Recursively defined (quasi) orders on terms

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Abstract

We study the problems involved in the recursive definition of (quasi) orders on terms, focussing on the question of establishing well-definedness, and the properties required for partial and quasi-orders: irreflexivity and transitivity, and reflexivity and transitivity, respectively. These properties are in general difficult to establish and this has in many cases come down in the literature as folklore results. Here we present a general scheme that allows us to show that relations are well-defined and represent partial or quasi-orders. Known path orders as semantic, recursive and lexicographic path order as well as Knuth-Bendix order fit into the scheme. Additionally we will also discuss how to obtain other properties commonly found in term orders (amongst which well-foundedness) as an integrated feature of the scheme.

Keywords: Path Orders, term rewriting, *semantic path order*, *recursive path order*, *lexicographic path order*, *Knuth-Bendix order* .

Contents

1 Introduction

Recursively defined (quasi) orders on terms (and in particular well-founded ones) play an important role in the theory of rewriting, from termination proofs to completion procedures. Most, if not all, of such orders found in the literature belong to the family of the so-called *path orders*. The basic idea behind path orders is the construction of orders on terms starting from a well-founded order on the signature $\mathcal{F}$ (usually called a *precedence*). In general a term s is greater than any term built from "smaller" terms connected together under a function symbol smaller, in the precedence, than the root (i. e., the top symbol) of s. Thus path orders compare the roots of the terms using the precedence and for equal or equivalent roots, subterms are compared recursively in some manner. The different ways of doing this subterm's comparison give rise to different path orders.

Path orders originated with the work of Plaisted (*path of subterms orderings* [20, 21]) and Dershowitz (*recursive path order* [2, 3]) at the end of the seventies. Since then other orders have been proposed and the original ones improved; examples of such orders include the *lexicographic path order* [13], the *recursive decomposition ordering* [12], the *path ordering* [14]. Based on the earlier examples, others have been proposed and a lot of work has been done on generalizing and improving existing ones (see [18, 23]). For an exhaustive account on path orders and their history, see [26].

When giving a recursive definition of a path order, several problems are posed. One of them is well-definedness of the order, i. e., one should see that an object of the sort that is being defined exists. Another important aspect concerns the properties that make a relation a partial or quasi-order, i. e., irreflexivity and transitivity, and reflexivity and transitivity, respectively. In general these properties, especially transitivity, are quite difficult to establish. Even though these are essential issues in the theory of recursively defined term orders, they haven't received the attention they deserve and need, and many results concerning path orders have come down in the literature either as folklore theorems or with unconvincing proofs. Furthermore the inexistence of an unified framework for the definition of such orders has the unpleasant consequence that every time a new order is proposed, even if that order strongly resembles known ones, all properties have to be shown anew.

Here we propose a remedy to this situation. We present a general scheme such that any recursive relation on terms defined according to our scheme will result in a well-defined partial or quasi-order on terms. The advantages of our approach are twofold:

- proving well-definedness, (ir)reflexivity and transitivity has to be done only once, namely for the scheme presented; then for any particular recursively defined relation on terms that we want to establish as a partial or quasi-order, we only need to check that the relation satisfies the properties required by the scheme, and this check is in general substantially simpler than establishing well-definedness, (ir)reflexivity and transitivity.

- the abstraction provided by the scheme allows for a better understanding of the mechanisms behind the definition of these orders and the scheme itself can be used for defining new path orders; furthermore, the scheme can be combined with results from [9, 10], in order to ensure well-foundedness of the orders.

Many if not all of the orders known in the literature are instances of the schemes presented. As an example we show how four of the most representative path orders, namely *recursive path order* [3, 4], *semantic path order* [13], *lexicographic path order* [13], and *Knuth-Bendix order* [16], fit in the scheme.

The rest of the paper is organized as follows. In sec. 2 we introduce some needed notions about Complete Partial Orders and most of the terminology/notions on terms that are used through out out the paper. In sec. 3 we introduce the scheme for recursive definitions of path quasi-orders and discuss the properties enjoyed by the scheme. We also show how *spo* and *rpo* can be seen as instances of our scheme. In sec. 4 we present another scheme for the recursive definitions of path partial orders on terms. Because quasi-orders and partial orders are essentially different, the schemes for their definition are also different. It would be possible to use a common framework for defining both schemes but the treatment would actually become more complicated. In the same section, and as was done for the quasi-order case, the properties of the scheme are discussed, and we also show how *lpo* and *kbo* can be obtained as instances of the scheme presented. We conclude in sec. 5.

