown in Figure 1.

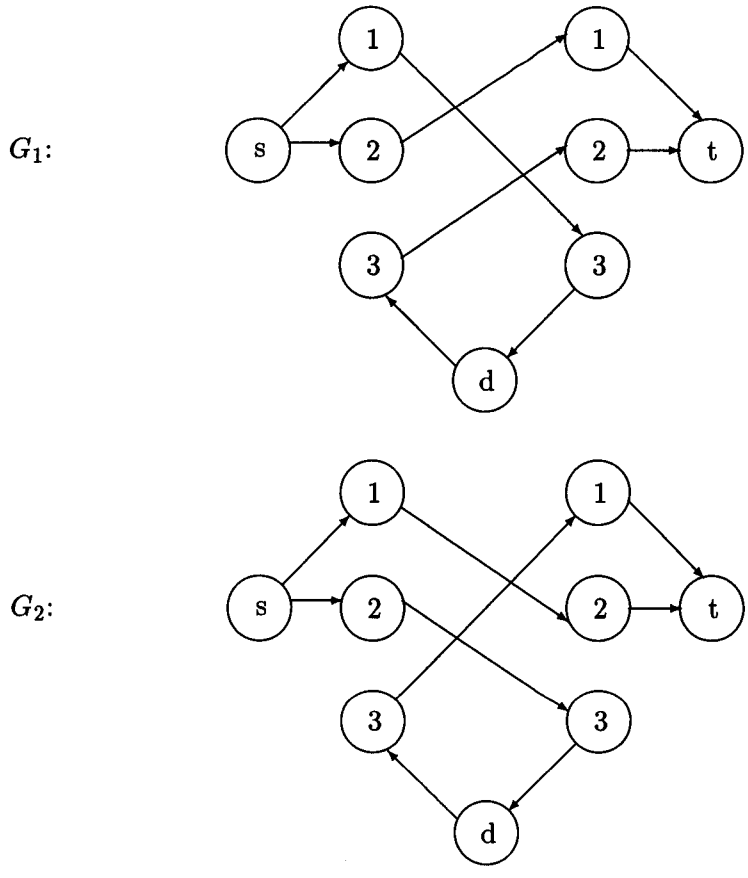

Figure 1: The graphs G_1 and G_2 for Example 1.

We call $f \in \mathbf{R}^{A_k}$ a *pseudo-feasible* flow for G_k if flow balance is maintained at each node in G_k except node d; that is, for each node $v \in N_k \setminus \{d\}$,

$$\sum_{u \in N_k:(u,v) \in A_k} f_{(u,v)} = \sum_{w \in N_k:(v,w) \in A_k} f_{(v,w)}.$$

Note that we allow the flow value for an arc to be either positive or negative. Let $\hat{G}_k$ be the subgraph of G_k induced by the node-set $\tilde{I}_k \cup \tilde{J}_k \cup \{d\}$.

Theorem 3 *$(x, y, z) \in \bar{X}_{\mathrm{MC}}$ is an extreme point of $\bar{X}_{\mathrm{MC}}$ if and only if*

(a) *for all $i \in I_1$, $z_i = \max_k \sum_{j \in J} x_{ijk}$,*

(b) *for all $j \in J_1$, $y_j = \max_k \sum_{i \in I} x_{ijk}$,*

(c) *for any k in K_1, the graph $\hat{G}_k$ does not contain any even undirected circuit,*

(d) *there is no non-zero vector $(\alpha', \beta') \in \mathbf{R}^{I_1} \times \mathbf{R}^{J_1}$ such that:*

$$\begin{aligned} &\text{for each } k \in K_1, \text{ there is a pseudo-feasible flow } f^k \text{ for } G_k \text{ with } f^k_{(s,i)} = \\ &\alpha'_i \text{ for all } i \in \tilde{I}_k \cap I_1 \text{ and } f^k_{(j,t)} = \beta'_j \text{ for all } j \in \tilde{J}_k \cap J_1. \end{aligned} \tag{8}$$

The vector (α', β') specifies a set of flow-values for the arcs incident to the source node s and sink node t that is held fixed for *all* the graphs $G_k, k \in K_1$. Condition (8) says that for *each* $k \in K_1$, we can assign flow-values to the remaining arcs in G_k to give a pseudo-feasible flow.

This result generalizes an analogous result for the uncapacitated facility location problem in Cornuéjols, Fisher and Nemhauser (1977). Since the proof is quite lengthy it is deferred to the Appendix.

2 Adapting Facets from the One-Level Uncapacitated Facility Location Problem

The set of feasible solutions to the uncapacitated facility location problem (UFL) is defined as:

$$\begin{aligned} X_{\mathrm{UFL}} = \{(x, y) \in R^{nq+n} : \sum_{j \in J} x_{jk} &= 1 && \text{for all } k \in K, \\ x_{jk} &\leq y_j && \text{for all } j \in J,\ k \in K, \\ x_{jk} &\geq 0 && \text{for all } j \in J,\ k \in K, \end{aligned}$$

$$\begin{array}{rll} 0 \le y_j & \le 1 & \text{for all } j \in J, \\ y_j \text{ integer} & \text{valued} & \text{for all } j \in J, \} \end{array}$$

where x_{jk} is the fraction of client k's demand served by facility j, and where $y_j = 1$ if facility j is open, and $y_j = 0$ otherwise. As before, we assume that n and q are both at least equal to 2. Let $P_{\text{UFL}} = \text{conv}(X_{\text{UFL}})$.

Proposition 4 (Cornuéjols, Nemhauser and Wolsey, 1990)
The dimension of P_{UFL} is $nq + n - q$.

Below we show that all nontrivial facets of UFL are also facets for P_{MC}, thus providing us with a large collection of facets for TUFL.

Theorem 5 *Let*

$$\sum_{j \in J} \sum_{k \in K} a_{jk} x_{jk} + \sum_{j \in J} b_j y_j \le d \tag{9}$$

represent a facet of P_{UFL} that is not a non-negativity constraint for some x_{jk}. Then

$$\sum_{j \in J} \sum_{k \in K} a_{jk} \sum_{i \in I} x_{ijk} + \sum_{j \in J} b_j y_j \le d$$

represents a facet of P_{MC}.

Proof. Let $\mathcal{F} \equiv \{(x, y, z) \in P_{\text{MC}} : \sum_{j \in J} \sum_{k \in K} a_{jk} \sum_{i \in I} x_{ijk} + \sum_{j \in J} b_j y_j = d\}$, and let $\alpha x + \beta y + \gamma z \le \delta$ be a valid inequality for P_{MC} such that for all $(x, y, z) \in \mathcal{F}$,

$$\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \alpha_{ijk} x_{ijk} + \sum_{j \in J} \beta_j y_j + \sum_{i \in I} \gamma_i z_i = \delta \tag{10}$$

Let $\mathcal{F}_1 \equiv \{(x, y) \in P_{\text{UFL}} : \sum_{j \in J} \sum_{k \in K} a_{jk} x_{jk} + \sum_{j \in J} b_j y_j = d\}$. We will show that there exists $\theta \in \mathbb{R}^1$ and $\eta_k \in \mathbb{R}^1, k = 1, \ldots, q$ such that

$$\begin{array}{rcll} \alpha_{ijk} & = & \theta a_{jk} + \eta_k & \text{for all } i \in I, j \in J, k \in K, \\ \beta_j & = & \theta b_j & \text{for all } j \in J, \\ \gamma_i & = & 0 & \text{for all } i \in I, \\ \delta & = & \theta d + \sum_{k \in K} \eta_k. & \end{array}$$

By Theorem 3.6 in Section I.4 of Nemhauser and Wolsey (1988), this would imply that the inequality $\alpha x + \beta y + \gamma z \le \delta$ defines a facet of P_{MC}.

First, let (x, y) be a point in $\mathcal{F}_1$. From this point, we can construct two points $(\bar{x}, y, \bar{z})$, $(\bar{x}, y, \tilde{z}) \in \mathcal{F}$ as follows:

- $\bar{z}_1 = 1, \bar{z}_i = 0$ for $i \neq 1$,
- $\tilde{z}_1 = 1, \tilde{z}_2 = 1, \tilde{z}_i = 0$ for $i \neq 1, 2$,
- the y_j-values are unchanged for all $j \in J$,
- for all $j \in J$ and $k \in K$, $\bar{x}_{1jk} = x_{jk}, \bar{x}_{ijk} = 0$ for $i \neq 1$.

These two points correspond to allocating all clients to z-facility 1. For the first point, only z-facility 1 is open; both z-facilities 1 and 2 are open for the second point. By evaluating (10) at the above points we obtain

$$\begin{aligned} \sum_{j \in J} \sum_{k \in K} \alpha_{1jk} x_{jk} + \sum_{j \in J} \beta_j y_j + \gamma_1 &= \delta, \\ \sum_{j \in J} \sum_{k \in K} \alpha_{1jk} x_{jk} + \sum_{j \in J} \beta_j y_j + \gamma_1 + \gamma_2 &= \delta. \end{aligned}$$

Hence, $\gamma_2 = 0$, and by symmetry, $\gamma_i = 0$ for all $i \in I$.

