A Complete Epistemic Logic for Multiple Agents: Combining Distributed and Common Knowledge

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Combining Distributed and Common Knowledge

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Abstract

We give one overall system for describing the knowledge in a group of m agents, in which distributed knowledge, everybody's knowledge and common knowledge can be dealt with at the same time. The canonical model for this complete epistemic logic for m agents appears to lack two desirable properties. We combine several validity-preserving techniques to transfer the satisfiability of an epistemic formula between classes of models; thus eventually proving completeness for the logic under consideration with respect to the class of models containing all the desired properties. Although the full procedure for achieving this seems, we admit, quite formidable, the method consists in essence of applying three standard techniques: firstly, we show how a filtration technique of Goldblatt can be used to gain one of the desired properties of the models. Next, we unravel this finite filtration, following ideas that were introduced by Sahlqvist for the mono-modal case. Finally, we use an equivalence relation to identify unravelled paths, the equivalence classes of this relation becoming the worlds that together constitute a new model. Thus, we eventually obtain a class of models for which, on the one hand, the given epistemic system is a sound and complete axiomatization, and on the other hand appealing mathematical properties can be proven. In passing, we prove the epistemic system to be decidable.

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1 Introduction

In the field of AI and Computer Science, the modal system $S5$ is a familiar logic to model knowledge. Although the system on the one hand models an idealized notion of knowledge (the epistemic agent is assumed to be fully introspective, for example, and $S5$ knowledge also suffers from the problem of logical omniscience—the agent knows all logical validities, and his knowledge is closed under logical consequence— see [20] for a discussion and relaxations of these properties), its nice mathematical properties, on the other hand, often motivate researchers to adopt this system in their first exploration of the field. Then, for specific purposes, such as decision- and game-theoretic applications, some or many of these idealizations concerning introspective properties or logical omniscience are given up or, sometimes, replaced by weaker assumptions about knowledge.

It is a well-known fact that the idealized modal system $S5$ exactly describes the valid formulas of Kripke models in which the accessibility relation is an *equivalence relation.* In the case of one agent, one may use a result about preservation of truth under taking so-called *generated sub-models* (the formulas that are true in a word w in a Kripke model M are exactly those that are true in w and M', where M' is the restriction of M to only those worlds that are accessible from w—in any number of steps) to conclude that we may even assume this relation to be *universal* (cf. [11, 5, 6]). The latter fact guarantees many pleasant properties of the logic of knowledge, like the existence of small models and the superfluity of iteration of modal operators (cf. [16]). However, since the logic $S5$ was first used as an epistemic logic, many extensions and adaptions of it have been proposed.

A first extension satisfies the need to have a logic that describes the knowledge in a *group* of, say, m agents (the operator K_i expressing what is known by agent i). A shift from the bare system $S5$ to $S5_m$ did not give rise to any logical complications, even though completeness with respect to universal models had to be sacrificed; the underlying accessibility relations R_i ($i \leq m$) may still be taken to be equivalences. Later, researchers interested in the kinds of knowledge that emerge in a group of agents proposed enriching the language with operators E ('Everybody knows'), C ('it is Common knowledge that') and D ('it is Distributed knowledge that') (cf. [5, 11]). (Initially, the operator I —for Implicit knowledge— was introduced instead of D, but to avoid confusion with Levesque's notions of Explicit and Implicit knowledge, it was later replaced by D, cf. [2].) Apart from these extensions, we may mention in passing various proposals for combining the notion of knowledge with that of *belief* ([8, 15]), *time* ([8]) or *probability* ([3]).

When they are treated as primitive operators, $K_1, \ldots K_m, C, D$ and E all are interpreted as necessity operators with respect to accessibility relations $R_1, \ldots,$ R_m, R_C, R_D and R_E in a Kripke model. The relations between the various kinds of knowledge, as expressed syntactically in the axioms of the logic, give rise to the enterprise of finding corresponding connections between the accessibility relations associated with these knowledge operators. The exact nature of the semantic counterparts of the modal axioms has been subject of folklore conjectures and assertions in the AI-community for some time now. It has also been claimed that several known results from the literature of modal logic apply. For instance, the relation between the operators E and C is much like a relation between some special modal operators studied in Dynamic Modal Logic by Goldblatt in [4]. His result concerning a semantical characterisation of these operators is often referred to when giving a completeness proof for $S5_m(EC)$, which is our notation for an '$S5$-like' modal logic for knowledge incorporating the operators E and C (such a completeness proof can be found in [6]). Claiming such a correspondence seems to invoke an implicit assumption of some 'modularity' principle, i.e. that the exact dependency between the E and K_i-operators does not undermine Goldblatt's technique of manipulating Kripke models.

Another example regards the system $S5_m(D)$, the logic that combines the knowledge of m agents with the notion of distributed knowledge. Halpern and Moses claimed ([5]) completeness with respect to models in which $R_D = R_1 \cap R_2 \cap \ldots \cap R_m$. Although, on the one hand, it turned out that this property was not modally definable (cf. [17]), on the other hand it appeared that the claim of Halpern and Moses could be proven using some non-standard techniques ([2, 17]).

However, where Goldblatt's technique was essentially successful since he was able to build a *finite* model, the techniques used to prove completeness for the case of distributed knowledge yielded *infinite* models, so that a naive combination of the techniques will not do the trick. This, of course, raises the question of the completeness of $S5_m(CDE)$ which, as far as we know, has not been stated—let alone solved—explicitly in the literature[1].

Taking into account the claims and partial proofs we mentioned above, this paper is written to solve the following questions. Firstly, it *does* provide a completeness proof for $S5_m(CDE)$, the epistemic logic that deals with Common, Distributed and Everybody's Knowledge in a group of m agents. Secondly, it shows how the techniques used by Goldblatt can be applied in the $S5$ environment. Thirdly, as a consequence of the techniques we apply, we also obtain *finite* models for the cases in which distributed knowledge is involved, and from that, we easily get decidability of $S5_m(D)$ and $S5_m(CDE)$. Finally, we consider this paper as an application of the modular approach of [17], where it was claimed that one can use one uniform 'unravelling-and-rewriting technique' to transform models, keeping track of various properties of the relations in the underlying Kripke model (although, in the present paper, the rewrite technique is implicitly represented by an equivalence relation on paths, for reasons of space). It will appear, on the one hand, that this principle of modularity indeed takes

[1] We like to mention that Dimiter Vakarelov posed this question in a private communication, though.

care of keeping the models $S5$-like, but on the other hand, that to achieve the intended properties for the relations concerning the operators C, D and E, we have to make some additional decisions about how the equivalence relation is affected by the relations associated with those operators.

The rest of this paper is organised as follows. In Section 2 we define a rich system **L** for reasoning about the various kinds of knowledge in a group of m agents. In Section 3 we investigate the Kripke models for this logic. In 3.1 we show how, in general, one constructs a *canonical model* for a modal logic; we also argue how a modal scheme (or axiom) in such a logic affects the kind of models (and frames) for it. In Section 3.2, we compare the canonical model of **L** with the models that are usually considered the standard (or 'preferred') models for **L** and we notice two main differences. In Section 4 we give some techniques that transfer truth from one model to another one. In particular, in Sections 4.1 and 4.2 we apply several of such transfer techniques on the canonical model of Section 3 for **L**, thus showing how one achieves the completeness of **L** with respect to the preferred models. In Section 5 we argue that the order in which we applied our techniques to our models, is not arbitrary, after which we conclude the paper.

2 A system for knowledge of m agents

For convenience, we repeat the definition of the general logic for knowledge in a group of m agents, starting by giving its language.

Definition 2.1 Let P be a non-empty set of propositional variables, and $m \in I\!N$ be given. The *language* L is the smallest superset of P such that

$$\varphi, \psi \in \mathsf{L} \Rightarrow \neg\varphi, (\varphi \wedge \psi), K_i\varphi, C\varphi, D\varphi, E\varphi \in \mathsf{L} \ (i \leq m).$$

We also assume to have the usual definitions for $\vee, \leftarrow$ and $\leftrightarrow$ as logical connectives, as well as the special formula $\bot =_{\text{def}} (p \wedge \neg p)$. In the sequel, we will use $\Box$ as a variable over the operators $\mathsf{OP} = \{K_1, \ldots, K_m, C, D, E\}$. Indices i and j will range over $\{1, \ldots m\}$.

The intended meaning of $K_i\varphi$ is 'agent i knows φ', $D\varphi$ means 'φ is distributed knowledge', or 'φ is implicit knowledge of the m agents'. $E\varphi$ has to be read as 'everybody knows φ' and $C\varphi$ is 'it is common knowledge that φ'. However, we do not know of any presentation of a system that combines all these notions together. Although the separate notions are supposed to be familiar to the reader, (all the operators appear in [5, 6, 11]), let us briefly try and indicate the intended meaning of the operators.

Distributed knowledge is the knowledge that is implicitly present in a group, and which might become explicit if the agents were able to communicate (however, see also [18]). For instance, it is possible that no agent knows the assertion ψ, while at the same time $D\psi$ may be derived from $K_1\varphi \wedge K_2(\varphi \rightarrow \psi)$). A common example of distributed knowledge in a group is for instance the fact whether two members of that group have the same birthday. The meaning of 'everybody knows φ' is simply that all members of the group know that φ, and Common knowledge of φ is supposed to be $E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \ldots$. Suppose everybody at a meeting receives a note with the announcement φ. Then, of course, $E\varphi$ holds, but not $C\varphi$. If everybody then receives a second note saying that everybody has received the same first note, we have $EE\varphi$, but still not $C\varphi$. The latter formula may be obtained if somebody at the meeting rises and announces 'φ!' However, to really achieve $C\varphi$ in this case, everybody must be convinced that everybody has heard the announcement ($EE\varphi$), but also that everybody knows this latter fact ($EEE\varphi$),[2] ad infinitum.

Definition 2.2 The logic $S5_m(CDE)$, or **L** for short, has the following axioms:

$A1$	any axiomatization for propositional logic
$A2$	$(K_i\varphi \wedge K_i(\varphi \rightarrow \psi)) \rightarrow K_i\psi$
$A3$	$K_i\varphi \rightarrow \varphi$
$A4$	$K_i\varphi \rightarrow K_iK_i\varphi$
$A5$	$\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$
$A6$	$E\varphi \leftrightarrow (K_1\varphi \wedge \cdots \wedge K_m\varphi)$
$A7$	$C\varphi \rightarrow \varphi$
$A8$	$C\varphi \rightarrow EC\varphi$
$A9$	$(C\varphi \wedge C(\varphi \rightarrow \psi)) \rightarrow C\psi$
$A10$	$C(\varphi \rightarrow E\varphi) \rightarrow (\varphi \rightarrow C\varphi)$
$A11$	$K_i\varphi \rightarrow D\varphi$
$A12$	$(D\varphi \wedge D(\varphi \rightarrow \psi)) \rightarrow D\psi$
$A13$	$D\varphi \rightarrow \varphi$
$A14$	$D\varphi \rightarrow DD\varphi$
$A15$	$\neg D\varphi \rightarrow D\neg D\varphi$

[2] which means that it should be ruled out that agent 1 thinks it is possible that agent 2 thinks it is possible that agent 3 allows $\neg\varphi$ to be true

On top of that, we assume the following derivation rules:

$R1$ $\vdash \varphi, \vdash \varphi \to \psi \;\Rightarrow \vdash \psi$

$R2$ $\vdash \varphi \;\Rightarrow \vdash K_i\varphi$, for all $i \leq m$

$R3$ $\vdash \varphi \;\Rightarrow \vdash C\varphi$

In words, we assume a logical system $(A1, R1)$ for rational agents, (that the agents are taken to be rational, is perhaps best reflected by the fact that we have the properties $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$ for all $\Box \in \mathsf{OP}$ —which follows from $A2, A9, A12$ and, in the case of E, with a simple calculation using $A6$). Individual knowledge, common knowledge and distributed knowledge are all supposed to be *veridical* ($A3$, $A7$ and $A13$, respectively). The agents are assumed to be *fully introspective*: they are supposed to have *positive* ($A4$) as well as *negative* ($A5$) introspection; properties we also ascribe to distributed knowledge ($A14$ and $A15$, respectively). Both properties of introspection can be shown to hold for common knowledge, as we will see in an example derivation in Proposition 2.4 below. Axiom $A6$ can be understood as the definition of E, whereas $A8$ says that all common knowledge is known by everybody as such. Axiom $A10$ is also known as the *induction axiom*; this terminology will become clearer when reading the proof of Lemma 3.12. The axiom tells how one can derive that φ is common knowledge: by deriving φ itself together with some common knowledge about $\varphi \to E\varphi$. An illustration of a derivation using this induction axiom is provided in Proposition 2.4 below. Finally, the rules $R2$ and $R3$ express another rationality principle of the (group of) agents we consider: it guarantees that **L**-derivable formulas give rise to the derivability in **L** of the same formula, prefixed by any of the operators from OP:

Lemma 2.3 The following are derivable in **L** ($\alpha, \beta \in \mathsf{L}$), $\Box \in \mathsf{OP}$:

1 $\vdash \alpha \;\Rightarrow \vdash \Box\alpha$

2 $\vdash \alpha \to \beta \;\Rightarrow \vdash \Box\alpha \to \Box\beta$

3 $\vdash \alpha \to \beta \;\Rightarrow \vdash \neg\Box\neg\alpha \to \neg\Box\neg\beta$

4 $\vdash \alpha \to \gamma, \vdash \gamma \to \beta \;\Rightarrow \vdash \alpha \to \beta$

5 $\vdash K_i\neg K_i\neg K_i\alpha \to K_i\alpha$

6 $\vdash C\alpha \to E\alpha$

Proof: We give informal proofs:

1 For $\Box \in \{K_1, \ldots K_m, C\}$ this is expressed by $R2$ and $R3$. For E, this follows from $R2$ and $A6$, for D, from $R2$ and $A11$.