2 Preliminaries

In this section we introduce the notions over orders, CPO's and the algebra of terms needed to comprehend the rest of the paper.

2.1 Orderings, CPO's and monotone functions

We begin by introducing the concepts of quasi and partial orders.

Definition 2.1. A binary relation Θ on a set S is said to be

- *(ir)reflexive* if $(\neg)s\Theta s$, for all $s \in S$,
- *symmetric* if Θ satisfies $(u\Theta v \Rightarrow v\Theta u)$, for all $u, v \in S$,
- *transitive* if Θ satisfies $(u\Theta v \wedge v\Theta w \Rightarrow u\Theta w)$, for all $u, v, w \in S$.

Definition 2.2. A binary relation θ on a set S is said to be:

- an *equivalence relation* if it is reflexive, symmetric and transitive. The set $S/\theta = \{\langle|s|\rangle_\theta |\ s \in S\}$ is the quotient of S modulo θ and $\langle|s|\rangle_\theta$ is the θ-equivalence class of the element $s \in S$, i. e., $\langle|s|\rangle_\theta = \{x \in S|\ x\theta s\}$.[1]
- a (strict) *partial order*, or simply *order* if it is transitive and irreflexive. We use the terminology *poset* meaning a set with a partial order.
- a *quasi-order* if it is transitive and reflexive; we denote such relations in general by $\succeq$.

Quasi-orders also appear in the literature under the name *pre-orders*. Any quasi-order $\succeq$ defines an equivalence relation, namely $\succeq \cap \preceq$, and a partial order, namely $\succeq \setminus \preceq$ (or its inverse $\preceq \setminus \succeq$). We usually denote the induced equivalence relation by $\sim$ and the induced partial order by $\succ$. But when need arises, we will also use the following notation:

Definition 2.3. If $\geq$ is a quasi-order over a set S then $\texttt{ord}(\geq) = \geq \setminus \leq$ and $\texttt{eq}(\geq) = \geq \cap \leq$, i. e., $\texttt{ord}(\geq)$ represents a partial order contained in $\geq$, and $\texttt{eq}(\geq)$ represents the equivalence relation contained in $\geq$.

Conversely, given a partial order $\succ$ and an equivalence $\sim$, their union does not always define a quasi-order (the transitive closure of their union does). However if $\succ$ and $\sim$ satisfy

$$(\sim\ \circ\ \succ\ \circ\ \sim)\ \subseteq\ \succ \tag{1}$$

where $\circ$ represents composition, then $\succ\ \cup\ \sim$ is a quasi-order, of which $\succ$ is the strict part and $\sim$ the equivalence part.

From now on if we characterize a quasi-order via $\succ \cup \sim$, we assume that the condition (1) is satisfied. Also we take as partial order defined by a quasi-order $\succeq$ the relation $\succ\ =\ \succeq\ \setminus\ \preceq$. Note that if $\succ$ and $\sim$ satisfy condition 1, then $\succ \cap \sim = \emptyset$, as we want it to be: if this condition is not satisfied we have that $a \succ b \sim a$, for some elements a, b, and this conflicts either with irreflexivity or condition 1.

[1] Note that the equivalence class of an element does not depend on the element chosen for its representative.

Definition 2.4. Given a quasi-order $\succeq$ over S and the quotient $S/\sim$ consisting of the ($\sim$-) equivalence classes of $\sim$ (which are are denoted by $\langle\!|\ |\!\rangle$), we can extend $\succ$ to $S/\sim$ in a natural way, namely by defining $\langle\!|s|\!\rangle \sqsupset \langle\!|t|\!\rangle$ if and only if $s \succ t$.

The following lemma is not difficult to prove.

Lemma 2.5. In the conditions of definition 2.4, the relation $\sqsupset$ on $S/\sim$ is well-defined. Furthermore $\sqsupset$ is a partial order over $S/\sim$.