From now on, all the points $(x, y, z) \in \mathcal{F}$ that we consider are such that $z_i = 1$, for all i. Since we assume that (9) is not a nonnegativity constraint then, for any y-facility j' and any client k', there is a point in $\mathcal{F}_1$ with $x_{j'k'} = 1$. For simplicity, let (x, y) be a point in $\mathcal{F}_1$ with $x_{11} = 1$. From this point we can construct m points, $(x^l, y, \mathbf{1}) \in \mathcal{F}$, for $l = 1, \ldots, m$, as follows:

- the y_j-values are unchanged for all $j \in J$,
- $x^l_{i11} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{otherwise,} \end{cases}$ for all $i \in I$,

  $x^l_{ij1} = 0,$ for all $i \in I, j \in J, j \neq 1$,

  $x^l_{ijk} = \begin{cases} x_{jk} & \text{if } i = m, \\ 0 & \text{otherwise,} \end{cases}$ for all $j \in J, k \in K, k \neq 1$.

In the l-th point, client 1 is allocated to y-facility 1 and z-facility l. All other clients are allocated to the same y-facilities as indicated by (x, y), and to the last z-facility m. By evaluating (10) for $(x^1, y, \mathbf{1}), (x^2, y, \mathbf{1}), \ldots, (x^m, y, \mathbf{1})$ we obtain for $l = 1, \ldots, m$:

$$\alpha_{l11} + \sum_{j \in J} \sum_{\substack{k \in K \\ k \neq 1}} \alpha_{mjk} x_{jk} + \sum_{j \in J} \beta_j y_j = \delta.$$

Hence, α_{l11} is equal to some constant α_{11} for $l = 1, \ldots, m$. By symmetry, we obtain that for all $j \in J$ and $k \in K$, $\alpha_{ijk} = \alpha_{jk}$ for all $i \in I$.

Finally, let (x, y) be any point in $\mathcal{F}_1$ and construct the point $(x, y, \mathbf{1}) \in \mathcal{F}$ as follows:

- the y_j-values are unchanged for all $j \in J$,
- $x_{1jk} = x_{jk}$, for all $jinJ, k \in K$,
  $x_{ijk} = 0$, for all $i \in I, i \neq 1, j \in J, k \in K$.

Here, all clients are assigned to z-facility 1. Evaluating (10) at $(x, y, \mathbf{1})$ gives

$$\sum_{j \in J} \sum_{k \in K} \alpha_{jk} x_{jk} + \sum_{j \in J} \beta_j y_j = \delta.$$

Since this holds for any point in $\mathcal{F}_1$ and since $\mathcal{F}_1$ is a facet of P_{UFL}, we know that there exists $\theta \in \mathbb{R}^1$ with $\theta > 0$, and $\eta_k \in \mathbb{R}^1$ such that

$$\begin{aligned} \alpha_{jk} &= \theta a_{jk} + \eta_k \qquad \text{for all } j \in J, k \in K, \\ \beta_j &= \theta b_j \qquad \text{for all } j \in J, \\ \delta &= \theta d + \sum_{k \in K} \eta_k, \end{aligned}$$

which completes the proof. ∎

Next, we show that some facets of the polytope for the two-level problem also induce facets for the one-level problem. More precisely, a facet of P_{MC} is also facet-defining for P_{UFL} as long as the coefficients of all z_i-variables are equal to zero, and as long as the coefficients of the variables x_{ijk} are independent of i.

Theorem 6 *Every facet of P_{MC} of the form*

$$\sum_{j \in J} \sum_{k \in K} a_{jk} \sum_{i \in I} x_{ijk} + \sum_{j \in J} b_j y_j \leq d, \tag{11}$$

where the coefficient of z_i is zero, for all $i \in I$, and the coefficient of x_{ijk} is independent of i for all $i \in I$, $j \in J$, $k \in K$, induces a facet for P_{UFL}.

Proof. Let p denote the dimension of P_{MC}. If (11) represents a facet of P_{MC}, then we can construct a $p \times (mnq + m + n + 1)$ matrix of rank p of the form $[\mathbf{M}|\mathbf{1}]$ where the elements in the last column are all 1's, and each row of $\mathbf{M}$ is the vector defining a point in P_{MC} lying on the facet represented by (11). Since the rank of a matrix is unchanged by column operations, the rank of this matrix is unchanged if, for each j and each k, the column corresponding to the variable x_{1jk} is replaced by the sum of the columns corresponding to x_{ijk} for $i = 1, \ldots, m$.

Now consider a submatrix $[\mathbf{M}'|\mathbf{1}]$ obtained by deleting the columns corresponding to z_i for $i = 1, \ldots, m$ and x_{ijk} for $i = 2, \ldots, m$, $j = 1, \ldots, n$, and $k = 1, \ldots, q$. The rank of this matrix is at least $p - (nq(m-1) + m) = nq + n - q$, since $p = mnq + m + n - q$. Each row of this submatrix corresponds to a point in X_{UFL} on the facet defined by (9). ∎

Since the inequalities

$$\begin{aligned} x_{jk} &\leq y_j \qquad \text{for all } j \in J, k \in K, \\ \text{and} \quad y_j &\leq 1 \qquad \text{for all } j \in J, \end{aligned}$$

are facet-defining for P_{UFL} (See Cornuéjols, Nemhauser and Wolsey, 1990), applying Theorem 5 proves parts (a) and (d) of Proposition 2. Furthermore, since the roles of the z- and the y-facilities are symmetric, an analogous collection of facets can be derived by interchanging the roles of the indices i and j, and correspondingly the variables z_i and y_j, thus proving parts (b) and (e) of Proposition 2.

In addition to identifying the trivial facets of P_{MC}, Theorem 5 can be exploited more generally to yield many more facet-defining inequalities for P_{MC}. Cornuéjols et al. (1977), Guignard (1980), Cornuéjols and Thizy (1982) introduced several classes of facets for P_{UFL}. Later, Cho et al. (1983a,b) developed a general class of facet-defining inequalities that subsumes all previously-known facets of P_{UFL}. By applying Theorem 5 to these results, we obtain a large collection of facets for P_{MC} involving the clients and the y-facilities, and an analogous collection of facets for P_{MC} involving the clients and the z-facilities.

3 New Facets for TUFL

Here we introduce two new classes of facet-defining inequalities for TUFL. We first describe, in Section 3.1, facets of P_{MC}, and introduce, in Section 3.2, facets of a slightly modified polytope. Both classes involve either z- and x-variables or y- and x-variables. Since the roles of z- and y-facilities are interchangeable, such classes of facets always come in pairs.

3.1 New Facets of P_{MC}

3.1.1 The Valid Inequalities

Consider p clients, and assume that precisely one of the p clients is assigned to every y-facility. If only one y-facility, say j_1 is open, then all clients, including the designated client $k(j_1)$, have to be assigned to facility j_1, since all demand must be met, leading to the following valid inequality.

Proposition 7 *Assume that $q \geq n$ and let $S = \{k(1), k(2), \ldots, k(n)\} \subseteq K$ be any sequence of n distinct clients. Then*

$$\sum_{j \in J} (y_j + \sum_{i \in I} x_{ijk(j)}) \geq 2 \tag{12}$$

is valid for X_{MC}.

Theorem 8 *If $q \geq n \geq 3$, then inequality (12) defines a facet of P_{MC}.*

Proof. We will prove that (12) represents a facet using the indirect method. Let $\alpha x + \beta y + \gamma z \leq \delta$ be valid for P_{MC}, and assume that

$$\begin{aligned} \mathcal{F} &\equiv \{(x,y,z) \in P_{\mathrm{MC}} : \sum_{j\in J}(y_j + \sum_{i\in I} x_{ijk(j)}) = 2\} \\ &\subseteq \{(x,y,z) \in P_{\mathrm{MC}} : \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \alpha_{ijk}x_{ijk} + \sum_{j\in J}\beta_j y_j + \sum_{i\in I}\gamma_i z_i = \delta\}. \end{aligned}$$

To prove that inequality (12) defines facets of P_{MC} when $q \geq n \geq 3$, we need to show that

$$\begin{aligned} \alpha_{ijk} &= -\theta + \eta_k && \text{for all } i \in I, j \in J, k(j) \in S, \\ \alpha_{ijk} &= \eta_k && \text{for all } i \in I, j \in J, k \in K, k \neq k(j), \\ \beta_j &= -\theta && \text{for all } j \in J, \\ \gamma_i &= 0 && \text{for all } i \in I, \\ \delta &= -2\theta + \sum_{k\in K} \eta_k \end{aligned}$$

for some constants $\theta > 0$ and $\eta_k,\ k \in K$.