2 From item 1, it follows from $\vdash \alpha \rightarrow \beta$ that $\vdash \Box(\alpha \rightarrow \beta)$. We already observed that we also have $\vdash \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$, and then the conclusion follows with $R1$.

3 Using $A1$ and $R1$, we obtain from $\vdash \alpha \rightarrow \beta$ that $\vdash \neg\beta \rightarrow \neg\alpha$. Item 2 now yields $\vdash \Box\neg\beta \rightarrow \Box\neg\alpha$; by $A1$ and $R1$ we then obtain $\vdash \neg\Box\neg\alpha \rightarrow \neg\Box\neg\beta$

4 Use the assumption and $R1$, applied to the following $A1$-principle: $\vdash (\alpha \rightarrow \gamma) \rightarrow ((\gamma \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta))$

5 We use the principle that $\vdash (\varphi \wedge \neg\psi) \rightarrow \bot$ implies $\vdash \varphi \rightarrow \psi$ (which is validated by $A1$ and $R1$), so that it is sufficient to prove $\vdash (K_i\neg K_i\neg K_i\alpha \wedge \neg K_i\alpha) \rightarrow \bot)$: which is done by applying the property of item 4 to the following sequence:
$\vdash (K_i\neg K_i\neg K_i\alpha \wedge \neg K_i\alpha) \rightarrow (\neg K_i\neg K_i\alpha \wedge \neg K_i\alpha)$ (use $A1, R1$ and $A3$)
$\vdash (\neg K_i\neg K_i\alpha \wedge \neg K_i\alpha) \rightarrow (\neg K_i\neg K_i\alpha \wedge K_i\neg K_i\alpha)$ (use $A1, R1$ and $A5$)
$\vdash (\neg K_i\neg K_i\alpha \wedge K_i\neg K_i\alpha) \rightarrow \bot$ (use $A1$)

6 By applying item 2 to $A7$, we obtain $\vdash EC\alpha \rightarrow E\alpha$; to this, and $A8$ one applies item 4 to obtain the desired result.

■

Proposition 2.4 In **L**, common knowledge satisfies both positive and negative introspection, i.e.:

1. $\vdash C\varphi \rightarrow CC\varphi$
2. $\vdash \neg C\varphi \rightarrow C\neg C\varphi$

Proof:

1. Positive introspection for C:

a	$\vdash C(C\varphi \rightarrow EC\varphi) \rightarrow (C\varphi \rightarrow CC\varphi)$	$(A10)$
b	$\vdash C(C\varphi \rightarrow EC\varphi)$	$(R2, A8)$
c	$(C\varphi \rightarrow CC\varphi)$	$(R1, a, b)$

2. Negative introspection for C:

a	$\vdash \neg K_i \neg C\alpha \rightarrow K_i \neg K_i \neg C\alpha$	$(A5)$
b	$\vdash C\alpha \rightarrow K_i C\alpha$	$(A6, A8,$ 2.3(4))
c	$\vdash \neg K_i \neg C\alpha \rightarrow \neg K_i \neg K_i C\alpha$	(b, 2.3(3))
d	$\vdash K_i \neg K_i \neg C\alpha \rightarrow K_i \neg K_i \neg K_i C\alpha$	(c, 2.3(2))
e	$\vdash \neg K_i \neg C\alpha \rightarrow K_i \neg K_i \neg K_i C\alpha$	(a, d, 2.3(4))
f	$\vdash K_i \neg K_i \neg K_i C\alpha \rightarrow K_i C\alpha$	(2.3(5))
g	$\vdash \neg K_i \neg C\alpha \rightarrow K_i C\alpha$	(e, f, 2.3(4))
h	$\vdash \neg K_i \neg C\alpha \rightarrow C\alpha$	(g, $A3$, 2.3(4))
i	$\vdash \neg C\alpha \rightarrow K_i \neg C\alpha$	(h, $A1, R1$)
j	$\vdash C(\neg C\alpha \rightarrow K_i \neg C\alpha)$	(i, $R3$)
k	$\vdash C(\neg C\alpha \rightarrow K_i \neg C\alpha) \rightarrow (\neg C\alpha \rightarrow C \neg C\alpha)$	$(A10)$
l	$\vdash (\neg C\alpha \rightarrow C \neg C\alpha)$	($R1$, j, k)

■

3 Semantics

Now we will present a Kripke semantics for **L**. It will emerge that there are in fact several classes of such models which can be related to the logic **L**.

3.1 General Kripke models

Definition 3.1 A (general) *Kripke model* M is a tuple $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E \rangle$, where

- W is a non-empty set (of *worlds*)
- V is a truth assignment of the propositional atoms, for each world: $V : W \times \mathsf{P} \rightarrow \{true, false\}$
- $R_1, \ldots, R_m, R_C, R_D, R_E \subseteq W \times W$ are called the accessibility relations
- Truth of a formula φ (written $M, w \models \varphi$) is defined on pairs M, w straightforwardly. For any $\Box \in \mathsf{OP}$ we let $R_\Box$ be the 'corresponding' relation: if $\Box = K_i$ then $R_\Box = R_i$; if $\Box = C$ then $R_\Box = R_C$ and so forth. The modal case then reads

$$M, w \models \Box\varphi \Leftrightarrow \forall v (R_\Box wv \Rightarrow M, v \models \varphi)$$

We refer to this general class of models as $\mathcal{K}^m_{CDE}$, or, when $m = 1$ as $\mathcal{K}$. A model in $\mathcal{K}^m_{CDE}$ with $k = m + 3$ relations is called *k-dimensional.*

Although it is standard to take the pair M, w as the semantic unit, there is some variety concerning this, according to the level of truth and validity under study:

Definition 3.2 Let $\langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E \rangle$ be given.

- We say that φ is true in M, written $M \models \varphi$, if $M, w \models \varphi$ for all $w \in W$
- φ is *valid* on a class of models $\mathcal{C}$, if φ is true in all models M of $\mathcal{C}$. We write $\models_{\mathcal{C}} \varphi$ in such a case
- Abstracting away from the particular valuation V, we obtain a *frame* $F = \langle W, R_1, \ldots, R_m, R_C, R_D, R_E \rangle$. We then write $F, V, w \models \varphi$ as an abbreviation for $\langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E \rangle \models \varphi$. The truth of φ in F, w is defined as: $F, w \models \varphi$ if for all V, $F, V, w \models \varphi$. Validity on a frame ($F \models \varphi$) and on a class of frames $\mathcal{F}$ ($\models_{\mathcal{F}} \varphi$) are defined as straightforward generalizations of the corresponding notions for models.

The models of Definition 3.1 are too general for our logic of Section 2; in fact, one easily proves that validity on the class of models of Definition 3.1 finds a sound and complete axiomatization in a logic that consists of the axiom $A1$ of $\mathbf{L}$, together with $A2$ for all the operators $\Box$ in OP and the rules $R1$ and $R2$— the latter for all $\Box$ in OP. Following terminology of Chellas ([1]), we will call a modal logic that satisfies $A1, A2$ and $R1, R2$ for its operators, a *normal modal logic* (we write $\mathbf{K}^m$ for the logic that exactly satisfies the axioms and rules mentioned or, in the case of only one operator, $\mathbf{K}$).

Such a completeness proof is generally obtained by constructing a *canonical model* for the logic under consideration. In such a case the semantic units are constituted from *maximal consistent sets*[3].

Definition 3.3 A set of formulas Σ is a *maximal consistent* set (m.c.s.) if:

- Σ is consistent
- For all $\varphi \in \mathsf{L}$: $\Sigma \cup \{ \varphi \}$ is consistent $\Leftrightarrow$ $\varphi \in \Sigma$

The next lemmas show why an m.c.s. is suitable to play the role of our semantic unit.

Lemma 3.4

- Every consistent set of formulas Γ can be extended to a m.c.s. $\Sigma \supseteq \Gamma$.
- Let Σ be maximal consistent, $\varphi_1, \varphi_2, \varphi \in \mathsf{L}$ Then:
    1. $(\varphi_1 \wedge \varphi_2) \in \Sigma \;\Leftrightarrow\; \varphi_1 \in \Sigma$ and $\varphi_2 \in \Sigma$
    2. $(\varphi_1 \vee \varphi_2) \in \Sigma \;\Leftrightarrow\; \varphi_1 \in \Sigma$ or $\varphi_2 \in \Sigma$
    3. $\varphi_1 \in \Sigma$ and $(\varphi_1 \to \varphi_2 \in \Sigma) \Rightarrow \; \varphi_2 \in \Sigma$

[3] although we should define these notions with respect to some logic $\mathbf{X}$, we sometimes omit reference to $\mathbf{X}$ here.

4. $\neg\varphi \in \Sigma \Leftrightarrow \varphi \notin \Sigma$

5. $\varphi \in \Sigma \Leftrightarrow \Sigma \vdash \varphi$

Lemma 3.5 Let **X** be a modal logic, that is sound with respect to some class of models $\mathcal{X}$, i.e. for which we have $\vdash_{\mathbf{X}} \varphi \Rightarrow \models_{\mathcal{X}} \varphi$. Then, for any model $M \in \mathcal{X}$, and world $w \in W$, the set $Th(\langle M, w\rangle) = \{\psi | M, w \models \psi\}$ is maximal consistent with respect to **X**.

Proof: First, $Th(\langle M, w\rangle)$ is of course consistent, otherwise we would have $Th(\langle M, w\rangle) \vdash_{\mathbf{X}} \bot$, and, by soundness, $M, w \models \bot$, which is absurd. Next, suppose $Th(\langle M, w\rangle) \cup \{\varphi\}$ is consistent, for some φ. If $\varphi \notin Th(\langle M, w\rangle)$, then $M, w \not\models \varphi$, so $M, w \models \neg\varphi$ and thus $\neg\varphi \in Th(\langle M, w\rangle)$, contradicting our assumption that $Th(\langle M, w\rangle) \cup \{\varphi\}$ is consistent. ∎

Definition 3.6 Let **X** be some modal logic in a language L' with a set of modal operators $\mathsf{OP}' = \{O_1, \ldots, O_k\}$. Then, the canonical model for **X** is the model $M^c_{\mathbf{X}} = \langle W^c, V^c, R^c_1, \ldots, R^c_k\rangle$, where

- $W^c = \{\Sigma \subseteq \mathsf{L}' | \Sigma \text{ is maximal consistent in } \mathbf{X}\}$
- $R^c_s \Sigma\Delta \Leftrightarrow$ for all $O_s\varphi \in \mathsf{L}'$, $(O_s\varphi \in \Sigma \Rightarrow \varphi \in \Delta)$
- for all atoms p: $V^c(\Sigma)(p) = \text{true} \Leftrightarrow p \in \Sigma$

Crucial in the completeness proof for **X** is then the following lemma:

Lemma 3.7 Truth Lemma ([7, 11]) For all $\varphi \in \mathsf{L}'$, $\varphi \in \Sigma \Leftrightarrow M^c_{\mathbf{X}}, \Sigma \models \varphi$

Now, to show that a modal logic **X** is complete with respect to validity in some class $\mathcal{X}$ of models, i.e. that for all φ, $\models_{\mathcal{X}} \varphi \Rightarrow \vdash_{\mathbf{X}} \varphi$, the standard argument runs as follows: suppose that $\not\vdash_{\mathbf{X}} \varphi$, i.e. $\neg\varphi$ is consistent. Then, by Lemma 3.4 there is some m.c.s. Σ with $\neg\varphi \in \Sigma$. By the Truth Lemma then, $M^c_{\mathbf{X}} \models \neg\varphi$. If we moreover can prove that $M^c_{\mathbf{X}} \in \mathcal{X}$ we find $\not\models_{\mathcal{X}} \varphi$. Note that the whole procedure runs by a standard argument on m.c.s.'s, and the main non-standard part consists of proving that the canonical model $M^c_{\mathbf{X}}$ is of the right kind, i.e. a member of $\mathcal{X}$.