Note that well-definedness means that $\sqsupset$ does not depend on the class representative and is a consequence of the fact that $\succ$ and $\sim$ satisfy condition (1). When the extension $\sqsupset$ is well-defined we abusively write $\succ$ instead of $\sqsupset$.

Definition 2.6. Given a partial order $\succ$ (respectively quasi-order $\succeq$) over some set S, we say that $\succ$ (respectively $\succeq$) is

- *well-founded* if and only if $\succ$ (respectively $\succeq$) has no infinite descending sequences, i. e., there are no sequences of the form $s_0 \succ s_1 \succ s_2 \succ \ldots$.
- *total* if and only if for any elements $u, v \in S$ we have either $u = v$ (resp. $s \sim v$) or $u \succ v$ or $v \succ u$.

We consider two useful extensions of partial orders, namely the *multiset* and *lexicographic* extensions. First we have to define the domain of these extensions.

Definition 2.7. Let S be any set. A *finite multiset* over S is a function $\rho : S \to \mathbb{N}$ such that the set $\{s \in S |\ \rho(s) \neq 0\}$ is finite. The set of all finite multisets over S is denoted by $\mathcal{M}(S)$.

Intuitively a finite multiset is a finite set where elements can be repeated finitely many times. For any $s \in S$, $\rho(s)$ just gives the frequency (number of occurrences) of the element s in the multiset.

We will use a set-like notation $\{\!\{\ \}\!\}$ to denote a multiset. Operations similar to the ones applied on sets (e. g. $\in$, $\cup$, $\subseteq$ etc.) are also applied to multisets. We will use round symbols to denote operations on sets (e. g. $\subseteq$) and similar squared symbols for the same operation on multisets (e. g. $\sqsubseteq$), whenever possible. Some operations, like $\in$, $\setminus$, will be denoted ambiguously by the same symbol. In the following we abbreviate finite multiset to multiset.

Definition 2.8. Let S be any set and $n \in \mathbb{N}$, fixed. Then S^n represents the set of sequences of elements of S of size exactly n. $S^* = \bigcup_{k \geq 0} S^k$ represents all possible sequences over S, where S^0 contains only the empty sequence ϵ. We use the notation $S^{\leq n}$ for the set $\bigcup_{k=0}^{n} S^k$. Elements of S^k, for any k, are denoted by $\langle s_1 \cdots s_k \rangle$, where "$\cdot$" denotes concatenation.

We now consider posets and define the multiset and lexicographic extension of the orders. The following definition is due to Dershowitz and Manna [6].

Definition 2.9. Let $(S, >)$ be a poset. The multiset extension of $>$ over $\mathcal{M}(S)$ is denoted by $>_{mul}$ and defined as follows: $X >_{mul} Y$ if and only if there are multisets $X_0, Y_0 \in \mathcal{M}(S)$ satisfying

- $X_0 \neq \emptyset$ and $X_0 \subseteq X$,
- $Y = (X \setminus X_0) \cup Y_0$,
- $\forall y \in Y_0\ \exists x \in X_0 :\ x > y$.

The following lemma is proven in [6].

Lemma 2.10. If $(S, >)$ is a poset then $(\mathcal{M}(S), >_{mul})$ is also a poset. Furthermore, $>$ is well-founded (respectively total) on S if and only if $>_{mul}$ is well-founded (respectively total) on $\mathcal{M}(S)$.

Definition 2.11. Let $(S, >)$ be a poset. The *lexicographic extension* of $>$ over S^n, $S^{\leq n}$ (for some fixed $n \in \mathbb{N}$) or S^* is defined as follows:

$$u_1 \cdots u_k >_{lex} v_1 \cdots v_m \iff \begin{cases} m < k \wedge \forall 1 \leq j \leq m :\ u_j = v_j, \text{ or} \\ \exists 1 \leq j \leq min\{m, k\} : (u_j > v_j) \wedge (\forall 1 \leq i < j : u_i = v_i) \end{cases}$$

Note that when restricted to S^n, the first condition is irrelevant.

We have a result similar to lemma 2.10.

Lemma 2.12. If $(S, >)$ is a poset then $(S^n, >_{lex})$, $(S^{\leq n}, >_{lex})$ and $(S^*, >_{lex})$ are also posets. Furthermore, $>$ is well-founded on S if and only if $>_{lex}$ is well-founded on S^n or $S^{\leq n}$ and $>$ is total on S if and only if $>_{lex}$ is total on S^n, $S^{\leq n}$ or S^*.