Choose arbitrarily one z-facility i_1, two y-facilities j_1, j_2 and consider the following tight point in $\mathcal{F}$ with:

$$\begin{aligned} z_{i_1} &= 1 \\ y_{j_1} = y_{j_2} &= 1 \\ x_{i_1 j_1 k(j_2)} &= 1 \\ x_{i_1 j_2 k(j_1)} &= 1 \\ x_{i_1 j_1 k} &= 1 \qquad \text{for all } k \in K, k \neq k(j_1) \text{ or } k(j_2), \end{aligned}$$

and all other elements with value zero. We refer to the above point as *point 1.* Consider a second point with $z_{i_2} = 1$, $i_2 \neq i_1$, and other elements as in point 1. By evaluating the expression

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \alpha_{ijk}x_{ijk} + \sum_{j\in J}\beta_j y_j + \sum_{i\in I}\gamma_i z_i = \delta \tag{13}$$

at these two points, we obtain $\gamma_{i_2} = 0$. Varying over all possible choices of i_2 gives $\gamma_i = 0$ for all $i \in I$, $i \neq i_1$ and since i_1 was chosen arbitrarily we get

$$\gamma_i = 0 \text{ for all } i \in I.$$

Next, consider the modification of point 1 where $z_{i_2} = 1$ as well for $i_2 \neq i_1$, and where one arbitrarily chosen client k_1 is assigned to z-facility i_2. Evaluating (13) at this point and at point 1 gives $\alpha_{i_1 j_1 k_1} = \alpha_{i_2 j_1 k_1}$. Since i_1, j_1 and k_1 were chosen arbitrarily we have

$$\alpha_{ijk} = \alpha_{jk}, \text{ for all } i \in I, j \in J, k \in K.$$

We now consider point 1 and choose an arbitrary client k_1, different from clients $k(j_1), k(j_2)$, and we re-assign client k_1 to y-facility j_2. Evaluate (13) at the new point and at point 1, and compare the expressions. This gives $\alpha_{j_1 k_1} = \alpha_{j_2 k_1}$. Knowing that j_1 and j_2 were arbitrarily chosen and that k_1 is any client different from client $k(j_1)$ and $k(j_2)$ yields

$$\alpha_{jk} = \eta_k \text{ for all } j \in J, k \in K, \ k \neq k(j). \tag{14}$$

Next consider a point in $\mathcal{F}$ with the same structure as point 1, but where a y-facility $j_3 \neq j_1$ or j_2 takes the role of y-facility j_2, i.e.

$$\begin{array}{rcll} z_{i_1} & = & 1 & \\ y_{j_1} = y_{j_3} & = & 1 & \\ x_{i_1 j_1 k(j_3)} & = & 1 & \\ x_{i_1 j_3 k(j_1)} & = & 1 & \\ x_{i_1 j_1 k} & = & 1 & \text{for all } k \in K, k \neq k(j_1) \text{ or } k(j_3), \end{array}$$

and all other variables are equal to zero. By evaluating (13) at the above point and at point 1, we get $\alpha_{j_2 k(j_1)} - \alpha_{j_3 k(j_1)} + \beta_{j_2} - \beta_{j_3} = 0$. Since $\alpha_{j_2 k(j_1)} = \alpha_{j_3 k(j_1)} = \eta_{k(j_1)}$ we have $\beta_{j_2} = \beta_{j_3}$, and as j_2 and j_3 can be chosen arbitrarily we obtain

$$\beta_j = -\theta \text{ for all } j \in J, \tag{15}$$

and since (12) is not an improper face we must have $\theta > 0$.

Finally, consider the following point in $\mathcal{F}$ that we refer to as *point 2*. Choose i_1 and j_1 as in point 1, and let

$$\begin{array}{rcll} z_{i_1} & = & 1 & \\ y_{j_1} & = & 1 & \\ x_{i_1 j_1 k(j_1)} & = & 1 & \\ x_{i_1 j_1 k(j_2)} & = & 1 & \\ x_{i_1 j_1 k} & = & 1 & \text{for all } k \in K, k \neq k(j_1) \text{ or } k(j_2). \end{array}$$

Comparing point 1 to point 2 and using (14), and (15) gives $\alpha_{j_1 k(j_1)} = -\theta + \eta_{k(j_1)}$. Given that j_1 is an arbitrary y-facility we obtain

$$\alpha_{jk} = -\theta + \eta_{k(j)} \text{ for all } j \in J, \text{ and } k(j) \in S.$$

Evaluating (13) at any point in $\mathcal{F}$ gives

$$\delta = -2\theta + \sum_{k \in K} \eta_k,$$

which concludes our proof. ∎

Corollary 9 *Assume that $q \geq m$ and let $S = \{k(1), k(2), \ldots, k(n)\} \subseteq K$ be any sequence of n distinct clients. Then inequality*

$$\sum_{i \in I} (z_i + \sum_{j \in J} x_{ijk(j)}) \geq 2 \tag{16}$$

is valid for X_{MC} and defines a facet of P_{MC} if $q \geq m \geq 3$.

3.1.2 Separation

Inequality (12), and similarly (16), can be separated by solving a minimum-weight maximum-cardinality matching problem. For any point $(\bar{x}, \bar{y}, \bar{z})$, we consider the complete bipartite graph $G = (J \cup K, J \times K)$ with edge-weights

$$w_{jk} = \bar{y}_j + \sum_{i \in I} \bar{x}_{ijk} \qquad \forall\, (j,k) \in J \times K.$$

Let the minimum-weight matching among all matchings of cardinality n be

$$M = \{(1, k(1)), (2, k(2)), \ldots, (n, k(n))\}.$$

If the total weight of the matching M is less than 2, then inequality (12), with the sequence S as defined by the edges of the matching M above, is a valid inequality that cuts off the point $(\bar{x}, \bar{y}, \bar{z})$.

3.2 New Facets for a Modified Formulation

The formulation MC allows a facility to be open even when no clients are assigned to it, whereas in many applications we may require that a facility be closed when not used. When all the fixed costs are positive, the objective function ensures that this will be the case. For a more general cost structure, it may be necessary to explicitly model this requirement, thereby obtaining a different formulation of TUFL which we shall refer to as formulation MC′.

Formulation MC′ is obtained by adding the following constraints to formulation MC:

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} \geq z_i \quad \text{for all } i \in I, \tag{17}$$

$$\sum_{i\in I}\sum_{k\in K} x_{ijk} \geq y_j \quad \text{for all } j \in J. \tag{18}$$

Constraints (17) and (18) ensure that no facility is opened if no client is assigned to it. For inequalities (17) and (18) to be valid we need to explicitly state that the demand should not be split between clients. In formulation MC′ we therefore replace constraint (7) by

$$x_{ijk}, y_j, z_i \text{ integer valued for all } i \in I,\ j \in J,\ k \in K. \tag{7′}$$

Let

$$X_{\mathrm{MC}'} = \{(x,y,z) \in R^{mnq+m+n} : (x,y,z) \text{ satisfies } (1)-(6), (7'),\ (17)-(18)\}.$$

Analogous to the the definitions of $\bar{X}_{\mathrm{MC}}$ and P_{MC} we define $\bar{X}_{\mathrm{MC}'}$ and $P_{\mathrm{MC}'}$ as

$$\bar{X}_{\mathrm{MC}'} = \{(x,y,z) \in R^{mnq+m+n} : (x,y,z) \text{ satisfies } (1)-(6),\ (17)-(18)\}.$$

and

$$P_{\mathrm{MC}'} = \mathrm{conv}(X_{\mathrm{MC}'}).$$

3.2.1 Dimension and Trivial Facets of $P_{\mathrm{MC}'}$

Here, we state the results on the dimension and trivial facets for $P_{\mathrm{MC}'}$. Since the proof techniques used for these results are standard they are omitted here.