It is well-known, that adding some properties for $\Box$ to a modal logic (like $\Box\varphi \to \Box\Box\varphi$), often implies that, semantically, one has to put some additional constraints on $R_\Box$ in order to regain completeness (like transitivity of $R_\Box$). This is a good place to make the following distinction:

Definition 3.8 Let φ be a modal scheme, and Φ some (generally first order) condition on frames.

- We say that φ *corresponds* to Φ ($\varphi \sim_{cor} \Phi$) if for all frames F, $F \models \varphi \Leftrightarrow F$ satisfies Φ. In such a case Φ is called *definable* (in the modal language at hand).
  There is also a notion of correspondence and definability *relative* to some class of frames $\mathcal{F}$: φ corresponds to Φ relative to $\mathcal{F}$, if for all frames $F \in \mathcal{F}$, $F \models \varphi \Leftrightarrow F$ satisfies Φ. We then say that φ defines Φ *on the class* $\mathcal{F}$.

- φ is *canonical* for Φ ($\varphi \sim_{can} \Phi$) if the canonical model for the logic containing the axiom φ has property Φ

- φ is *complete* for Φ ($\varphi \sim_{com} \Phi$) if $\mathbf{K} + \varphi$ is complete with respect to the class of models that satisfies property Φ.

To give some examples of correspondences (for a thorough treatment, cf. [14]) we have $\Box\varphi \to \varphi \sim_{cor} \forall x R_\Box xx$, $\neg\Box\varphi \to \Box\neg\Box\varphi \sim_{cor} \forall x, y, z((R_\Box xy \wedge R_\Box xz) \to R_\Box yz)$, and, to give some multi-modal examples (see also [11]), $\Box_1\varphi \to \Box_2\varphi \sim_{cor} R_{\Box_2} \subseteq R_{\Box_1}$, $(\Box_1\varphi \to \Box_3\varphi) \wedge (\Box_2\psi \to \Box_3\psi) \sim_{cor} R_{\Box_3} \subseteq (R_{\Box_1} \cup R_{\Box_2})$ and $\Box_1\varphi \leftrightarrow (\Box_2\varphi \wedge \Box_3\varphi) \sim_{cor} R_{\Box_1} = R_{\Box_2} \cup R_{\Box_3}$. We have $\varphi \sim_{can} \Phi \Rightarrow \varphi \sim_{com} \Phi$, but it is important to notice that these notions are really different. To give some examples, let φ be $\Box\psi \to \Box\Box\psi$ and Φ transitivity of $R_\Box$; then $\varphi \sim_{cor} \Phi$, $\varphi \sim_{can} \Phi$ and $\varphi \sim_{com} \Phi$. Taking the modal formula $\varphi = A3 \wedge A4 \wedge A5$ and Φ is $\forall x, y R_i xy$ then $\varphi \not\sim_{cor} \Phi$ $\varphi \not\sim_{can} \Phi, \varphi \sim_{com} \Phi$. The latter modal formulas *does* correspond to a first order formula: we have $\varphi \sim_{cor}$ (R_i is an equivalence relation).

Related to these different notions is the fact that some logic $\mathbf{X}$ can be complete with respect to validity in several classes of models. As an example, we could take the normal modal logic $\mathbf{K}$. This logic is complete with respect to the class of arbitrary Kripke models $\mathcal{K}$, but is also sound and complete with respect to the class of *irreflexive* Kripke models $\mathcal{I} \subseteq \mathcal{K}$. The argument for this runs as follows: suppose $\not\vdash \varphi$, then $\neg\varphi$ is consistent in $\mathbf{K}$, and, by completeness, there is a model $M \in \mathcal{K}$ for which $M, w \models \neg\varphi$. This model then can be unravelled (cf. Section 4.2 and [12]) into an irreflexive model M' for which $M', w' \models \neg\varphi$, and hence $\not\models_{\mathcal{I}} \varphi$. Similarly, an argument about *generated sub-models* (cf. Lemma 4.1) can be put to work to show that $S5$ is sound and complete both with respect to the class of models in which R is an equivalence relation, and the class in which R is universal.

In the sequel, we will see several cases where an axiom φ of $\mathbf{L}$ raises, on semantic grounds, the expectation that it is canonical for some property Φ, but it is not; and neither does φ correspond to Φ. For those cases, we will apply some validity-preserving technique in order to derive completeness for a class of models satisfying Φ.
This is how our completeness proof in the next Sections will proceed:

- We give classes of models $\mathcal{C} \subseteq \mathcal{K}^m_{CDE}$, for which we provide arguments to support the claim that $\mathbf{L}$ is sound or complete for them. In particular, these models satisfy a number of first-order properties Φ_x (to be introduced in Definition 3.11) so that we also will write $\mathcal{K}^m_{CDE}$ ({Φ_1, Φ_2, Φ_{3a}, Φ_{3b}, Φ_{4a}, Φ_{4b}}) for $\mathcal{L}$. This class $\mathcal{L} \subseteq \mathcal{K}^m_{CDE}$ is the class of models satisfying all the properties Φ_x that are relevant here, and for which we aim to prove that **L** is both sound and complete for it.

- We then start reasoning from $M^c_{\mathbf{L}}$, showing that this canonical model for **L** is in fact not a member of $\mathcal{L}$; it is a member of a class $\mathcal{K}^m_{CDE}$ ({Φ_1, Φ_2, Φ_{3b}, Φ_{4b}}) $\supset \mathcal{L}$. For this class we show that **L** is complete for it, but at the same time, it does lack vital properties Φ_{4a} and Φ_{3a}, concerning R_C and R_D, respectively.
- We then show that, as far as completeness and soundness are concerned, one might restrict oneself to a class of models $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$, with $\mathcal{L} \subseteq \mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\}) \subseteq \mathcal{K}^m_{CDE}$ ({Φ_1, Φ_2, Φ_{3b}, Φ_{4b}}), thus obtaining models that *do* have the desired property for R_C with respect to R_E. We also show that in fact the subclass $\mathcal{FIN}(\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\}))$ of *finite* models of $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ validates exactly the **L**-derivable formulas, implying decidability of **L**.
- The class $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ appears to still suffer from not being intended with respect to R_D. We show that we may 'throw away' the 'bad' models from this class, ending up in $\mathcal{L}$, whereas this procedure does not affect the valid formulas of these classes.

As will be explained in Section 5 the order in which we apply the respective techniques, seems to be crucial.

3.2 Models for L

In Section 3.1 we mentioned that one can obtain completeness for a modal logic if one inspects the canonical model for that logic (cf. our remarks following the Truth Lemma). It belongs to the folklore on epistemic modal logic that the axioms $A3 - A5$ enforce the relations R^c_i to be reflexive, transitive Euclidean relations: i.e. the relations R^c_i are *equivalence relations*. Moreover, it is easily seen that $A6 \sim_{can} R^c_E = R^c_1 \cup \ldots \cup R^c_m$, so the model $M^c_{\mathbf{L}}$ is such that R^c_E is the union of the R^c_i's. Let us consider R^c_C and R^c_D. The argument that follows applies to arbitrary relations that are associated with the operators C, D and E; therefore, for the moment, we will not consider them as relations in the canonical model; we will discuss for instance R_C instead of R^c_C.

Definition 3.9 Let R be a binary relation on W. Then we define, for any $w \in W$, $R(w) = \{v \mid Rwv\}$. Moreover, we define R^0xy iff $x = y$; R^nxy iff there is a z for which Rxz and $R^{n-1}zy$ ($n > 0$). The *transitive closure* R^* of R is the set $\{(x,y) \mid \exists n : R^nxy\}$. For any model $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E \rangle$ and world w in M, we let M_w be the model *generated* by w: it is the restriction of M to those worlds that are $(R_1 \cup \ldots \cup R_m \cup R_C \cup R_D \cup R_E)^*$-reachable from w. For arbitrary k-dimensional models $M = \langle W, V, R_1, \ldots R_k \rangle$, the worlds in M_w are those that are $(R_1 \cup \ldots \cup R_k)^*$-reachable from w. The definition of R^n also has a syntactical counterpart: we define $\Box^0 \varphi =_{\text{def}} \varphi$ and $\Box^n \varphi =_{\text{def}} \Box^{n-1} \Box \varphi$.

Lemma 3.10 Let R be an equivalence relation on W. Then, for all $w, v \in W$:

$$Rwv \Rightarrow R(w) = R(v)$$

Axiom $A9$ tells us that we might hope that C is indeed the necessity operator for some relation R_C. Axiom $A8$ guarantees that $C\varphi \to (E\varphi \wedge EE\varphi \wedge EEE\varphi \ldots)$. In fact, the intended meaning of the C-operator is that it equals all possible nestings of E-operators. So, if we would allow for infinite conjunctions, we have the following:

$$C\varphi \leftrightarrow (E\varphi \wedge EE\varphi \wedge EEE\varphi \ldots) \quad (1)$$

In fact, Halpern and Moses ([6]) impose this semantically by defining $M, w \models C\varphi$ iff for all $k, M, w \models E^k \varphi$. If we would have the displayed equivalence (1), the axioms $A8$ and $A10$ would easily follow. Keeping this is in mind, we can unravel the truth definition for $C\varphi$:

$$\begin{array}{lll} M, w \models C\varphi & \Leftrightarrow & M, w \models (E\varphi \wedge EE\varphi \wedge EEE\varphi \ldots) \\ & \Leftrightarrow & \forall v \text{ with } R_E wv, (M, v \models \varphi \\ & & \text{and } \forall u \text{ with } R_E vu, (M, u \models \varphi \\ & & \text{and } \forall t \text{ with } R_E ut, (\ldots))) \\ & \Leftrightarrow & \forall s \text{ with } R_E^* ws,\ M, s \models \varphi \end{array}$$

Thus, a reasonable guess seems to be that $R_C = R_E^*$.

Finally, we consider R_D. Halpern and Moses ([5]) approach distributed knowledge semantically: they define $D\varphi$ to be true at w iff φ is true in all worlds v such that $(R_1 \cap \ldots \cap R_m)wv$. In other words, φ is distributed knowledge iff φ is true in all those worlds that are epistemic alternative for all the agents. Using terminology of Halpern and Moses, whereas we may relate the E-operator to the knowledge of 'any fool' (since all the agents know φ if $E\varphi$ is true), we may relate the D-operator to the 'wise man' (since in the truth definition of D, any epistemic alternative is given up as soon as one of the agents rejects this alternative). This notion of distributed knowledge is often related to *communication* in the group: φ should be distributed knowledge iff φ would be a conclusion if all the agents put their knowledge together. One way to formalize this is the following:

$$M, w \models D\varphi \Leftrightarrow \exists \varphi_1, \ldots \varphi_m : M, w \models K_i \varphi_i \text{ and } \models (\varphi_1 \wedge \cdots \wedge \varphi_m) \to \varphi \quad (2)$$

However, we demonstrated in [18] that, with the semantical definition of D above, the '$\Rightarrow$'-direction of (2) cannot be guaranteed. So, when thinking of D-knowledge as the knowledge of the 'wise man', this wise man may know more than the agents 'know together'. For more technical aspects of this issue, we refer to our full paper [18]. Now, we return to the semantics of **L**, since we are ready to define our ideal, or intended class of models.