Note that if $>$ is well-founded, $>_{lex}$ is not necessarily well-founded on S^*, as the following example shows.

Example 2.13. Let $S = \{a, b\}$ with $a > b$. Then we have the infinite descending chain

$$a >_{lex} ba >_{lex} bba >_{lex} bbba >_{lex} \ldots$$

This problem can easily be avoided if we take the length of the sequence into consideration, i. e., if we define

$$u_1 \cdots u_k >^*_{lex} v_1 \cdots v_m \iff \begin{cases} k > m, \text{or} \\ m = k \text{ and } u_1 \cdots u_k >_{lex} v_1 \cdots v_m \end{cases}$$

We have that $>^*_{lex}$ is a partial order whenever S is a partial order. Furthermore $>^*_{lex}$ is well-founded (respectively total) if and only if $>$ is well-founded (respectively total).

Sometimes we are also interested in the lexicographic combination of orders over possibly different sets.

Definition 2.14. Given $n \geq 1$ posets $(A_i, >_i)$, then $\succ$, the lexicographic product of the orders $>_i$, $1 \leq i \leq n$, over the set $A_1 \times \ldots \times A_n$, is defined as

$$(u_1, \ldots, u_n) \succ (v_1, \ldots, v_n) \iff \begin{cases} \exists 1 \leq j \leq n : & (u_j >_j v_j \text{ and} \\ & (\forall 1 \leq i < j : u_i = v_i)) \end{cases}$$

It is not difficult to see that $\succ$ is a partial order over $A_1 \times \ldots \times A_n$. Furthermore $\succ$ is well-founded (respectively total) over $A_1 \times \ldots \times A_n$ if and only if $>_i$ is well-founded (respectively total) over A_i, for all $1 \leq i \leq n$.

We can also define the multiset and lexicographic extensions and lexicographic product for quasi-orders. Direct definitions similar to the definitions 2.9, 2.11 and 2.14, can be given, but the simplest way of defining these concepts is, in our view, to consider the equivalence classes.

Definition 2.15. Let $\geq \; = \; > \cup \sim$ be a quasi-order over S and let $\langle\!|a|\!\rangle$ denote the $\sim$-equivalence class of the element $a \in S$. Let $\sqsupset$ denote the extension of $>$ to the quotient $S/\!\sim$ of the $\sim$-equivalence classes, $\sqsupset_{mul}$ its multiset extension on $\mathcal{M}(S/\!\sim)$, and $\sqsupset_{lex}$ its lexicographic extension on $(S/\!\sim)^*$ $((S/\!\sim)^n$, $(S/\!\sim)^{\leq n}$, for some $n)$ The multiset extension of $\geq$ is denoted by $\geq_{mul}$ and defined as follows:

$$\{\!\{a_1, \cdots, a_m\}\!\} \; \mathtt{eq}(\geq_{mul}) \; \{\!\{b_1, \cdots, b_n\}\!\} \iff \{\!\{\langle\!|a_1|\!\rangle, \cdots, \langle\!|a_m|\!\rangle\}\!\} = \{\!\{\langle\!|b_1|\!\rangle, \cdots, \langle\!|b_n|\!\rangle\}\!\}$$

$$\{\!\{a_1, \cdots, a_m\}\!\} \; \mathtt{ord}(\geq_{mul}) \; \{\!\{b_1, \cdots, b_n\}\!\} \iff \{\!\{\langle\!|a_1|\!\rangle, \cdots, \langle\!|a_m|\!\rangle\}\!\} \sqsupset_{mul} \{\!\{\langle\!|b_1|\!\rangle, \cdots \langle\!|b_n|\!\rangle\}\!\}$$