Proposition 10 *The dimension of the polytope $P_{\mathrm{MC}'}$ is*

(a) $mnq + n + n - q - 2$ *if* $m = n = q = 2$,

(b) $mnq + m + n - q - 1$ *if* $q = 2$ *and* either $m \geq 3,\ n = 2$ or $m = 2,\ n \geq 3$,

(c) $mnq + m + n - q$ *in all other cases.*

In the cases when $P_{\mathrm{MC}'}$ is not full-dimensional, all points in $P_{\mathrm{MC}'}$ satisfy the following equations in addition to the demand constraints (1):

$$\text{If } n = 2 \text{ and } q = 2: \qquad y_1 - \sum_{i\in I}\sum_{k\in K} x_{i1k} = y_2 - 1. \tag{19}$$

$$\text{If } m = 2 \text{ and } q = 2: \qquad z_1 - \sum_{j\in J}\sum_{k\in K} x_{1jk} = z_2 - 1. \tag{20}$$

The trivial facets of parts (a), (b) and (c) of Proposition 2 are also facet defining with respect to $P_{\mathrm{MC}'}$. In addition the following holds:

Proposition 11

(a) *If $q \geq 3$, $z_i \leq 1$ represents a facet of $P_{\mathrm{MC}'}$ for all $i \in I$.*

(b) *If $q \geq 3$, $y_j \leq 1$ represents a facet of $P_{\mathrm{MC}'}$ for all $j \in J$.*

(c) *For any $j \in J$, $\sum_{i \in I} \sum_{k \in K} x_{ijk} \geq y_j$ defines a facet of $P_{\mathrm{MC}'}$ unless $n = 2$ and $q \geq 3$.*

(d) *For any $i \in I$, $\sum_{j \in J} \sum_{k \in K} x_{ijk} \geq z_i$ defines a facet of $P_{\mathrm{MC}'}$ unless $m = 2$ and $q \geq 3$.*

Note that none of the inequalities $x_{ijk} \leq 1$, $y_j \geq 0$ and $z_i \geq 0$ represent facets of $P_{\mathrm{MC}'}$ as they are implied by the above inequalities. Also note that $x_{ijk} \leq y_j$ and $x_{ijk} \leq z_i$ are never facet defining.

3.2.2 The Valid Inequalities

The new inequalities introduced below are designed to capture structural properties involving y-facilities and clients as well as z-facilities and clients.

Theorem 12 *Let $J' \subseteq J$ with $1 \leq |J'| < q$.*

$$\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in I} \sum_{k \in K} x_{ijk} + (q - |J'|)(1 - \sum_{j \notin J'} y_j) \leq 0 \tag{21}$$

is valid for $P_{\mathrm{MC}'}$ and represents a facet of $P_{\mathrm{MC}'}$ if $|J'| < \min\{n, q\}$.

Note that when $n = q = 2$ and $|J'| = 1$, (21) represents an *improper* facet of $P_{\mathrm{MC}'}$; that is, all points in $P_{\mathrm{MC}'}$ satisfy (21) with equality.

Proof of validity:

Case 1: $\sum_{j \notin J'} y_j = 0$.

$$\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in I} \sum_{k \in K} x_{ijk} + (q - |J'|) = \sum_{j \in J'} y_j - |J'| \leq 0,$$

where the equality follows from $\sum_{j \in J'} \sum_{i \in I} \sum_{k \in K} x_{ijk} = q$, which holds since all demand has to be met.

Case 2: $\sum_{j \notin J'} y_j \geq 1$.

$$\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in I} \sum_{k \in K} x_{ijk} + (q - |J'|)(1 - \sum_{j \notin J'} y_j) \leq (q - |J'|)(1 - \sum_{j \notin J'} y_j) \leq 0,$$

where the first inequality follows from a combination of inequalities (18), and the second inequality follows from the assumption that $(q - |J'|) \geq 0$.

Proof that (21) is facet defining:

Here we use the same technique as in the proof of Theorem 8. Let $|J'| < \min\{n, q\}$. Without loss of generality, assume that $J' \ni 1$. Let $\alpha x + \beta y + \gamma z \leq \delta$ be a valid inequality for $P_{\text{MC}'}$ such that

$$\begin{aligned} \mathcal{F} \equiv & \{(x,y,z) \in P_{\text{MC}'} : \sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in I} \sum_{k \in K} x_{ijk} + (q - |J'|)(1 - \sum_{j \notin J'} y_j) = 0\} \\ & \subseteq \{(x,y,z) \in P_{\text{MC}'} : \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \alpha_{ijk} x_{ijk} + \sum_{j \in J} \beta_j y_j + \sum_{k \in K} \gamma_k z_k = \delta\}. \end{aligned}$$

For any $i', i'' \in I,\ i' \neq i'',\ j' \notin J'$, and $k' \in K$, consider the following two points in $\mathcal{F}$:

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

$$z_i = \begin{cases} 1 & \text{if } i = i', \\ 0 & \text{otherwise;} \end{cases}$$

and

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i'',\ j = j' \text{ and } k = k', \\ 1 & \text{if } i = i',\ j = j' \text{ and } k \neq k', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$z_i = \begin{cases} 1 & \text{if } i = i' \text{ or } i'', \\ 0 & \text{otherwise.} \end{cases}$$

By comparing the two points we can conclude that

$$\alpha_{i'j'k'} = \alpha_{i''j'k'} + \gamma_{i''} \qquad \text{for any } i', i'' \in I,\ i' \neq i'', \text{ any } j' \notin J', \text{ and any } k' \in K. \tag{23}$$

Similarly for any $j' \in J'$, and for any arbitrary $i' \neq i''$ and any arbitrary k', consider a solution where all clients are assigned to z-facility i' and y-facilities in J' such that all $|J'|$ facilities are used, and compare it to the solution where client k' is switched from z-facility i' to i''. We see that (23) also holds for $j \in J'$.

Next for $j' \notin J'$, comparing point (22) and

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i'' \text{ and } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$z_i = \begin{cases} 1 & \text{if } i = i'', \\ 0 & \text{otherwise,} \end{cases}$$

for any arbitrary $i' \neq i''$, and using (23) we see that

$$-(q-1)\gamma_{i''} = \gamma_{i'} \qquad \text{for any } i' \neq i''. \tag{24}$$

If $m \geq 3$ or $q \geq 3$, applying (24) to all pairs of distinct i' and i'' shows that $\gamma_i = 0$ for all i, and hence $\alpha_{i'jk} = \alpha_{i''jk}$ for any $i \neq i''$, any j, and any k. When $m = q = 2$, we have $\gamma_2 = -\gamma_1$.

Next, for any client pair $k' \neq k''$, and any $j' \neq 1$, consider a point in $\mathcal{F}$ where client k' is assigned to y-facility 1, client k'' is assigned to y-facility j' and all clients are assigned to z-facility 1. Comparing this point to the point in $\mathcal{F}$ where the y-facilities for the clients k' and k'' are switched, we get

$$\begin{aligned} & \alpha_{11k'} + \alpha_{1j'k''} &&= \alpha_{1j'k'} + \alpha_{11k''} \\ \text{or} \quad & \alpha_{1j'k'} - \alpha_{11k'} &&= \alpha_{1j'k''} - \alpha_{11k''} \qquad \text{for any } k' \neq k''. \end{aligned}$$

Therefore, $\alpha_{1jk} = \alpha_{11k} + \eta_j$ for some constant η_j, for any $j \neq 1$ and any k.

Next, let $i' = 1$. For any k', compare (22) for $j' \notin J'$ and the following point:

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i',\ j = j'' \text{ and } k = k', \\ 1 & \text{if } i = i',\ j = j' \text{ and } k \neq k', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } j'', \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

$$z_i = \begin{cases} 1 & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

For $j'' = 1$, we see that $\eta_j = \beta_1$ for all $j \notin J'$. Compare (25) for $j'' = 1$ and $j'' \in J' \setminus \{1\}$, we see that $\eta_j = \beta_1 - \beta_j$ for all $j \in J'$.

For any client k', consider a point in $\mathcal{F}$ where all clients are assigned to z-facility i and y-facilities in J' such that all $|J'|$ facilities are used, and such that client k' and at least one other client are assigned to y-facility 1. Compare this solution to the solution where client k' is re-assigned to some other y-facility $j' \in J'$. We can then conclude that $\alpha_{i1k} = \alpha_{ijk}$ for any i, any $j \in J'$, and any k. Therefore $\eta_j = 0$, and hence $\beta_j = \beta_1$, for any $j \in J'$.

Lastly, let $i' = 1$. For any $j_1 \notin J'$ and any $j_2 \notin J'$ with $j_1 \neq j_2$, comparing (22) for $j' = j_1$ and $j' = j_2$, we see that $\beta_{j_1} = \beta_{j_2}$. Comparing (22) with $i' = 1, j' \notin J'$ and a point in $\mathcal{F}$ where all clients are assigned to z-facility 1 and y-facilities in J', we see that

$$\beta_j = -(q - |J|)\beta_1 \qquad \text{for all } j \notin J'.$$

Substitution into (22) shows that $\delta = \sum_{k \in K} \alpha_{11k} + |J'|\beta_1 + \gamma_1$. Therefore, $\alpha x + \beta y + \gamma z \leq \delta$ can be written as

$$\sum_{k \in K} (\alpha_{11k} + \beta_1) \sum_{j \in J} \sum_{k \in K} x_{ijk} - \gamma_1 \left(z_2 - \sum_{j \in J} \sum_{k \in K} x_{2jk} \right)$$

$$+\beta_1 \left(\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in I} \sum_{k \in K} x_{ijk} + (q - |J'|)(1 - \sum_{j \notin J'} y_j) \right)$$

$$\leq \sum_{k \in K} (\alpha_{11k} + \beta_1) - \gamma_1 (z_1 - 1).$$

When $m \geq 3$ or $q \geq 3$, $\gamma_1 = 0$. Hence (21) represents a facet of $P_{\mathrm{MC}'}$. ∎

Corollary 13 *Let $I' \subseteq I$ with $1 \leq |I'| < q$. The inequality*

$$\sum_{i \in I'} z_i - \sum_{i \in I'} \sum_{j \in J} \sum_{k \in K} x_{ijk} + (q - |I'|)(1 - \sum_{i \notin I'} z_i) \leq 0 \tag{26}$$

is valid for $P_{\mathrm{MC}'}$ and represents a facet if $|I'| < \min\{m, q\}$.