Definition 3.11 We define the following first order properties on the class of models $\mathcal{K}^m_{CDE}$ with typical elements $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E\rangle$

Φ_1: The relations R_i are equivalence relations ($i \in \{1, \ldots, m, D\}$)

Φ_2: $R_E = R_1 \cup \ldots \cup R_m$

Φ_{3a}: $R_D \supseteq R_1 \cap \ldots \cap R_m$

Φ_{3b}: $R_D \subseteq R_1 \cap \ldots \cap R_m$

Φ_{4a}: $R_C \subseteq R_E^*$

Φ_{4b}: $R_C \supseteq R_E^*$

For any subclass $\mathcal{C} \subseteq \mathcal{K}_{m,C,D,E}$, and any subset $F \subseteq \{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\}$ the class $\mathcal{C}(F) \subseteq \mathcal{C}$ is the subset of models from $\mathcal{C}$ that satisfy the properties of F. The class of models $\mathcal{K}_{m,C,D,E}(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$, i.e. the models that satisfy *all* the properties denoted above, is abbreviated by $\mathcal{L}$.

Lemma 3.12 (Soundness) L is sound for $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$: $\vdash_{\mathbf{L}} \varphi \ \Rightarrow \models_{\mathcal{K}^m_{CDE}(\{\Phi_1,\Phi_2,\Phi_{3b},\Phi_{4a},\Phi_{4b}\})} \varphi$

Proof: As an illustration, we show validity of the induction axiom ($A10$). Let M be a model in $\mathcal{L}$, and suppose $M, w \models C(\varphi \to E\varphi) \wedge \varphi$. Let v be any world for which $R_C wv$, i.e. $R_E^* wv$. This means that there is a sequence $v_0, \ldots v_n$ such that $w = v_0, v = v_n$ and $R_E v_k v_{k+1}$ $(k < n)$. All these v_k's are R_C-related to w, so in all of them, $\varphi \to E\varphi$ is true. By assumption, φ is true in w. By the observation above, $E\varphi$ must be true in w, and hence φ in v_1. Repeating this argument n times, we end up with φ being true in $v_n = v$. Since this was an arbitrary R_C-successor of w, we have $M, w \models C\varphi$. ■

That the property Φ_{4a} is necessary for soundness was observed in [9]; in Theorem 3.14 we will see that it is not needed for completeness. In the rest of this paper we are concerned with proving completeness of **L** with respect to $\mathcal{L}$. First we note, that the canonical model $M^c_{\mathbf{L}}$ is not of the proper kind:

Proposition 3.13 Consider the canonical model $M^c_{\mathbf{L}}$ for **L**. This model has all the properties of being an $\mathcal{L}$-model, except for the properties Φ_{3a} and Φ_{4a}

Proof: That $M^c_{\mathbf{L}}$ satisfies the properties Φ_1, Φ_2 and Φ_{3b} is due to the fact that $(A3 \wedge A4 \wedge A5)$ and $(A13 \wedge A14 \wedge A15)$ are canonical for Φ_1; $A6 \sim_{can} \Phi_2$ and $A11 \sim_{can} \Phi_{3b}$. These facts belong to the folklore on modal logic, the reader may consult [14] for a general treatment, or [11] for these specific cases. Let us now show that $M^c_{\mathbf{L}}$ also satisfies Φ_{4b}. To do so, suppose that for $\Gamma, \Delta \in W^c$ we have $R^*_E\Gamma\Delta$; it means that there are $\Sigma_1 = \Gamma, \Sigma_2, \ldots, \Sigma_n = \Delta$ such that for each Σ_k with $k < n$ there is an $i_k \leq m$ such that $R^c_{i_k}\Sigma_k\Sigma_{k+1}$. We have to show that $R^c_C\Gamma\Delta$; so suppose $C\varphi \in \Gamma$, we now have to demonstrate that $\varphi \in \Delta$. Using $A6$ and $A8$ we see that $\vdash_{\mathbf{L}} C\psi \to K_{i_k}C\psi$. This implies that $K_{i_k}C\varphi \in \Sigma_k$ whenever $C\varphi \in \Sigma_k$. Since $C\varphi \in \Sigma_1$, we have $K_{i_1}C\varphi \in \Sigma_1$ and, since $R^c_{i_1}\Sigma_1\Sigma_2$, we have $C\varphi \in \Sigma_2$. Repeating this argument, we obtain $C\varphi \in \Sigma_n = \Delta$. By axiom $A7$, we conclude that $\varphi \in \Delta$.
We will now show that $M^c_{\mathbf{L}}$ does *not* satisfy Φ_{3a} and Φ_{4a}.

Φ_{3a} Let us consider a system for two agents ($m = 2$) and consider the following model $M \in \mathcal{L}$.

- $W = \{x_1, x_2, y_1, y_2\}$
- $V(x_1) = V(x_2); V(x_1)(p) =$ *true* and $V(y_1) = V(y_2); V(y_1)(p) =$ *false*
- The equivalence classes for R_1 are $\{x_1, y_1\}$ and $\{x_2, y_2\}$, for R_2 they are $\{x_1, y_2\}$ and $\{y_1, x_2\}$. Furthermore, $R_D = \{(w, w) | w \in W\}$, and $R_E = R_1 \cup R_2$; $R_C = (R_E)^*$.

Obviously, M is a model from $\mathcal{L}$. Moreover, an easy induction on modal formulas shows that

$$Th(\langle M, x_1\rangle) = Th(\langle M, x_2\rangle) \text{ and } Th(\langle M, y_1\rangle) = Th(\langle M, y_2\rangle)$$

Combining the Lemmas 3.5 and 3.12, the sets $\Sigma = Th(\langle M, x_1\rangle)$ and $\Delta = Th(\langle M, y_1\rangle)$ are maximally consistent with respect to $\mathbf{L}$. Hence, they appear in the canonical model M^c for $\mathbf{L}$ as two worlds, with the property that $R^c_1\Sigma\Delta$, $R^c_2\Sigma\Delta$, but *not* $R^c_D\Sigma\Delta$. The latter holds because $Dp \in \Sigma$, but $p \notin \Delta$.

Φ_{4a} The fact that M^c does not satisfy Φ_{4a} was already observed by Goldblatt in [4]; for a formal verification of the following argument we refer to [11]. Consider the set

$$\Gamma = \{Ep, EEp, EEEp, \ldots\} \cup \{\neg Cp\}$$

This set is consistent in our logic $\mathbf{L}$; we showed in [11] that all of its finite subsets are $\mathcal{K}^m_{CDE}$ ($\{\Phi_1$, Φ_2, Φ_{3b}, $\Phi_{4b}\}$)-satisfiable, and hence consistent. By the first item of Lemma 3.4, there must be a m.c. set $\Sigma \supseteq \Gamma$. Now, consider the set $\Omega = \{\neg p\} \cup \{\psi | \ C\psi \in \Sigma\}$. Consistency of Σ implies that Ω is also consistent, so, again with Lemma 3.4 we find a m.c. set

$\Delta \supseteq \Omega$. By the definition of the canonical model, (cf. Definition 3.6) the sets Σ, Δ are worlds in W^c for which $R^c_C \Sigma\Delta$, but not $(R^c_E)^* \Sigma\Delta$, so that the canonical model for **L** does not satisfy Φ_{4a}. That we do not have this is seen as follows. Suppose that $(R^c_E)^* \Sigma\Delta$ would hold, then, for some n we would have $(R^c_E)^n \Sigma\Delta$. But, as one easily verifies, the fact that $E^n p \in \Sigma$ then implies that p is in Δ, which is a contradiction, since $\Delta \supseteq \{\neg p\}$ is a consistent set.

■

Theorem 3.14 Let $\mathcal{K}^m_{CDE}(F)$ be among the classes of models of Definition 3.11 in $\mathcal{L}$ and **L** as before. Then we have:

1. Soundness:
$F \supseteq \{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\} \Leftrightarrow (\vdash_{\mathbf{L}} \varphi \Rightarrow \models_{\mathcal{K}^m_{CDE}(F)} \varphi)$

2. Completeness:
$F \subseteq \{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\} \Rightarrow (\vdash_{\mathbf{L}} \varphi \Leftarrow \models_{\mathcal{K}^m_{CDE}(F)} \varphi)$

Proof:

1. Let $G = \{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\}$. To prove '$\Rightarrow$', if $F \supseteq G$, then $\models_{\mathcal{K}^m_{CDE}(G)} \varphi \Rightarrow \models_{\mathcal{K}^m_{CDE}(F)} \varphi$ and the conclusion follows from Lemma 3.12. Conversely, if $F \not\supseteq G$, we can find a model $M \in \mathcal{K}^m_{CDE}(F)$ that does not satisfy one of the axioms of **L**. Here, the examples of correspondences that we gave in section 3.1 following Definition 3.8 are helpful. For, suppose that for a $\Phi_x \in G \setminus F$ we know that there is a modal formula φ_x such that $\varphi_x \sim_{cor} \Phi_x$ and $\vdash_{\mathbf{L}} \varphi_x$. By correspondence, we find a model $M \in \mathcal{K}^m_{CDE}(F)$, such that $M \not\models \varphi_x$, and this is sufficient to disprove soundness of **L** with respect to $\mathcal{K}^m_{CDE}(F)$. Since we know that the conjunction $\varphi_1 = A3 \wedge A4 \wedge A5$ corresponds to Φ_1, (in case $i \leq m$; if $i = D$, we take $\varphi_1 = A13 \wedge A14 \wedge A15$), if $\Phi_1 \in G \setminus F$, we find a model $M \in \mathcal{K}^m_{CDE}(F)$ such that $M \not\models \varphi_1$. Similarly, if $\Phi_2 \in G \setminus F$, we find $\not\models_{\mathcal{K}^m_{CDE}(F)} A6$, and if $\Phi_{3b} \in G \setminus F$, we have $\not\models_{\mathcal{K}^m_{CDE}(F)} A11$. Finally, if $\Phi_{4a} \in G \setminus F$, by an argument of [9] we have $\not\models_{\mathcal{K}^m_{CDE}(F)} A10$ and, if $\Phi_{3b} \in G \setminus F$ then one may choose $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E\rangle$ such that $R_C = \emptyset$, and M satisfies all the properties of F. Obviously, one easily chooses V such that $M \not\models A7$, so $\not\models_{\mathcal{K}^m_{CDE}(F)} A7$.

2. Suppose $\not\vdash_{\mathbf{L}} \varphi$, that is, $\neg\varphi$ is **L**-consistent and thus $\neg\varphi \in \Sigma$ for some maximal **L**-consistent set Σ. From Proposition 3.13 we know that there is a model in $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$, i.e. the canonical model $M^c_{\mathbf{L}}$, such that $M^c_{\mathbf{L}}, \Sigma \models \neg\varphi$. Obviously, this model $M^c_{\mathbf{L}}$ is a $\mathcal{K}^m_{CDE}(F)$ model, for any $F \subseteq \{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\}$, and hence, for each such F, we have $\not\models_{\mathcal{K}^m_{CDE}(F)} \varphi$.

■

Our conclusion so far is, that the epistemic logic **L** can be proven to be complete with respect to those $\mathcal{K}^m_{CDE}$- Kripke models that need not satisfy Φ_{3a} and Φ_{4a}, since the canonical model for **L** does not satisfy those properties. This implies that no subset of axioms of **L** is canonical for Φ_{3a} or Φ_{4a}. In the sequel (Corollary 4.29) we will also learn that none of the properties Φ_{3a} and Φ_{4a} is *definable.*

4 Transferring Truth

We know from Theorem 3.14 that our epistemic logic **L** is complete with respect to the class of models $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$. In particular, the theorem implies that every **L**-consistent formula φ is satisfied in some model $M \in \mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$, which means that for some world w in M, we have $M, w \models \varphi$.

Lemma 4.1 Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{K}^m_{CDE}$ be two classes of Kripke models, and φ a modal formula. Suppose that for every M_1, w_1 (with $M_1 \in \mathcal{C}_1$) for which $M_1, w_1 \models \varphi$, we can find a pair M_2, w_2 (with $M_2 \in \mathcal{C}_2$) such that $M_2, w_2, \models \varphi$. Then any logic **J** that is complete with respect to $\mathcal{C}_1$ is also complete with respect to $\mathcal{C}_2$.