The lexicographic extension of $\geq$ is denoted by $\geq_{lex}$ and defined as follows:

$$\begin{array}{lcl} \langle a_1 \cdot \ldots \cdot a_m \rangle \; \mathtt{eq}(\geq_{lex}) \; \langle b_1 \cdot \ldots \cdot b_n \rangle & \iff & \langle \langle\!|a_1|\!\rangle \cdot \ldots \cdot \langle\!|a_m|\!\rangle \rangle = \langle \langle\!|b_1|\!\rangle \cdot \ldots \cdot \langle\!|b_n|\!\rangle \rangle \\ \langle a_1 \cdot \ldots \cdot a_m \rangle \; \mathtt{ord}(\geq_{lex}) \; \langle b_1 \cdot \ldots \cdot b_n \rangle & \iff & \langle \langle\!|a_1|\!\rangle \cdot \ldots \cdot \langle\!|a_m|\!\rangle \rangle \sqsupset_{lex} \langle \langle\!|b_1|\!\rangle \cdot \ldots \cdot \langle\!|b_n|\!\rangle \rangle \end{array}$$

It is important to note that both $\mathtt{ord}(\geq_{lex})$ and $\mathtt{ord}(\geq_{mul})$ are different from the lexicographic and multiset extensions, respectively, of $>$, the strict part of $\geq$. Consider the set $S = \{a, b\}$ and the quasi-order $\geq$ satisfying reflexivity and $a \geq b$ and $b \geq a$. Then $>$ is the empty relation. We have that $\langle a \cdot a \rangle \; \mathtt{ord}(\geq_{lex}) \; \langle b \rangle$ and $\{\!\{a, a\}\!\} \; \mathtt{ord}(\geq_{mul}) \; \{\!\{b\}\!\}$, while $\langle a \cdot a \rangle \not>_{lex} \langle b \rangle$ and $\{\!\{a, a\}\!\} \not>_{mul} \{\!\{b\}\!\}$.

The relations $\succeq_{lex}$ and $\succeq_{mul}$ are themselves quasi-orders, satisfying condition 1 and preserving both well-foundedness and totality. More precisely:

Lemma 2.16. In the conditions of definition 2.15,

- $\geq_{mul} = \mathtt{ord}(\geq_{mul}) \cup \mathtt{eq}(\geq_{mul})$ is a quasi-order satisfying condition 1. Furthermore $\geq$ is well-founded (respectively total) over a set S if and only if $\geq_{mul}$ is well-founded (respectively total) over $\mathcal{M}(S)$.

- $\geq_{lex} = \texttt{ord}(\geq_{lex}) \cup \texttt{eq}(\geq_{lex})$ is a quasi-order satisfying condition 1. Furthermore $\geq$ is well-founded over a set S if and only if $\geq_{lex}$ is well-founded over S^n (or $S^{\leq n}$), for a fixed $n \geq 1$. Also $\geq$ is total over a set S if and only if $\geq_{lex}$ is total over S^n ($S^{\leq n}$ or S^*).

The lexicographic product of $n \geq 1$ quasi-orders $(A_i, \succeq_i)$, $1 \leq i \leq n$, is defined similarly to the lexicographic product of partial orders, we only need to change equality in definition 2.14 to $\sim_i$, the equivalence relation contained in $\succeq_i$, while the equivalence relation associated to the lexicographic product is defined by using equality of the equivalence classes, as in definition 2.15.

The main purpose of this paper is the definition of recursive orderings on terms and recursive definitions are related to *fixed points*. As usual, given a function $f : A \to A$, a fixed point of f is an element $a \in A$ satisfying $f(a) = a$. Not all functions have fixed points, but it is possible to ensure the existence of fixed points if both the domain and the functions satisfy certain conditions. A possibility is to require A to be a *CPO* and f to be *continuous*. We now introduce these concepts. For more detailed information, see for example Davey and Priestley [1].

Definition 2.17. Let $(P, >)$ be a poset and let S be a subset of P. An element $p \in P$ is named an *upper bound* for S if it satisfies $p \geq s$, for all $s \in S$. The *supremum* of S, denoted by $\bigvee S$, when it exists, is the *least* upper bound of S, i. e.,

- $\bigvee S \geq s$, for all $s \in S$,
- if p is an upper bound for S then $p \geq \bigvee S$.

The supremum of P (when it exists) is named the *greatest element* or *top*.

We note that the notions of *lower bound*, *greatest lower bound* or *minimum* and *least element* (or *bottom*) have a dual definition.

Definition 2.18. Let D be a non-empty subset of a poset $(P, >)$. D is said to be *directed* if for any finite subset F of D there is an element $d \in D$ which is an upper bound for F.