Again, if $m = q = 2$, the inequality (26) for $|I'| = 1$ is an improper facet.

Example 2

As an illustration, when $I = J = K = \{1, 2, 3\}$, the following point defined by:

$$x_{121} = \frac{1}{2}, \quad x_{211} = \frac{1}{6}, \quad x_{311} = \frac{1}{3}, \quad x_{212} = \frac{2}{3}, \quad x_{312} = \frac{1}{3},$$
$$x_{113} = \frac{1}{2}, \quad x_{223} = \frac{1}{6}, \quad x_{323} = \frac{1}{3}, \quad \text{all other } x_{ijk} = 0,$$
$$z_1 = z_2 = 1, \quad z_3 = \frac{1}{3}, \quad y_1 = y_2 = 1, \quad y_3 = 0.$$

is in $\bar{X}_{\mathrm{MC}'}$ but does not satisfy (26) with $I' = \{1, 2\}$. ∎

We note that when $|I'| = q - 1$ and when $|J'| = q - 1$, the inequalities (21) and (26) are Chvátal-Gomory inequalities of rank 1 (see Nemhauser and Wolsey, 1988).

3.2.3 Separation

The separation problems based on the families of inequalities (21) and (26) are quadratic knapsack problems. However, for fixed size of the subsets $|I'|$ and $|J'|$ respectively, we obtain linear 0-1 knapsack problems. Let (x^*, y^*, z^*) denote a fractional solution to formulation MC$'$ and let $\pi_j = 1$ if $j \in J'$ and $\pi_j = 0$ otherwise. Below we state the separation problem based on inequalities (21) for fixed $|J'| = t$.

$$\text{(KS):} \qquad \max \sum_{j \in J} (y_j^* - \sum_{i \in I} \sum_{k \in K} x_{ijk}^*) \pi_j + (q - t)(1 - \sum_{j \in J} y_j^* (1 - \pi_j))$$

subject to

$$\begin{aligned} \sum_{j \in J} \pi_j &= t \\ \pi_j &\in \{0, 1\} \qquad j \in J. \end{aligned}$$

Since $y_j^* - \sum_{i \in I} \sum_{k \in K} x_{ijk}^* \leq 0$ for all $j \in J$ due to constraints (18), and since $q - t > 0$ due to the assumption of Theorem 12, it is easy to see that inequality (21) can be violated only if $\sum_{j \notin J'} y_j^* < 1$. Let J'' denote a maximum cardinality subset of J such that $\sum_{j \in J''} y_j^* < 1$. It then follows that we only have to solve the knapsack problems KS for $n - |J''| \leq t \leq q - 1$. Since the constraint coefficients of the π_j-variables are all equal to one, problem KS can be solved by a greedy algorithm.

4 New Facets for the One-Level Uncapacitated Facility Location Problem

All facet-defining inequalities for UFL previously known in the literature have 0-1 coefficients, except for facets obtained by simultaneous lifting (see Cornuéjols and Thizy, 1982 and Cho et al., 1983b). In both classes of facet-defining inequalities introduced in Theorems 12 and 13, the coefficients of y_j for $j \notin J'$ and z_i for $i \notin I'$ may be greater than one. If we consider the version of UFL where the constraints

$$\sum_{k \in K} x_{jk} \geq y_j \quad \text{for all } j \in J, \tag{27}$$

are added to the set of constraints defining X_{UFL}, we obtain the following results. Let $X_{\text{UFL}'}$ be the set of solutions (x, y) satisfying the constraints of X_{UFL} and constraint (27). Also, let $P_{\text{UFL}'} = \text{conv}(X_{\text{UFL}'})$.

Theorem 14 *Let*

$$\sum_{j\in J}\sum_{k\in K} a_{jk}x_{jk} + \sum_{j\in J} b_j y_j \leq d, \tag{28}$$

represent a facet of $P_{\mathrm{UFL}'}$ *that is not a non-negativity constraint for some* x_{ij}. *Then*

$$\sum_{j\in J}\sum_{k\in K} a_{jk}\sum_{i\in I} x_{ijk} + \sum_{j\in J} b_j y_j \leq d, \tag{29}$$

represents a facet of $P_{\mathrm{MC}'}$.

Theorem 15 *Every facet of* $P_{\mathrm{MC}'}$ *of the form*

$$\sum_{j\in J}\sum_{k\in K} a_{jk}\sum_{i\in I} x_{ijk} + \sum_{j\in J} b_j y_j \leq d,$$

where the coefficient of z_i *is zero for all* $i \in I$, *and the coefficient of* x_{ijk} *is independent of* i *for all* $i \in I,\ j \in J,\ k \in K$, *induces a facet for* $P_{\mathrm{UFL}'}$.

The proofs of Theorems 14 and 15 are similar to the proofs of Theorems 5 and 6 and therefore omitted here.

By applying Theorem 15 to the facet-defining inequalities (21) and (26), we can obtain new facets for $P_{\mathrm{UFL}'}$, the modified formulation of the the one-level uncapacitated facility location problem.

5 An Alternative Flow Formulation

When solving integer or mixed-integer problems using LP-based solution techniques, such as the cutting plane approach, the choice of initial formulation plays an important role. It is crucial to obtain a good initial bound from the linear relaxation. On the other hand, one wants to avoid too large formulations, since it is practically inevitable that at least a few branch-and-bound nodes are needed to verify the optimal solution. If the formulation is very large, the time gained from a very sharp LP-bound may be lost by the time consumed by solving a large linear program in every branch-and-bound node. A solution to this dilemma can be the use of a weaker, smaller, initial formulation and then augmenting the formulation by identifying and adding violated inequalities, thus in some way compensating for the lack of strength in certain parts of the initial formulation.

Computational experiments have shown that the linear programming bound produced by formulation MC in general is very sharp (see Rardin and Choe, 1979). However, the formulation grows rapidly with the size of the instance. Therefore, we introduce in the following subsection a weaker flow formulation, as an alternative to the multi-commodity formulation MC. In Section 5.2 we discuss how the flow formulation can be strengthened by adding dicut collection inequalities (see Rardin and Wolsey, 1993).

5.1 The Single-Commodity Flow Formulation

Let w_{ij} be the quantity sent from z-facility i to y-facility j and v_{jk} the quantity sent from y-facility j to client k. Note that $v_{jk} = \sum_{i \in I} x_{ijk}$ and $w_{ij} = \sum_{k \in K} x_{ijk}$. Without loss of generality we assume that each client's demand is one unit. The set of feasible solutions to the flow formulation, F, is given by the following sets of constraints:

$$
\begin{aligned}
(\mathrm{F}): \quad & \sum_{j \in J} v_{jk} = 1 && \text{for all } k \in I, && (30)\\
& v_{jk} \le y_j && \text{for all } j \in J,\ k \in K, && (31)\\
\sum_{k \in K} v_{jk} - & \sum_{i \in I} w_{ij} = 0 && \text{for all } j \in J, && (32)\\
& \sum_{j \in J} w_{ij} \le q z_i && \text{for all } i \in I, && (33)\\
& v_{jk} \ge 0 && \text{for all } j \in J,\ k \in K, && (34)\\
& w_{ij} \ge 0 && \text{for all } i \in I,\ j \in J, && (35)\\
0 \le y_j & \le 1 && \text{for all } j \in J, && (36)\\
0 \le z_i & \le 1 && \text{for all } i \in I, && (37)\\
z_i, y_j, \text{ integer} & \text{ valued} && \text{for all } i \in I,\ j \in J. && (38)
\end{aligned}
$$

Let

$$X_{\mathrm{F}} = \{(v, w, y, z) \in R^{nq+mn+m+n} : (v, w, y, z) \text{ satisfies (30) – (38)}\},$$

$$\bar{X}_{\mathrm{F}} = \{(v, w, y, z) \in R^{nq+mn+m+n} : (v, w, y, z) \text{ satisfies (30) – (37)}\},$$

and let $P_{\mathrm{F}} = \mathrm{conv}(X_{\mathrm{F}})$.

Proposition 16 $\bar{X}_{\mathrm{MC}} \subset \bar{X}_{\mathrm{F}}$.