Proof: Easy: completeness of **J** with respect to $\mathcal{C}_1$ simply means that every **J**-consistent formula is satisfied at some pair M_1, w_1, with $M_1 \in \mathcal{C}_1$. The conditions of the lemma then immediately yield that there is a pair M_2, w_2 with $M_2 \in \mathcal{C}_2$ that satisfies φ, which implies that **J** is complete with respect to $\mathcal{C}_2$. ■

So our strategy will be, given a $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$-model M_1 that satisfies a formula φ at some world w_1, to find a model $M_2 \in \mathcal{L}$ with a world w_2, that satisfies φ as well. In this way, we establish completeness of **L** with respect to $\mathcal{L}$: we already know from the first item of Theorem 3.14 that **L** is also sound with respect to $\mathcal{L}$. In fact we implement the strategy mentioned above in a number of steps; we will find models M in several classes, that all preserve the truth of φ. To prove such preservation, we will often use the following construct:

Definition 4.2 A function f between two models $M_1 = \langle W_1, V_1, R_1 \rangle$ and $M_2 = \langle W_2, V_2, R_2 \rangle$ is called a *p-morphism* if it satisfies the following conditions:

- f is surjective
- $\forall w_1, v_1 \in W_1$: $R_1 w_1 v_1 \Rightarrow R_2 w_2 v_2$
- $\forall w_1 \in W_1, v_2 \in W_2$: $R_2 f(w_1) v_2 \Rightarrow \exists v_1 \in W_1$: $R_1 w_1 v_1$ and $f(v_1) = v_2$
- $\forall p \in \mathsf{P}, \forall w_1 \in W_1 : V_1(w_1, p) = V_2(f(w_1), p)$

Segerberg ([13] introduced the notion of p-morphism (for 'pseudo-morphism') for standard modal logic. Later on, this notion was generalized to p-relation or *zigzag-connection*, which, on its turn, is a special case of a *bi-simulation* (see [14]), a familiar notion in the field of process-algebra. The notion of p-morphism is straightforwardly extended to k-dimensional Kripke models: from now on, when we say that f is a p-morphism between two models M_1 and M_2, we assume that the two models have the same number of accessibility relations $R_{i_1}, \dots R_{i_k}$ ($i \in \{1,2\}$), and that f satisfies the conditions above for each pair R_{1_j}, R_{2_j} ($j \leq k$).

Lemma 4.3 Let f be a p-morphism between M_1 and M_2. Then, for all formulas φ:

1. for all $w_1 \in W_1$: $M_1, w_1 \models \varphi \;\Leftrightarrow\; M_2, f(w_1) \models \varphi$
2. $M_1 \models \varphi \;\Leftrightarrow\; M_2 \models \varphi$

Proof: With induction on the (multi-) modal formula φ; see for instance [14]. ■

Before we start off, we give one more truth preserving result on models that we will need in the sequel:

Lemma 4.4 Let M be a multi-modal model with some world w, and let M_w be the model generated by w (Cf. 3.9). Then, for all formulas φ:

1. $M, w \models \varphi \;\Leftrightarrow\; M_w, w \models \varphi$
2. $M \models \varphi \;\Rightarrow\; M_w \models \varphi$

4.1 Filtrations

Definition 4.5 Let M be a general Kripke model, and $\Sigma \subseteq \mathsf{L}$ a set of formulas. The relation $\equiv_\Sigma$ on W defined as

$$w \equiv_\Sigma v \Leftrightarrow \text{ for all } \sigma \in \Sigma : (M, w \models \sigma \Leftrightarrow M, v \models \sigma)$$

defines an equivalence on W; we denote its equivalence classes with $[w]_\Sigma$; we will often omit the subscript Σ, though. The model $M^f = \langle W^f, V^f, R_1^f, \dots, R_m^f, R_C^f, R_D^f, R_E^f \rangle$ is a *filtration of M through* Σ if

- $W^f = \{[w] \mid w \in W\}$
- Each $R_\Box^f$ satisfies the properties Min^f and Max^f:

Min^f For all $[w], [v] \in W^f$, if $R_\Box wv$ then $R_\Box^f[w][v]$

Max^f If $R_\Box^f[w][v]$ then, for all $\Box\psi \in \Sigma(M, w \models \Box\psi \;\Rightarrow\; M, v \models \psi)$

- $V^f([w], p) = V(w, p)$

Lemma 4.6 Let M^f be a filtration through Σ of M. Then, for all $[w] \in W^f$ and all $\sigma \in \Sigma$,

$$M, w \models \sigma \Leftrightarrow \mathcal{M}^f, [w] \models \sigma$$

Definition 4.7 Let φ be a formula in L, and

- $\Sigma_1 = \{\psi, \neg\psi \mid \psi \text{ is a sub-formula of } \varphi\}$
- $\Sigma_2 = \{K_1\psi, \neg K_1\psi \mid E\psi \in \Sigma_1\}$
- $\Sigma_3 = \{EC\psi, \neg EC\psi, K_1C\psi, \neg K_1C\psi \mid C\psi \in \Sigma_1\}$
- $\Sigma_4 = \{DK_i\psi, \neg DK_i\psi \mid K_i\psi \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3, i \leq m\}$

Then $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_4$ is called the *suitable* set of formulas for φ (or just: 'Σ is suitable for φ').

Lemma 4.8 Let Σ be suitable for φ. Then:

- Σ is finite
- Σ is 'semi-closed' under negation: if $\psi \in \Sigma$ is not of the form $\neg\psi'$ then $\neg\psi \in \Sigma$.
- For all formulas ψ, $i \leq m$:
    1. $C\psi \in \Sigma \Rightarrow EC\psi \in \Sigma$
    2. $K_i\psi \in \Sigma \Rightarrow DK_i\psi \in \Sigma$
    3. $E\psi \in \Sigma \Rightarrow K_1\psi \in \Sigma$

Definition 4.9 Let $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E\rangle$ be a model in $\mathcal{K}^m_{CDE}$ ($\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\}$). Furthermore, let φ be a formula and Σ suitable for φ. We define $M^+ = \langle W^+, V^+, R_1^+, \ldots, R_m^+, R_C^+, R_D^+, R_E^+\rangle$ as follows

- $W^+ = \{[v] \mid v \in W\}$
- The relations in M^+ are defined as follows:
    1. $R_\Box^+[w][v] \Leftrightarrow$ for all $\Box\psi \in \Sigma : (M, w \models \Box\psi \Leftrightarrow M, v \models \Box\psi,)$ for all $R_\Box \in \{R_1, \ldots R_m, R_D\}$
    2. $R_E^+ = (R_1^+ \cup \ldots \cup R_m^+)$
    3. $R_C^+ = (R_E^+)^*$
- $V^+([w])(p) = V(w)(p)$

Lemma 4.10 Let M^+ be defined as in Definition 4.9 above. Then, for each $A \subseteq W^+$ there is a formula σ_A such that for all $[w] \in W^+$;

$$M, w \models \sigma_A \;\Leftrightarrow\; [w] \in A$$

Proof: Let $Form(w)$ be the conjunction of formulas in Σ that are true in M, w. From the definition of [.], it is clear that we have $M, v \models Form(w) \;\Leftrightarrow\; [w] = [v]$. Now, let σ_A be the disjunction of all $Form(a)$ for which $[a] \in A$; it is easily seen that this σ_A satisfies the lemma. ■

Lemma 4.11 Let M be a model in $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$ and M^+ as in Definition 4.9. Then M^+ is a filtration through Σ.

Proof: We have to show that each $R_\Box$ satisfies the properties Min^f and Max^f, respectively.

- Suppose $R^+_\Box \in \{R^+_1, \ldots, R^+_m, R^+_D\}$.
  To prove Min^f, let $[w], [v] \in W^+$ and $R_\Box wv$, and suppose $\Box\psi \in \Sigma$. In order to prove that $R^+[w][v]$, we have to show that $M, w \models \Box\psi \;\Leftrightarrow\; M, v \models \Box\psi$. But this follows immediately from the fact that $R_\Box(w) = R_\Box(v)$ which on its turn follows from Lemma 3.10.
  To prove Max^f for the $R^+_\Box$'s under consideration, suppose $R^+_\Box[w][v]$, and $M, w \models \Box\psi$. By definition of $R^+_\Box$, we have $M, v \models \Box\psi$. Since all the $R_\Box$'s under consideration are reflexive, we get $M, v \models \psi$.

- Consider R^+_E. To prove Min^f, suppose $R_E wv$. Since $R_E = R_1 \cup \ldots \cup R_m$, for some $i \leq m$ we have $R_i wv$. the Min^f condition for R_i guarantees that $R^+_i[w][v]$ and hence, by definition of R^+_E, also $R^+_E[w][v]$.
  To check Max^f, suppose $R^+_E[w][v]$, and $E\psi \in \Sigma$. If $M, w \models E\psi$ then, by definition of R_E, we also have $M, w \models K_1\psi$. By Lemma 4.8 we also have $K_1\psi \in \Sigma$. The Max^f condition for R_1 then guarantees that $M, v \models \psi$.

- The claim that Min^f holds for R^+_C is essentially an argument given by Goldblatt in [4]. We give the argument, modified to our set-up: suppose $R_C wv$. Let A be the set

  $$A = \{[u] \mid R^+_C[w][u]\}$$

  Let σ_A be the formula that satisfies the condition of Lemma 4.10. Now we claim that

  (*) $M, w \models C\sigma_A$

  In that case we are done: for (*) implies that $M, v \models \sigma_A$ and hence, that $[v] \in A$, which expresses that $R^+_C[w][v]$. To prove claim (*), we use the fact that the model M is an element of the class $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$, in particular, it satisfies the induction axiom, so that we have:

(1) $M, w \models C(\sigma_A \to E\sigma_A) \to (\sigma_A \to C\sigma_A)$

We now observe that the antecedent of the formula given at (1) is also true:

(2) $M, w \models C(\sigma_A \to E\sigma_A)$

To see (2), suppose $R_C wx$, and $M, x \models \sigma_A$. We have to show $M, x \models E\sigma_A$. For this, suppose $R_E xy$. Since $[x] \in A$, we have $R_C^+[w][x]$, or, equivalently, $(R_E^+)^*[w][x]$. The latter says that there is some n such that $(R_E^+)^n[w][x]$. Since the condition Min^f holds for R_E^+, we have $R_E^+[x][y]$. All together, we have $(R_E^+)^{n+1}[w][y]$, so $R_C^+[w][y]$ and thus $[y] \in A$. The latter, together with Lemma 4.10 implies that $M, y \models \sigma_A$, which proves (2).
Next, since $R_C^+[w][w]$, we also have

(3) $M, w \models \sigma_A$

Obviously, the items (1), (2) and (3) are sufficient to prove (*).

To see that Max^f holds for R_C^+, suppose $R_C^+[w][v]$ and $M, w \models C\psi$ for some $\psi \in \Sigma$. By definition of R_C^+, we have that there are $[w_1] = [w], [w_2], \ldots, [w_n] = [v]$ with $R_E^+[w_i][w_{i+1}]$, $(i < n)$. M satisfies axiom $A8$, so that we have $M, w_i \models C\psi \Rightarrow \mathcal{M}, w_i \models EC\psi$. Since Σ is suitable, we obtain, given $C\psi \in \Sigma$ that $EC\psi \in \Sigma$. The relation R_E^+ is known to satisfy Max^f, so that $M, w_i \models EC\psi$ implies $M, w_{i+1} \models C\psi$. Observing that $C\psi$ is preserved in the sequence of w_i's and $v = w_n$, we eventually get $M, v \models C\psi$. Since M satisfies $A7$, we get $M, v \models \psi$.

Corollary 4.12 Let M be in $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$, and M^+, Σ as above. Then, for all $\sigma \in \Sigma, w \in W$:

$$M, w \models \sigma \Leftrightarrow M^+, [w] \models \sigma$$

Proof: Combine 4.6 with 4.11 ∎

Not only is the model M^+ a filtration through M, it is, in some sense, also a nicer model than M.