Proof. By using $v_{jk} = \sum_{i \in I} x_{ijk}$ and $w_{ij} = \sum_{k \in K} x_{ijk}$, we can easily verify that $\bar{X}_{\mathrm{F}}$ is equivalent to the set of solutions defined by the constraints (1),(3), (4)-(6). Let client k be assigned to z-facility i and y-facility j, and assume that all other clients are assigned to different facilities. $z_i = 1/q$ is feasible in $\bar{X}_{\mathrm{F}}$, whereas one of the constraints (2) in $\bar{X}_{\mathrm{MC}}$ would require $z_i = 1$. ∎

From the proof of Proposition 16 we observe that the flow formulation does not model the fixed-charge structure of the z-facility level as well as the MC formulation due to the weaker constraint set (33). In the next section we shall describe a class of inequalities that can be used to strengthen the flow formulation.

5.2 Dicut Collection Inequalities

Rardin and Wolsey (1993) derived a class of valid inequalities, called dicut collection inequalities, for general uncapacitated fixed-charge problems, from the projection of a multicommodity formulation onto the space of the flow variables. In fact, they showed that the class of dicut collection inequalities precisely describes this projection. They also showed that these inequalities subsume many of the previously-known inequalities developed for more specific network problems, such as uncapacitated lot sizing.

Here we describe a subclass, called simple dicut collection inequalities. Without loss of generality, Rardin and Wolsey (1993) consider a directed graph with only one source node s and a set of sink nodes T, where the set of arcs are partitioned into a set of continuous arcs C and a set of fixed-charge arcs F. A t-dicut for $t \in T$ is a set R of arcs whose removal will block the flow from s to t. We assume that each t-dicut is minimal.

Let $\Gamma = \{\Gamma^t\}_{t \in T}$ be a collection of sets of dicuts, with no more that one dicut in each set for each sink. Let $\gamma^t = |\Gamma^t|$; hence, $\gamma^t \leq 1$ for all $t \in T$. Moreover, let $\gamma_e = 1$ if arc e is in some dicut in Γ and $\gamma_e = 0$ otherwise. Furthermore, let

$$x_e = \text{the flow on arc } e \in C,$$

$$y_e = \begin{cases} 1 & \text{if arc } e \in F \text{ is open,} \\ 0 & \text{otherwise,} \end{cases}$$

and let d_t denote the demand of sink t. The corresponding simple dicut collection inequality is:

$$\sum_{e \in C} \gamma_e x_e + \sum_{e \in F} \sum_{t \in T} d_t \gamma_e y_e \geq \sum_{t \in T} \gamma^t d_t. \tag{39}$$

To represent the two-level facility location problem as a fixed-charge network of the form considered by Rardin and Wolsey (1993), we introduce a source node s and arcs from s to every z-facility i. We also split vertices representing facilities i and j into two vertices i, i' and j', j respectively. The set of sink vertices is the set K of clients. The arcs (j', j) for all $j \in J$ and (i, i') for all $i \in I$ are the fixed-charge arcs of the network. The arcs (s, i) for all $i \in I$, (i', j') for all $i \in I, j \in J$, and (j, k) for all $j \in J, k \in K$ are the continuous arcs. The values of v_{jk} and w_{ij} in formulation F represent flows on arcs (j, k) and (i', j') respectively. The structure of the digraph relevant to TUFL is shown in Figure 2.

One subclass of the simple dicut collection inequalities that seems particularly useful for formulation F of TUFL is the so-called fixed-charge path inequalities (see Van Roy and Wolsey, 1987). These inequalities involve variables for *both* z- and y-facilities as well as clients.

Consider the path $(\tilde{i}, \tilde{j}, \tilde{k})$ from z-facility $\tilde{i}$, through y-facility $\tilde{j}$ to client $\tilde{k}$. The demand of the nodes in the path is equal to zero except for client $\tilde{k}$. Hence, if we choose a subset

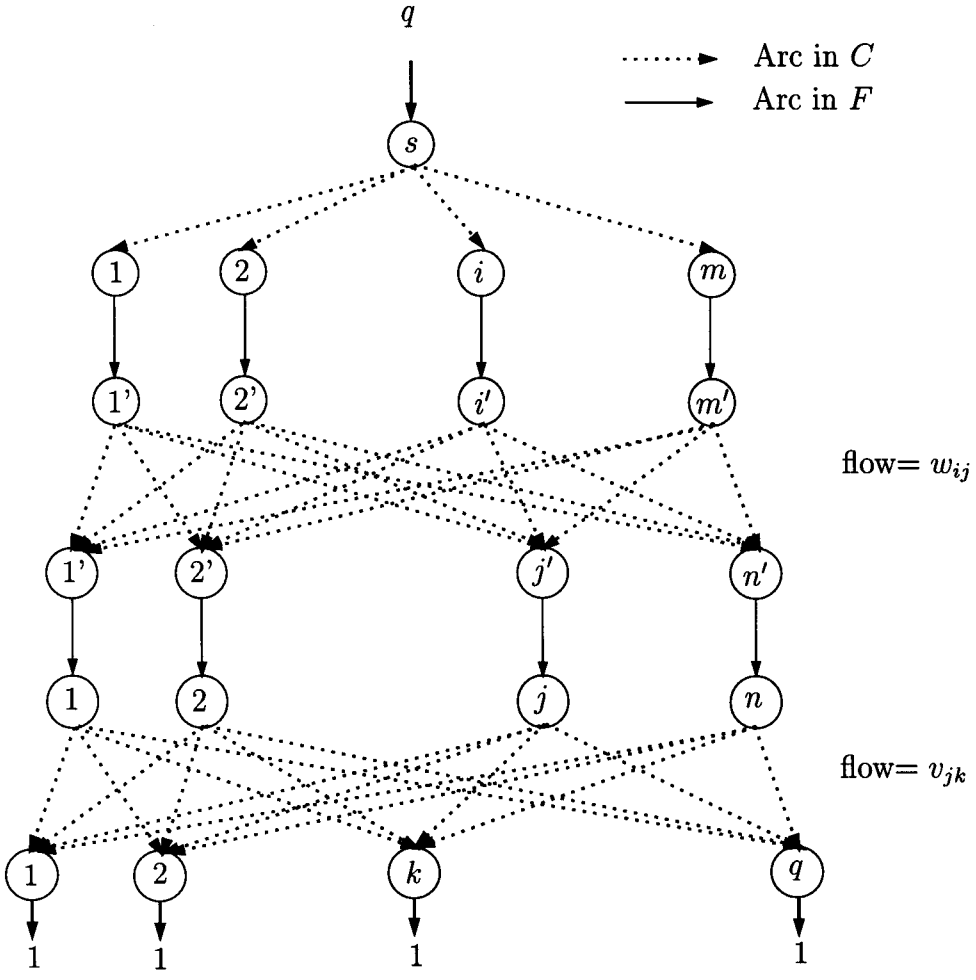

Figure 2: TUFL as a fixed-charge network flow problem.

of arcs supplying any of the nodes on the path, then the total flow on these arcs has to be less than or equal to the demand of client $\tilde{k}$ plus all possible outflow from that node and all nodes further down on the path. For the inflow arcs we make use of the fixed-charge structure. The path structure is shown in Figure 3.

Let $I^+ \subseteq I \setminus \{\tilde{i}\}$, and $J^+ \subseteq J \setminus \{\tilde{j}\}$. Also, let w_i denote the flow from the source s to z-facility i. The fixed-charge path inequality for path $(\tilde{i}, \tilde{j}, \tilde{k})$ is:

$$w_{\tilde{i}} + \sum_{i \in I^+} w_{i\tilde{j}} + \sum_{j \in J^+} v_{j\tilde{k}} \leq d_{\tilde{k}} z_{\tilde{i}} + \sum_{i \in I^+} d_{\tilde{k}} z_i + \sum_{j \in J^+} d_{\tilde{k}} y_j + \sum_{j \in (J \setminus \{\tilde{j}\})} w_{\tilde{i}j} + \sum_{k \in K \setminus \{\tilde{k}\}} v_{\tilde{j}k} \quad (40)$$

Note that $d_k = 1$ for all $k \in K$ in our case. The following example shows how an inequality of type (40) can be used to cut off a fractional solution of $\bar{X}_{\mathbf{F}}$.