Theorem 4.13 Let M be a model in $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$. Then M^+ is a model in $\mathcal{K}_{m,C,D,E}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$

Proof:

Φ_1 The relations R_i^+ are equivalence relations, as follows immediately from the definition of R_i^+, $i \in \{1, \ldots, m, D\}$

Φ_2 Immediately from the definition of R_E^+

Φ_{3b} Suppose $R_D^+[w][v]$. Then, for all $D\psi \in \Sigma$, we have $M, w \models D\psi \Leftrightarrow \mathcal{M}, v \models D\psi$. Now suppose that for some $i < m$, we have $K_i\psi \in \Sigma$ and $M, w \models K_i\psi$. By construction of Σ (Cf. Σ_4), we then also have $DK_i\psi \in \Sigma$. Since M satisfies $A4$, we have $M, w \models K_iK_i\psi$ and hence, using $A11$, $M, w \models DK_i\psi$. Thus, $M, v \models DK_i\psi$. Since M satisfies $A13$, we have $M, v \models K_i\psi$. Thus, we have $R_i^+[w][v]$.

$\Phi 4a$ Immediately from the definition of R_C^+

$\Phi 4b$ Like the proof of Φ_{4a}

■

Remark 4.14 Note the asymmetry in the definitions of the relations $R_1^+, \ldots, R_m^+, R_D^+$ on the one hand, and that of R_E^+ and R_C^+ on the other. By defining R_E^+ and R_C^+ as we did, the properties Φ_2, Φ_{4a} and Φ_{4b} of M^c are easily established, be it that it took more effort to prove Min^f and Max^f for them. Note that, since we do not have that R_E is an equivalence relation, a definition like

$$(*)\ R_\Box^+[w][v] \Leftrightarrow \text{ for all } \Box\psi \in \Sigma : (M, w \models \Box\psi \Leftrightarrow M, v \models \Box\psi)$$

would not have worked for $\Box = E$. To see this, let φ be $Ep \wedge K_1p$. Let M be a model in $\mathcal{K}_{CDE}^m$ ({Φ_1, Φ_2, Φ_{3b}, Φ_{4b}}) such that p is true in world w, x and y, but not in z. Suppose the equivalence classes for R_1 are $\{w, x\}, \{y\}$ and $\{z\}$, for R_2 they are $\{w, y\}, \{x, z\}$. Then, Ep is true at w, but not at x. So, although we do obtain $R_1^+[w][x]$ (we know that R_1^+ satisfies Min^f and we had R_1wx), using definition (*) for R_E^+, we would lose property Φ_2 for the filtration M^+. Moreover, once we define R_E^+ in terms of the R_i^+'s, rather than in terms of properties of the underlying model M, it is obvious that, when aiming for Φ_{4a}, Φ_{4b}, this is achieved most easily by imposing these properties immediately upon R_C^+. Finally, note that there is also an asymmetry in the definition of the suitable set Σ for φ in Definition 4.7, in the sense that it seems that the index 1 of K_1 plays a special role. In fact, any of the indices $1, \ldots, m$ would have done here: but one of them is necessary to prove the second item of Lemma 4.11. For the Max^f condition of R_E^+, we want to establish that v satisfies φ whenever w satisfies $E\varphi$ and $R_E^+[w][v]$. But, since this Max^f is already established for all $i \le m$, and we have $E\varphi \to K_i\varphi$ as a validity, the formula φ can be 'carried over' to v by Max^f of any of the R_i^+ 's, $i \le m$. In other words, the fact that we have chosen index 1 n Definition 4.7 is arbitrary, but sufficient.

We now have achieved, as for the conditions of Definition 3.11, that the model M^+ satisfies those of M, but it moreover satisfies the condition that $R_C^+ = (R_E^+)^*$.

Corollary 4.15 For every formula φ and every model $M_1 \in \mathcal{K}_{CDE}^m$ ({Φ_1, Φ_2, Φ_{3b}, Φ_{4b}}) with world w_1 such that $M_1, w_1 \models \varphi$, there is a model $M_2 \in \mathcal{K}_{CDE}^m$({$\Phi_1$, Φ_2, Φ_{3b}, Φ_{4a}, Φ_{4b}}) and a world w_2 such that $M_2, w_2 \models \varphi$.

Proof: Immediate from Corollary 4.12 and Theorem 4.13. ∎

Theorem 4.16 The logic **L** is sound and complete with respect to the class $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$

Proof: Soundness follows from Theorem 3.14. To prove completeness, we combine Corollary 4.15 with Lemma 4.1, with $\mathcal{C}_1 = \mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$ and $\mathcal{C}_2 = \mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$.

Corollary 4.17 The logic **L** is decidable

Proof: Let φ be a formula. It is satisfiable iff it is satisfied in the canonical model $\in \mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$. Let Σ be suitable respect to φ; we know from Corollary 4.12 that φ is satisfiable iff it is satisfied in the filtration M^+ through M^c. This filtration has at most $2^{|\Sigma|}$ worlds, so we only have to inspect finitely many models from $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ to check whether φ is satisfiable. ∎

4.2 Unravelling and Identifying

We now give a procedure to show that **L** is indeed also complete for those models in $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ that satisfy Φ_{3a}. We establish this by applying a technique that we developed in [17]; however, there we did not need to bother about keeping appropriate properties for R_E and R_C, and our presentation here will abstract away from the underlying rewrite techniques (a thorough embedding of the ideas explored here into the field of term-rewriting is planned in [19]). Roughly, our strategy is as follows. Consider the example model $N = \langle W, V, R_1, R_2, R_C, R_C, R_E\rangle$, where $W = \{w, a, b\}$, $R_1 = R_2 = W \times W$, and $R_D = \{(x, x) \mid x \in W\} \cup \{(a, b), (b, a)\}$, $R_E = W \times W = R_C$. Then, N is a $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$-model, that does not satisfy Φ_{3a}. We want to transform it into a model $N^{\equiv}_w$ that is a a model of $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ and such that there is a world w' in $N^{\equiv}_w$ such that for all φ, $(N, w \models \varphi \Leftrightarrow N^{\equiv}_w, w' \models \varphi)$. The first step in the procedure to construct $N^{\equiv}_w$ from N is by 'unravelling' all the paths in N, thus obtaining sequences of worlds and relations that connect them (Definition 4.18). Then, an equivalence on these paths is defined (also in 4.18). The equivalence classes thus obtained become the worlds in the new model $N^{\equiv}_w$ (Definition 4.21). In the sequel, we will come back to the example model N in order to clarify these two steps.

Definition 4.18 Let $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E\rangle \in \mathcal{K}^m_{CDE}$.

- A *path in M* is either a one-world sequence $\langle x\rangle$ with $x \in W$, or of the form $\vec{u} = \langle x_1, k^1, x_2, k^2, \ldots, k^{n-1}, x_n\rangle$, with $x_i \in W$ $(i \leq n)$ and $k^i \in I =$

$\{1,\ldots,m,D\}$ $(i < n)$ such that $R_{k^i}x_ix_{i+1}$ holds for all $i < n$. For such a path, we define $fst(\vec{u}) = x_1$, $lst(\vec{u}) = x_n$. If $fst(\vec{u}) = w$, $\vec{u}$ is called a *w-path*. Sets of paths of this form, with $x_i \in W$ and the indices k^i in some set I are denoted with $P(M,I)$. The set of all w-paths in $P(M,I)$ is denoted with $P_w(M,I)$.

- For two paths $\vec{u} = \langle x_1, k^1, x_2, k^2, \ldots, k^{n-1}, x_n\rangle$ and $\vec{v} = \langle y_1, m^1, \ldots, m^{r-1}, y_r\rangle$ for which $x_n = y_1$ we define their *concatenation* as $\vec{u};\vec{v} = \langle x_1, k^1, \ldots k^{n-1}, x_n, m^1, y_2, \ldots m^{r-1}, y_r\rangle$. If $lst(\vec{u}) \neq fst(\vec{v})$, concatenation is not defined. If $lst(\vec{u}) = fst(\vec{u})$, $\vec{u}^n$ is just n concatenations of $\vec{u}$. We will slightly abuse notation when writing a concatenation, for instance $\langle \vec{u}; k^i, y\rangle$ just means $\vec{u}$, concatenated with $\langle lst(\vec{u}), k^i, y\rangle$.

- We define an equivalence relation on a set of paths $P(M,I)$ in three steps:

  $\equiv_1$: We stipulate $\vec{u} \equiv_1 \vec{v}$ and $\vec{v} \equiv_1 \vec{u}$ iff one of the following cases holds ($i \in \{1,\ldots,m,D\}$):
  eq $\quad \vec{u} = \vec{v}$
  Ref(i) $\quad \vec{u} = \langle x, i, x\rangle$ and $\vec{v} = \langle x\rangle$
  Te(i) $\quad \vec{u} = \langle x, i, y, i, z\rangle$ and $\vec{v} = \langle x, i, z\rangle$
  D(i) $\quad \vec{u} = \langle x, i, y\rangle$ and $\vec{v} = \langle x, D, y\rangle$

  $\equiv_2$: $\vec{u} \equiv_2 \vec{v}$ iff for some $\vec{x}, \vec{y}, \vec{u}', \vec{v}' \in P(M,I)$ $\vec{u} = \vec{x};\vec{u}';\vec{y}$, $\vec{v} = \vec{x};\vec{v}';\vec{y}$ and $\vec{u}' \equiv_1 \vec{v}'$.

  $\equiv$: Finally, we put $\equiv$ to be the transitive closure of $\equiv_2$. The relation $\equiv$ is easily seen to be an equivalence on $P(M,I)$; we denote its classes with $|\vec{u}|, \vec{u} \in P(M,I)$

From now on, we will use k^i, l^i, m^i or just i and j for typical elements of $I = \{1,\ldots,m,D\}$, and x,y,z,u,v,w as typical elements of W, and, finally, we use $\vec{s}, \vec{t}, \vec{u}, \vec{v}$ and $\vec{w}$ as typical elements of $P(M,I)$. Note that a path is an alternating sequence of worlds from W and indexes of the relations $R_1,\ldots,R_m$, R_D in M. In fact, the technique of unravelling was introduced by Sahlqvist ([12]) for the mono-modal case. He then uses such paths as worlds to construct a new model verifying the same formulas as M. We will use the *equivalence classes* of the paths to construct a new model $M_w^{\equiv}$ (4.21). A few words on this equivalence relation are in order here. Firstly, for $\vec{u}$ and $\vec{v}$ to be equivalent, it is necessary that they are both members of the set of paths $P(M,I)$. Let us consider our example model N again: elements of $P(N,I)$ are $\langle w\rangle$, $\langle w,1,w\rangle$, $\langle w,2,w\rangle$, $\langle w,D,w\rangle$, $\langle w,1,a\rangle$, $\langle w,1,b\rangle$, $\langle w,2,a\rangle$, $\langle w,1,a,1,b\rangle$, $\langle w,1,a,2,b\rangle$ and $\langle w,1,a,D,b\rangle$. When defining the equivalence relation on those paths, we followed the properties of the relations in M 'as much as possible': in the example model, we obtain, since R_1 is reflexive, that $\langle w\rangle \equiv \langle w,1,w\rangle$ and, by transitivity of R_1 that $\langle w,1,a,1,b\rangle \equiv \langle w,1,b\rangle$. On the other hand, note that we can use the rule D(i) to prove that $\langle w,1,a,2,b\rangle \equiv \langle w,1,a,D,b\rangle$, but *not* $\langle w,1,a\rangle \equiv \langle w,D,a\rangle$, since the latter path, $\langle w,D,a\rangle$ is not an element of $P(N,I)$!

Observation 4.19 As indicated before, we feel that the best mathematical context to neatly prove properties of $\equiv$, is by conceiving the $\equiv_2$ pairs as *rewrite-instructions*, thus obtaining a rewrite system on paths. With this alternative view on $\equiv$, a variety of tools from rewrite theory becomes available. However, we think that this alternative presentation of $\equiv$ is too far from the current context. Conceptually, when deciding whether $\vec{u} \equiv \vec{v}$ for paths $\vec{u}, \vec{v} \in P(M, I)$, it may be helpful to think of each path having a *normal form*, induced by $\equiv$. Roughly speaking, such a normal form $nf(\vec{u})$ of a path $\vec{u}$ is a specific shortest path in M from $fst(\vec{u})$ to $lst(\vec{u})$, such that all worlds that occur in $nf(\vec{u})$, also occur in $\vec{u}$. To be more precise, we stipulate that $nf(\langle w \rangle) = \langle w \rangle$ (base case), $nf(\langle \vec{u}; i, lst(\vec{u}) \rangle) = nf(\vec{u})$ (omitting loops); and $nf(\langle \vec{u}; i, y, i, z \rangle) = nf(\langle \vec{u}; i, z \rangle)$ (taking short cuts). Finally, if none of the previous clauses is applicable, we stipulate the inductive clauses $nf(\langle \vec{u}; i, y \rangle) = nf(\langle nf(\vec{u}); D, y \rangle)$ if in M we have $R_D lst(\vec{u}) y$, and $nf(\langle \vec{u}; i, y \rangle) = nf(\langle nf(\vec{u}), i, y \rangle)$ else. These two clauses make sure that in a path in normal form, a D-step is taken between two worlds, whenever there is one in the model M. One may then prove that nf is well defined, and that $\vec{u} \equiv \vec{v}$ iff $nf(\vec{u}) = nf(\vec{v})$.

Before using the $\equiv$-classes to build our new model $M_w^{\equiv}$, we record some properties of $\equiv$ in Lemma 4.20.

Lemma 4.20 Let $\vec{u}, \vec{v}$ be paths, $i, j \in I, i \neq D$, $x, y \in W$ and $\equiv$ as defined above. Then:

(i) $\langle \vec{u}; i, y \rangle \in P(M, I) \Leftrightarrow \vec{u} \in P(M, I)$ & $R_i lst(\vec{u}) y$

(ii) $\vec{u} \equiv \vec{v} \Rightarrow lst(\vec{u}) = lst(\vec{v})$

(iii) $\vec{u} \equiv \vec{v} \Rightarrow \langle \vec{u}; i, y \rangle \equiv \langle \vec{v}; i, y \rangle$, provided that $\langle \vec{u}; i, y \rangle \in P(M, I)$ or $\langle \vec{v}; i, y \rangle \in P(M, I)$

(iv) $\langle \vec{u}; i, y \rangle \equiv \langle \vec{u}; j, y \rangle,\ i \neq j \Leftrightarrow \langle \vec{u}; i, y \rangle \equiv \langle \vec{u}; D, y \rangle,\ i \neq j$

Proof:

(i) Immediately from the definition of paths: a path can only be extended if there is another R_i-step in M.

(ii) Note that there is no $\equiv_1$-clause (and hence no $\equiv$-clause) that affects the last world in a path: only 'in-between' worlds may be removed (by Te(i)).

(iii) If $\vec{u} \equiv \vec{v}$, there must be a number of $\equiv_2$-steps that 'transfer' $\vec{u}$ into $\vec{v}$. We already know from item (ii) that the last world of $\vec{u}$ is not affected here. But then, we can mimic this $\equiv_2$-transfer, starting with $\langle \vec{u}; i, y \rangle$ leading to $\langle \vec{v}; i, y \rangle$, and thus we have $\equiv$-equivalence between those two extended paths.

(*iv*) From right to left is obvious: if $\langle\vec{u}; i, y\rangle \equiv \langle\vec{u}; D, y\rangle$ then in M, we must have $R_D lst(\vec{u})y$ and hence $R_j lst(\vec{u})y$ so that $\langle\vec{u}; j, y\rangle \in P(M, I)$ and, by D(i) and D(j), we have $\langle\vec{u}; i, y\rangle \equiv \langle\vec{u}; D, y\rangle \equiv \langle\vec{u}; j, y\rangle$. Although the converse is also clear at first sight (the basic observation being that a change of index in a path can only be effected by the D(i)-clause of the definition of $\equiv_1$), we feel that a full mathematical proof here needs an inductive argument. That is, such a proof is best given if one replaces our definition of $\equiv$ by a more cautious approach using rewrite rules, that together establish those equivalences. We think that introducing the machinery of rewriting systems is beyond the scope of the present paper: such an approach is taken in [19]., where one may also find a thorough proof of this item.

■

As an immediate consequence of Lemma 4.20.(*ii*) we observe that the function $Lst(|\vec{u}|) = lst(\vec{u})$ is well-defined.

Definition 4.21 Let $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E\rangle$, $w \in W$ and $I = \{1, \ldots, m, D\}$. Then the model $M_w^{\equiv} = \langle W', V', R'_1, \ldots, R'_m, R'_C, R'_D, R'_E\rangle$ is the following model:

- $W' = \{U | U = |\vec{u}|$ for some w-path $\vec{u}$ in $P(M, I)\}$
- $R'_i UV$ iff $\forall \vec{u} \in U \exists \vec{v} \in V \exists x \in W : \vec{v} = \langle\vec{u}; i, x\rangle$
- $R'_E = R'_1 \cup \ldots \cup R'_m$
- $R'_C = (R'_E)^*$
- $V'(U)(p) = V(Lst(U))(p)$

Let us informally see how the definition of $\equiv$, together with that of $M_w^{\equiv}$ guarantees that our example model $N_w^{\equiv}$ satisfies all the properties of $\mathcal{K}_{CDE}^m$ $(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$, knowing that $N \in \mathcal{K}_{CDE}^m(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$. First of all, the relations R'_C and R'_E are okay by definition. To see that R'_1 is reflexive, take any world U of $N_w^{\equiv}$, induced by a path $\vec{u} \in P(N, I)$. Let $\vec{u} \in U$ be arbitrary. Since R_1 in N is reflexive, we know that the path $\vec{v} = \langle\vec{u}; 1, lst(\vec{u})\rangle$ is also a path in $P(N, I)$. By Ref(i), we have $\vec{v} \equiv \vec{u}$ and hence $\vec{v} \in U$. By definition of R'_1, we have $R'_1 UU$. The cases of transitivity and Euclidicity are similar, cf. the proof of Lemma 4.25. We also see that we get $R'_D \subseteq R'_1$: suppose that $R'_D |\vec{u}|\ |\vec{v}|$. Then we know, that for every $\vec{s} \equiv \vec{u}$, there is a $\vec{t} \equiv \vec{v}$ and $x \in W$ with $\vec{t} = \langle\vec{s}; D, x\rangle$. But we know that $R_D \subseteq R_1$, so every such $\vec{t}$ is equivalent to a $\vec{t_1} = \langle\vec{s}; 1, x\rangle$. The definition of R'_1 then guarantees that $R'_1 |\vec{u}|\ |\vec{v}|$. Note that we do *not* have $R'_1 \subseteq R'_D$: although in $N_w^{\equiv}$ we have $R'_1 |\langle w\rangle|\ |\langle w, 1, a\rangle|$, we do not have $R'_D |\langle w\rangle|\ |\langle w, 1, a\rangle|$ which is seen as follows. Suppose we would have $R'_D |\langle w\rangle|\ |\langle w, 1, a\rangle|$. By definition of R'_D, it would mean that for some $x \in W$, we have a $\vec{v} \in P(N, I)$ with $\vec{v} \in |\langle w, 1, a\rangle|$ and $\vec{v} = \langle w, D, x\rangle$. Applying Lemma 4.20.($ii$) we see that $\vec{v} \equiv \langle w, D, a\rangle$, and we already argued (just before Observation 4.19) that then $\vec{v} \notin P(N, I)$.

The following lemma eases the reasoning about the relations R' in $M_w^{\equiv}$.

Lemma 4.22 Let $M_w^{\equiv}$ and I be as before, $i \in I$. The following are equivalent:

(i) $R_i'UV$

(ii) $\forall \vec{u} \in U \exists \vec{v} \in V \exists x \in W : \vec{v} = \langle \vec{u}; i, x\rangle$

(iii) $\forall \vec{u} \in U \exists \vec{v} \in V \exists x \in W : \vec{v} \equiv \langle \vec{u}; i, x\rangle$

(iv) $\exists \vec{u} \in U \exists \vec{v} \in V \exists x \in W : \vec{v} \equiv \langle \vec{u}; i, x\rangle$

(v) $\exists \vec{u} \in U \exists \vec{v} \in V \exists x \in W : \vec{v} = \langle \vec{u}; i, x\rangle$

Proof: We have (i) $\Leftrightarrow$ (ii) by definition of R_i' and (ii) $\Rightarrow$ (iii) by definition of $\equiv$. Since the equivalences classes U are not empty, we also have (iii) $\Rightarrow$ (iv). The implication (iv) $\Rightarrow$ (v) is also easy: if $\vec{v} \equiv \langle \vec{u}; i, x\rangle$ and $\vec{v} \in V$, then also $\vec{v}_1 = \langle \vec{u}; i, x\rangle \in V$, thus we have $\vec{u} \in U, \vec{v}_1 \in V$ with $\vec{v}_1 = \langle \vec{u}; i, x\rangle$. We finally show that also (v) $\Rightarrow$ (ii) holds. Suppose that $\vec{u} \in U, \vec{v} \in V$ are such that $\vec{v} = \langle \vec{u}; i, x\rangle$. In particular, this means that in M, we have $R_i lst(\vec{u})x$. Let $\vec{s} \in U$. By Lemma 4.20.(iii) we have $\vec{v} \equiv \langle \vec{s}; i, x\rangle$. Thus, for arbitrary $\vec{s} \in U$ we find $\vec{t} = \langle \vec{s}; i, x\rangle \in V$. ■

Lemma 4.23 Let $\mathcal{K}_{CDE}^m$ ($\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\}$)-model $M = \langle W, V, R_1, \ldots, R_m, R_C, R_D, R_E\rangle$, and suppose M is generated by w. Let the model $M_w^{\equiv}$, be as defined in Definition 4.21. Then the function $Lst : W' \to W$, defined above is a p-morphism.

Proof:

- First of all, Lst is obviously surjective if M is generated by a world w: then, every v is reachable from w, giving rise to a path $\langle w, \ldots, v\rangle$ in $P_w(M, I)$, and hence to a class $U = |\ \langle w, \ldots, v\rangle\ |$ in $M_w^{\equiv}$, with $Lst(U) = v$.
- We have to check, that for every $R_\Box \in \{R_1, \ldots, R_m, R_D, R_E, R_C\}$ the following holds:
  For all $U, V \in W'$; $R_\Box' UV \Rightarrow R_\Box Lst(U) Lst(V)$.
  Let $U = |\vec{u}|, V = |\vec{v}|$.
  - First, suppose $i \in \{1, \ldots, m, D\}$. Since $R_i'UV$, we must have some $\vec{v}' \in V, x \in W$ with $\vec{v}' = \langle \vec{u}; i, x\rangle$. By Lemma 4.20.($i$) we have $R_i lst(\vec{u}) lst(\vec{v}')$ and, since $lst(\vec{v}') = lst(\vec{v})$ (4.20.(ii)) and by definition, $Lst(|\vec{v}|) = lst(\vec{v})$, we have $R_i Lst(U) Lst(V)$.
  - Suppose $R_E'UV$. By definition of R_E' this means that for some $i \leq m$, $R_i'UV$. We already checked that then we have $R_i Lst(U) Lst(V)$. Since M is a model in $\mathcal{K}_{CDE}^m$ ($\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\}$), we have $R_i \subseteq R_E$ and hence $R_E Lst(U) Lst(V)$.

- If $R'_C UV$, then, by definition of R'_C, there must be a sequence $U_1 = U, \ldots, U_n = V$ with $R'_E U_i U_{i+1} (i \leq n)$. We already know that then we also have $R_E Lst(U_i) Lst(U_{i+1}) (i \leq n)$, and, since M satisfies $R_C = (R_E)^*$, we conclude $R_C Lst(U) Lst(V)$.

- Now we must verify that for all $R_\Box \in \{R_1, \ldots, R_m \; R_D, R_E, R_C\}$ the following holds:
  Suppose we have a $U = |\langle w, \ldots, u \rangle| \in W'$ with $R_\Box Lst(U)v$. Then there must be a world $V \in W'$ for which both $R'_\Box UV$ and $Lst(V) = v$.
  - We start by assuming $i \in \{1, \ldots, m, D\}$. We know that $R_i uv$. But then, the path $\vec{v} = \langle w, \ldots, u, i, v \rangle$ is an element of W'. Let $V = |\vec{v}|$, then we have $R'_i UV$ and $Lst(V) = v$.
  - Suppose $R_E Lst(U)v$; then there must an $i \leq m$ with $R_i Lst(U)v$; in the previous item we have found a $V \in W'$ with $R'_i UV$ and $Lst(V) = v$. By definition of R'_E then, we also have $R'_E UV$.
  - If $R_C Lst(U)v$, then there must be a sequence $v_1, \ldots v_n$ with $v_1 = Lst(U), v_n = v$ and $R_E v_i v_{i+1} (i < n)$. We already know that we then must have a sequence $V_1 = U, \ldots, V_n = V$ with $R'_E V_i V_{i+1} (i < n)$ and $Lst(V_i) = v_i (i \leq n)$. By definition of R'_C, we find a V with $R'_C UV$ and $Lst(V) = v$.