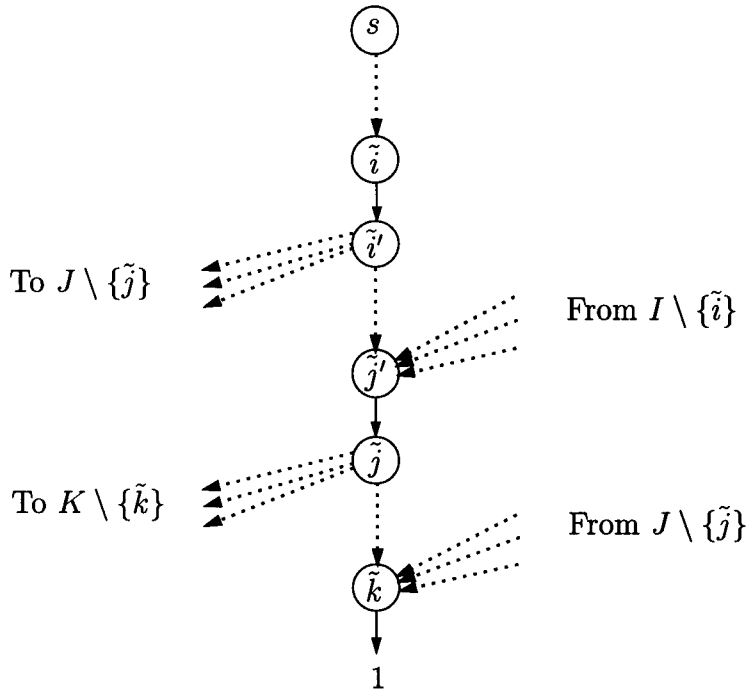

Figure 3: Structure of a path inequality.

Example 3

Let $I = J = K = \{1,2,3\}$ and consider the path $(\tilde{i} = 1, \tilde{j} = 1, \tilde{k} = 1)$ with $I^+ = \{3\}$ and $J^+ = \{3\}$. Then the point in $\bar{X}_{\mathrm{F}}$ defined by:

$$\begin{aligned} &z_1 = \frac{1}{3}, \quad z_2 = \frac{2}{3}, \quad z_3 = 0, \qquad y_1 = y_2 = 1, \quad y_3 = 0, \\ &w_{11} = 1, \quad w_{22} = 2, \qquad \text{all other } w_{ij} = 0, \\ &v_{11} = 1, \quad v_{22} = 1, \quad v_{23} = 1, \qquad \text{all other } v_{jk} = 0, \end{aligned}$$

does not satisfy the path inequality:

$$\underbrace{w_{11} + w_{12} + w_{13}}_{w_1} + w_{31} + v_{31} \leq z_1 + z_3 + y_3 + w_{12} + w_{13} + v_{12} + v_{13}.$$

To see that the inequality (40) is a simple dicut inequality, we use the following equation derived from flow balance along the path $(\tilde{i}, \tilde{j}, \tilde{k})$:

$$w_{\tilde{i}} + \sum_{i \in I^+} w_{i\tilde{j}} + \sum_{j \in J^+} v_{j\tilde{k}} = d_{\tilde{k}} + \sum_{j \in J \setminus \{\tilde{j}\}} w_{\tilde{i}j} + \sum_{k \in K \setminus \{\tilde{k}\}} v_{\tilde{j}k} - \sum_{i \in (I \setminus \{\tilde{i}\}) \setminus I^+} w_{i\tilde{j}} - \sum_{j \in (J \setminus \{\tilde{j}\}) \setminus J^+} v_{j\tilde{k}}.$$

Substituting in (40) gives

$$\sum_{i\in(I\setminus\{\tilde{i}\})\setminus I^+} w_{i\tilde{j}} + \sum_{j\in(J\setminus\{\tilde{j}\})\setminus J^+} v_{j\tilde{k}} + d_{\tilde{k}} z_{\tilde{i}} + \sum_{i\in I^+} d_{\tilde{k}} z_i + \sum_{j\in J^+} d_{\tilde{k}} y_j \geq d_{\tilde{k}}, \tag{41}$$

which is exactly the simple dicut collection inequality where $\Gamma^k = \emptyset$ for all $k \in K \setminus \{\tilde{k}\}$ and

$$\begin{aligned}\Gamma^{\tilde{k}} \;=\; & \{(i,i') : i \in \{\tilde{i}\} \cup I^+\} \cup \{(j',j) : j \in J^+\} \\ & \cup \{(i',\tilde{j}) : i \notin \{\tilde{i}\} \cup I^+\} \cup \{(\tilde{i},j) : j \notin \{\tilde{j}\} \cup J^+\}.\end{aligned}$$

Since the flow on the arc from a y-facility to a client cannot exceed the client's demand, we can extend the path inequality to the following valid inequality:

$$\begin{aligned} w_{\tilde{i}} + \sum_{i\in I^+} w_{i\tilde{j}} + \sum_{j\in J^+} v_{j\tilde{k}} \;\leq\; & \Big(\sum_{k\in K^-} d_k + d_{\tilde{k}}\Big) z_{\tilde{i}} + \sum_{i\in I^+} \Big(\sum_{k\in K^-} d_k + d_{\tilde{k}}\Big) z_{\tilde{i}} \\ & + \sum_{j\in J^+} d_{\tilde{k}} y_j + \sum_{j\in(J\setminus\{\tilde{j}\})} w_{\tilde{i}j} + \sum_{k\in(K\setminus\{\tilde{k}\})\setminus K^-} v_{\tilde{j}k} \end{aligned} \tag{42}$$

where $K^- \subseteq K \setminus \{\tilde{k}\}$.

The separation problem based on the dicut collection inequalities (39) can be solved by checking if a fractional solution (w^*, v^*, y^*, z^*) of the single-commodity flow formulation is feasible for the multi-commodity extended formulation. That is, if there is a solution (x^*, y^*, z^*) for MC such that $\sum_{i\in I} x^*_{ijk} = v^*_{jk}$ and $\sum_{k\in K} x^*_{ijk} = w^*_{ij}$ for all $i \in I$, $j \in J$ and $k \in K$. If there is, then all dicut collection inequalities are satisfied; if not, one has to examine the dual of the multi-commodity extended formulation, which is a large LP for any reasonable-sized problem, to identify a violated inequality. No combinatorial algorithm for solving the separation problem is known. A heuristic for finding violated fixed-charge path inequalities (40) and extended fixed-charge path inequalities (42) can be found in Van Roy and Wolsey (1987).

6 Extensions

Rardin and Choe (1979) have shown that the multi-commodity extended formulation models the fixed-charge behavior more carefully than single-commodity flow models. They presented empirical evidence showing that the LP-relaxation of the multi-commodity flow formulation is usually tight and is provably exact in some special networks. Since the model MC is *equivalent* to a multi-commodity flow formulation of the two-level uncapacitated location problem, we should expect its *LP*-relaxation to be relatively tight. Indeed, the largest gap that we have been able to construct for Euclidean problems as an example is 9 percent.

We plan to develop and test a cutting plane approach to solving the two-level uncapacitated facility location problem, using the models MC and MC'. One question to address is the effectiveness of the facets introduced in Section 3. Another issue to be explored is the tradeoff between the model size and solution quality between the stronger and weaker flow formulations.

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Appendix

Theorem 3 *$(x, y, z) \in \bar{X}_{\mathrm{MC}}$ is an extreme point of $\bar{X}_{\mathrm{MC}}$ if and only if*

(a) *For all $i \in I_1$, $z_i = \max_k \sum_{j \in J} x_{ijk}$.*

(b) *For all $j \in J_1$, $y_j = \max_k \sum_{i \in I} x_{ijk}$.*

(c) *for any k in K_1, the graph $\hat{G}_k$ does not contain any even undirected circuit,*

(d) *there is no non-zero vector $(\alpha', \beta') \in \mathbf{R}^{I_1} \times \mathbf{R}^{J_1}$ such that:*

$$\begin{aligned}&\text{for each } k \in K_1, \text{ there is a pseudo-feasible flow } f^k \text{ for } G_k \text{ with } f^k_{(s,i)} = \\ &\alpha'_i \text{ for all } i \in \tilde{I}_k \cap I_1 \text{ and } f^k_{(j,t)} = \beta'_j \text{ for all } j \in \tilde{J}_k \cap J_1.\end{aligned} \tag{8}$$

The vector (α', β') specifies a set of flow-values for the arcs incident to the source node s and sink node t that is held fixed for *all* the graphs $G_k, k \in K_1$. Condition (8) says that for *each* $k \in K_1$, we can assign flow-values to the remaining arcs in G_k to give a pseudo-feasible flow.

Proof. It is easy to check that (x, y, z) is not extreme if Conditions **(a)** and **(b)** are violated. Suppose condition **(c)** is violated and the graph $\tilde{G}_k$ contain an even undirected circuit. In the first case, suppose this circuit does not contain node d. Then, the circuit contains the arcs:

$$\{(i_1, j_1), (i_2, j_1), (i_2, j_2)(i_3, j_2) \ldots, (i_s, j_s), (i_1, j_s)\}$$

for some $s \geq 2$. Note that the values of the x_{ijk}'s corresponding to these arcs are strictly between 0 and 1. Construct a point (x', y, z) by alternately increasing and decreasing the corresponding x_{ijk}'s for the sequence of the arcs in this circuit, that is, let

$$\begin{aligned} x'_{i_h j_h k} &= x_{i_h j_h k} + \epsilon' && \text{for } h = 1, \ldots, s, \\ x'_{i_{h+1} j_h k} &= x_{i_{h+1} j_h k} - \epsilon' && \text{for } h = 1, \ldots, s, \text{ with } i_{s+1} = i_1, \\ \text{and all other elements} & \quad \text{of } (x', y, z) && \text{the same as } (x, y, z), \end{aligned}$$

for some $\epsilon' > 0$. A similar point (x'', y, z) can be constructed with $\epsilon'' = -\epsilon'$. Note that

$$\sum_{j \in J} x'_{ijk} = \sum_{j \in J} x''_{ijk} = \sum_{j \in J} x_{ijk} \qquad \forall\, i \in I, k \in K,$$

and

$$\sum_{i \in I} x'_{ijk} = \sum_{i \in I} x''_{ijk} = \sum_{i \in I} x_{ijk} \qquad \forall\, j \in J, k \in K.$$

For a sufficiently small ϵ', both points (x', y, z) and (x'', y, z) are in $\bar{X}_{\text{MC}}$, showing that (x, y, z) is not an extreme point of $\bar{X}_{\text{MC}}$.

In the second case, suppose the circuit contains node d. Since the circuit is even, the number of arcs not incident to d is also even. Again we can construct a point (x', y, z) by alternately increasing and decreasing by $\epsilon' > 0$ the x_{ijk}'s corresponding to the sequence of arcs of the circuit, skipping the arcs incident to d. An illustration is shown in Figure 4.

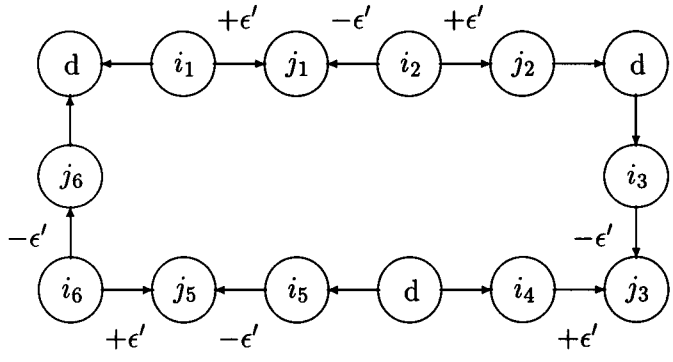

Figure 4: Using an even circuit to construct the point (x', y, z).

For node i incident from d in the circuit, $\sum_{j \in J} x'_{ijk}$ may differ from $\sum_{j \in J} x_{ijk}$, and for node j incident to d in the circuit, $\sum_{i \in I} x'_{ijk}$ may differ from $\sum_{i \in I} x_{ijk}$. However, since the arcs (d, i) and (j, d) in G_k indicate that the corresponding constraints (2) and (3) are slack, (x', y, z) will still be in $\bar{X}_{\text{MC}}$ if ϵ' is sufficiently small. As in the first case, we can construct a similar point (x'', y, z) with $\epsilon'' = -\epsilon'$. Since (x, y, z) is a convex combination of (x', y, z) and (x'', y, z), and by choosing $\epsilon' > 0$, but small enough, both (x', y, z) and (x'', y, z) will be in $\bar{X}_{\text{MC}}$, then (x, y, z) is not an extreme point.

Finally, we will show that if Conditions **(a)**, **(b)** and **(c)** are satisfied, (x, y, z) is an extreme point of $\bar{X}_{\text{MC}}$ if and only if Condition **(d)** holds.

$$\begin{aligned} \text{Let} \quad I_2 &= \{i : z_i = 0\}, \quad I_3 = \{i : z_i = 1\}, \\ J_2 &= \{j : y_j = 0\}, \quad J_3 = \{j : y_j = 1\}, \\ K_2 &= \{k : x_{ijk} \text{ integer-valued for all } i \text{ and all } j\}. \end{aligned}$$

Note that I_1, I_2 and I_3 partition I, J_1, J_2 and J_3 partition J, and K_1, K_2 partition K.

Suppose condition **(d)** is violated. Consider (x', y', z') and (x'', y'', z'') where

$$\begin{aligned} & z'_i = z_i + \alpha'_i, \quad y'_j = y_j + \beta'_j, \quad x'_{ijk} = x_{ijk} + \gamma'_{ijk}, \\ \text{and} \quad & z''_i = z_i - \alpha'_i, \quad y''_j = y_j - \beta'_j, \quad x''_{ijk} = x_{ijk} - \gamma'_{ijk}. \end{aligned} \tag{43}$$

By considering the upper and lower bounds on z_i, y_j and x_{ijk}, we see that $\alpha'_i = 0$ for $i \in I_2 \cup I_3$ and $\beta'_j = 0$ for $j \in J_2 \cup J_3$ in order for (x', y', z') and (x'', y'', z'') to be in $\bar{X}_{\mathrm{MC}}$. Furthermore, we must have

$$\begin{aligned} \gamma'_{ijk} = 0 \qquad & \text{for } i \in I_1 \cup I_2 \cup I_3, \quad j \in J_2, \quad k \in K_1, \\ & \text{for } i \in I_2, \quad j \in J_1 \cup J_2 \cup J_3, \quad k \in K_1, \quad \text{and} \\ & \text{for } i \in I_1 \cup I_2 \cup I_3, \quad j \in J_1 \cup J_2 \cup J_3, \quad k \in K_2. \end{aligned} \tag{44}$$

This leaves α'_i for $i \in I_1$, β'_j for $j \in J_1$ and γ'_{ijk} for $i \in I_1 \cup I_3, j \in J_1 \cup J_3, k \in K_1$ undetermined.

Let (α', β') satisfy (8), and let f^k for $k \in K_1$ be a corresponding set of pseudo-feasible flows. Then the values for γ'_{ijk} can be determined as follows. For $k \in K_1$, we define

$$\gamma'_{ijk} = \begin{cases} 0 & \text{if } x_{ijk} = 0 \text{ or } 1, \\ f^k_{(i,j)} & \text{if } 0 < x_{ijk} < 1. \end{cases}$$

Note that flow balance at the nodes $\tilde{I}_k \cap I_1$ and $\tilde{J}_k \cap J_1$ ensures that $\sum_{j \in J} \gamma'_{ijk} = \alpha'_i$ and $\sum_{i \in I} \gamma'_{ijk} = \beta'_j$ if the corresponding constraints (2) and (3) are tight. Flow balance at the nodes s and t ensures that $\sum_{\tilde{I}_k \cap I_1} \alpha'_i = 0$ and $\sum_{\tilde{J}_k \cap J_1} \beta'_j = 0$, which are also necessary for feasibility. Further, we note that the graphs G_k for $k \in K_1$ are of two types:

1. $\hat{G}_k$ contains no undirected cycles — whence any pseudo-feasible flow satisfies $\sum_{i \in I} \sum_{j \in J} \gamma'_{ijk} = 0$.
2. $\hat{G}_k$ contains an undirected odd cycle — in this case, by adjusting the flow on the sub-paths in the cycles that start and end with d and contain an odd number of arcs, a pseudo-feasible flow can be constructed with $\sum_{i \in I} \sum_{j \in J} \gamma'_{ijk} = 0$.

We also note that we can always assume that (α', β') is scaled such that

$$\begin{aligned} |\alpha'_i| &\leq \min\{z_i, 1 - z_i\} && \text{for all } i \in I_1, \\ |\beta'_j| &\leq \min\{y_j, 1 - y_j\} && \text{for all } j \in J_1, \\ \text{and} \quad |f^k_{(i',j')}| &\leq \min\{x_{i'j'k}, 1 - x_{i'j'k}\} && \text{for } (i', j') \text{ in the odd cycles in any } G^k. \end{aligned}$$

Thus, if condition **(d)** does not hold, then the values of α', β' and γ' can be set as above to show that (x, y, z) is a convex combination of two distinct points in $\bar{X}_{\mathrm{MC}}$.

Now suppose condition **(d)** holds, and suppose that the two points as defined by (43) are in $\bar{X}_{\mathrm{MC}}$. By considering the upper and lower bounds on the variables, we again see that $\alpha'_i = 0$ for $i \in I_2 \cup I_3$ and $\beta'_j = 0$ for $j \in J_2 \cup J_3$, and the conditions (44) hold.

Now, in order for the two points (x', y', z') and (x'', y'', z'') to be feasible for MC, we must have $\sum_{j \in J} \gamma'_{ijk} = \alpha'_i$ if the corresponding constraint (2) is tight; similarly, $\sum_{i \in I} \gamma'_{ijk} =$

β'_j for $j \in \tilde{J}_k \cap J_1$ if the corresponding constraint (3) is tight. Further, we must have $\sum_{i\in I}\sum_{j\in J}\gamma'_{ijk} = 0$ for every k, implying also $\sum_{i\in I}\alpha'_i = 0$ and $\sum_{j\in J}\beta'_j = 0$. We argue that α', β' and γ' must be zero, otherwise a non-zero vector (α', β') and corresponding pseudo-feasible flows can be constructed that violates **(d)**. ∎