- By definition of $M_w^{\equiv}$, we have $V'(U)(p) = V(Lst(U))(p)$

Lemma 4.24 Let M be a model in $\mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$ with a world w, and $M_w^{\equiv}$ defined for M as above. Then:

$$M, w \models \varphi \Leftrightarrow M_w^{\equiv}, |\langle w \rangle| \models \varphi$$

Proof: Combine Lemma 4.3 with Lemmas 4.23 and 4.4 ∎

Lemma 4.25 Let M be a model in $\mathcal{K}^m_{CDE}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ and $M_w^{\equiv}$ defined as in Definition 4.21. Then $M_w^{\equiv} \in \mathcal{K}^m_{CDE}$ $(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$.

Proof: The cases Φ_2 and Φ_{4a}, Φ_{4b} follow immediately from the definition of R'_E and R'_C, respectively; so let us consider the other cases.

Φ_1 We prove that $R'_i (i \leq m)$ is an equivalence relation by proving that R'_i is reflexive, transitive and Euclidean. That R'_D is an equivalence relation follows from the fact that all R'_i are equivalence relations $(i \leq m)$ and the fact that $M_w^{\equiv}$ satisfies Φ_{3a} and Φ_{3b}, to be proven below.

- We use Lemma4.22(ii). In order to prove reflexivity of R'_i, choose $U \in W'$, let $\vec{u}$ be an arbitrary element of U and suppose $Lst(U) = z$. We know that R_i is reflexive, hence $R_i zz$. Thus $\vec{u}; i, z\rangle \in P(M, I)$ and, by Ref(i), $\vec{u} \equiv \langle \vec{u}; i, z \rangle$, so that $\langle \vec{u}; i, z \rangle \in U$.

- To prove transitivity of R'_i, suppose R'_iUV and R'_iVT. Let $\vec{u} \in U$; then there must be a $\vec{v} \in V, x \in W$ with $\vec{v} = \langle \vec{u}; i, x \rangle$ and a $y \in W, \vec{t} \in T$ with $\vec{t} = \langle \vec{u}; i, x, i, y \rangle$. Let $\vec{s} = \langle \vec{u}; , i, y \rangle$. Using the rule Te($i$), we see that $\vec{s} \equiv \vec{t}$: and thus $\vec{s} \in T$. Since for arbitrary $\vec{u} \in U$ we have found an $\vec{s} \in T$ such that $\vec{s} = \langle \vec{u}; , i, y \rangle$, we may use Lemma 4.22($ii$) to conclude that R'_iUT.
- For Euclidicity, suppose R'_iUV, R'_iUT; to show that R'_iVT. Let $\vec{u}$ be an arbitrary representative of U; since V and T are both R'_i-successors of U, we find $\vec{v} \in V, \vec{t} \in T$ for which there are $x, y \in W$ with $\vec{v} = \langle \vec{u}; i, x \rangle$, $\vec{t} = \langle \vec{u}; i, y \rangle$. This implies that in M we have $R_i lst(\vec{u})x$ and $R_i lst(\vec{u})y$. Since R_i is Euclidean, we have R_ixy. Now consider $\vec{s} = \langle \vec{u}; i, x, i, y \rangle$. Using Te($i$), we see that $\vec{t} \equiv \vec{s}$, hence $\vec{s} \in T$, and, since $\vec{s} = \langle \vec{v}; i, y \rangle$, we have R'_iVT.

Φ_{3a} Suppose that for all $j \in \{1, \ldots, m\}$, R'_jUV. Let $\vec{u} \in U$. By definition of R'_i, we find $\vec{v}_1, \ldots \vec{v}_m$ and $x_j \in W$ with $\vec{v}_j = \langle \vec{u}; j, x_j \rangle$, for all $j \leq m$. By Lemma 4.20.(ii) we know that all x_j's are the same, say y. Hence, each $\vec{v}_i$ is of the form $\vec{v}_i = \langle \vec{u}; i, y \rangle$. All these $\vec{v}_i$'s are elements of the same equivalence class V, so that we have $\langle \vec{u}; 1, y \rangle \equiv \langle \vec{u}; 2, y \rangle$. From the definition of R'_D and Lemma 4.20(iv) we obtain R'_DUV.

Φ_{3b} This property is directly inherited from the fact that M satisfies Φ_{3b} and the rule D(i): Suppose that U and $V \in W'$ are such that R'_DUV, and let $\vec{u} \in U$. By definition of R'_D, there is a $\vec{v} \in V$ with $\vec{v} = \langle \vec{u}; D, y \rangle$. Since M satisfies $R_D \subseteq R_i$ we have $\langle \vec{u}; i, y \rangle \in P(M, I)$ and, by D(i), $\langle \vec{u}; D, y \rangle \equiv \langle \vec{u}; i, y \rangle$. Now, given $\vec{u} \in U$ we found a $\vec{v} \in V$ and $y \in W$ such that $\vec{u} \equiv \langle \vec{v}; i, y \rangle$ and hence, by Lemma 4.22(iii), we have R'_iUV.

■

Remark 4.26 Again, in the definition of the model $M_w^{\equiv}$, there is an asymmetry between the definition of the relations $R'_1, \ldots, R'_m, R'_D$ on the one hand, and R'_E and R'_C on the other. This is what would have gone wrong if we had involved rules for R_E in our equivalences. Let us go back to the example model N for this. Obviously, in order to have $R'_E = R'_1 \cup R'_2$, in $N_w^{\equiv}$, we should add a rule E(i) to Definition 4.18, saying

$$\vec{u} \equiv_1 \vec{v} \text{ and } \vec{v} \equiv_1 \vec{u} \text{ if } \vec{u} = \langle x, i, y \rangle \text{ and } \vec{v} = \langle x, E, y \rangle \tag{3}$$

It is clear that with the clause (3), in $P(N, I)$ we have $\langle w, 1, a \rangle \equiv \langle w, E, a \rangle \equiv \langle w, 2, a \rangle$. Thus, when writing U for $|\langle w \rangle|$ and V for $|\langle w, 1, a \rangle|$, in $N_w^{\equiv}$ we would obtain R'_1UV, and R'_2UV, but not R'_DUV, so that $N_w^{\equiv}$ would not satisfy Φ_{3a}. Note that we also do not *want* R'_DUV to hold: suppose that the atom p is true in N only at the world w. Then we have $N, w \models Dp$, but if R'_DUV would hold, we would also have $N_w^{\equiv}|\langle w\rangle| \models \neg Dp$, so that the process of going from N, w to $N_w^{\equiv}$ would not preserve truth.

We are now ready to state and prove our Main Theorem:

Theorem 4.27 The logic **L** is sound and complete with respect to validity in the class $\mathcal{K}_{CDE}^m$ $(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$.

Proof: For completeness, combine Lemma 4.1, with $\mathcal{C}_1 = \mathcal{K}_{CDE}^m(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$, and $\mathcal{C}_2 = \mathcal{K}_{CDE}^m$ $(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$, with Lemma's 4.24 and 4.25. For soundness, use Theorem 3.14. ∎

Remark 4.28 We already proved in Corollary 4.17 that our logic **L** is decidable. Doing so, we used the fact that one finds a finite model for any satisfiable formula. Note however, that this model is not one in the class that we are considering here: it need not satisfy property Φ_{3a}. We do not know whether the problem

'is any **L***-consistent formula satisfiable in* $\mathcal{K}_{CDE}^m$ $(\{\Phi_1, \Phi_2, \Phi_{3a}, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$?'

is decidable.

We conclude with the observation that the properties Φ_{3a} and Φ_{4a} are not modally definable. For any class of models $\mathcal{C}$, let $\mathcal{F}(\mathcal{C})$ be the class of frames on which the models of $\mathcal{C}$ are based.

Corollary 4.29

- The property Φ_{3a} is not modally definable, even not relative to the class $\mathcal{K}_{m,C,D,E}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$
- The property Φ_{4a} is not modally definable, not even relative to the class $\mathcal{K}_{m,C,D,E}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4b}\})$.

Proof: We show that Φ_{3a} is not modally definable relative to $\mathcal{K}_{m,C,D,E}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$, the other cases are similar.

Suppose there would be a formula φ_{3a} that relatively defines Φ_{3a}. Let F be a frame in $\mathcal{K}_{m,C,D,E}(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ that does not satisfy Φ_{3a}. Since φ_{3a} relatively corresponds to Φ_{3a}, there is a valuation V and a world w such that $F, V, w \models \neg\varphi_{3a}$. Let the model $M = \langle F, V\rangle$. We can now define the model corresponding model $M_w^{\equiv}$ (cf. Definition 4.21), to find a model in $\mathcal{K}_{CDE}^m(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ with the property that $M_w^{\equiv}, w' \models \neg\varphi_{3a}$. However, the frame F' based on $M_w^{\equiv}$ is a frame that satisfies Φ_{3a}, and hence we also have

$M_w^{\equiv}, w' \models \varphi_{3a}$, a contradiction. ■

Note that the the proof of Corollary 4.29 indicates that the undefinability of Φ_{3a} and Φ_{4a} is in some necessary to let our construction from $M_{\mathbf{L}}^c$ to $(M_{\mathbf{L}}^c)_w^{\equiv}$ be successful: we have $\varphi \not\sim_{can} \Phi_x$ & $\varphi \sim_{com} \Phi_x \Rightarrow \varphi \not\sim_{cor} \Phi_x (x \in \{3a, 4a\})$.

5 Conclusion

The canonical model for a complete epistemic logic for m agents appeared to lack two desirable properties. We combined several validity-preserving techniques to transfer the satisfiability of a multi-modal formula between classes of models, thus eventually proving completeness for the logic under consideration. Although the full procedure to achieve this seems, we admit, quite formidable, we essentially applied three existing techniques: firstly, we used a filtration technique of Goldblatt ([4]) to obtain a finite model from the canonical one. That filtration gained one of the essential properties that was needed for the completeness proof. Secondly, we unravelled the filtrated model, following ideas that were, we think, introduced by Sahlqvist ([12]). Finally, we used our rewrite technique ([17]) to identify worlds in this unravelled model to obtain a model in the class of models we were aiming for.

Let us spend some words on the *order* in which we applied the respective techniques. For instance, it looks tempting to do the filtration as a last step: it would solve the question we raised in Remark 4.28, for instance. However, we feel that doing filtration as a last step would not work out properly. To see this, let us reconsider the $\mathcal{K}_{CDE}^m(\{\Phi_1, \Phi_2, \Phi_{3b}, \Phi_{4a}, \Phi_{4b}\})$ model of Proposition 3.13, in the first item of the proof. Here, we had a situation in which the world x_1 had an R_1-successor y_1 and an R_2-successor y_2 that satisfied the same theory. This implies that, no matter the granularity of the filtration (one may even consider to filtrate through the whole language L), one obtains $[y_1] = [y_2]$. Since R_1^f and R_2^f in the filtration have to satisfy Min^f, we must have $R_i^f[x_1][y_1]$, for $i = 1, 2$. But then the filtration cannot satisfy Φ_{3a}, since, if Dp is a sub-formula of the filtration formula, we must prevent R_D^f to satisfy $R_D^f[x_1][y_1]$, since Dp was true in x_1, but not at y_1.

Although some of these techniques or related ones have been applied before in sub-logics of **L**—[6] builds finite models for the logic $S5_m(CE)$ in a way that is related to our filtration method and the same holds for [10]; furthermore, [2] and [17] give completeness proofs for $S5_m(D)$, both yielding infinite models— we do not know of any attempt that solves the problem for our full system. Doing so, we did, on the one hand, employ the modular approach of [17], enabling us to take care of various properties of the relations in the underlying Kripke model, but, on the other hand, we had to make some *ad hoc* decisions as well. Finally, there is a lesson to be learnt from the complexity of the proof of the completeness of $S5_m(CDE)$ and how it is related to the completeness proofs of

$S5_m(CE)$ and $S5_m(CE)$, respectively: in general, it seems hard to predict if, and if so, how, the methods of proving the sublogics complete can be used or combined to prove completeness of the whole system.

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