Definition 2.19. A poset $(P, >)$ is a *complete partial order*, abbreviated to *CPO*, if it satisfies the following conditions:

- P has a least element,
- every directed subset of P has a supremum.

Example 2.20. A very simple example of CPO is the powerset of P, for any set P, ordered by strict inclusion. The least element is the empty set and the supremum of any family of sets, and in particular a directed one, is the union of the elements in the family. This CPO also has a greatest element, namely P itself.

Definition 2.21. Let $(P, >)$ and $(Q, \succ)$ be two CPO's. A function $f : P \to Q$ is said to be:

- *(weakly) order-preserving* or *weakly monotone* if $x > y \Rightarrow f(x) \succeq f(y)$;
- *continuous* if for every directed set D of P we have $f(\bigvee_{>} D) = \bigvee_{\succ} f(D)$.

Note that in the definition above we do not need the existence of a least element neither in P nor in Q. In fact the definition of continuous function can be weakened to requiring that the condition for the supremums holds whenever they exist. Note also that if a function is continuous it is also weakly monotone since for any pair of elements x, y such that $x > y$, the set $\{x, y\}$ is directed and its supremum is x. Continuity now gives $f(\bigvee\{x, y\}) = f(x) \succeq f(y)$, by definition of supremum.

We now present the fixed-point result we need.

Theorem 2.22. Let $(P, >)$ be a CPO with least element $\bot$. Let $f : P \to P$ be any function. We have:

1. if f is order-preserving then f has a least fixed point. Furthermore if $\bigvee_{n \geq 0} f^n(\bot)$ is a fixed point then it is the least fixed-point.
2. if f is continuous then f has a least fixed point given by $\bigvee_{n \geq 0} f^n(\bot)$.

For a proof of these statements see [1]. Note that the set $\{f^n(\bot) | \; n \geq 0\}$ is a directed set and so the supremum is well-defined.

2.2 Terms and Rewriting Systems

We introduce some notions over terms and rewriting needed in the sequel. More complete information about term rewriting and its applications can be found in the surveys of Klop [15], Dershowitz and Jouannaud [5], and Plaisted [22].

Throughout this section (and the rest of the paper) we will use the following convention: whenever an object (relation, set, etc.) is defined inductively, we always have in mind the smallest object of the same type as the one being defined, satisfying the conditions specified in the definition, i. e., all other objects of the same type satisfying the conditions of the definition, will contain the object being defined.

Definition 2.23. A *signature* or *alphabet* $\mathcal{F}$ is a (non-empty) set of function symbols, each of which has associated an *arity* given by the function $\texttt{arity} : \mathcal{F} \to \mathbb{N}$. Elements of $\mathcal{F}$ with arity 0 are also called *constants*; constants are denoted usually by c instead of $c()$. We will also use the notation f/n meaning that symbol f has arity n.

It is not essential to consider that each function symbol has an associated fixed arity. Instead $\texttt{arity}(f)$ can be any non-empty subset of the natural numbers, i. e., $\texttt{arity}(f) \in$

$\mathcal{P}(\mathbb{N}) \setminus \emptyset$. If for at least one element $f \in \mathcal{F}$, $\texttt{arity}(f)$ contains more than one element, we speak of a *varyadic* signature. Otherwise we speak of a *fixed-arity* signature.

To define the set of terms we will also use variables. In the following $\mathcal{X}$ will represent a countable set of variables (whose elements we usually denote by letters $x, y, z, \ldots$). The function `arity` is extended to the elements of $\mathcal{X}$: they have arity 0.

Definition 2.24. Let $\mathcal{F}$ be a signature and let $\mathcal{X}$ denote a countable set of variables with $\mathcal{F} \cap \mathcal{X} = \emptyset$. The set of terms over $\mathcal{F}$ and $\mathcal{X}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and the set of ground terms over $\mathcal{F}$ by $\mathcal{T}(\mathcal{F})$; they are defined inductively as follows:

- $\mathcal{X}, \mathcal{F}_0 \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, $\mathcal{F}_0 \subseteq \mathcal{T}(\mathcal{F})$; where $\mathcal{F}_0$ represents the set of constants,
- $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ (respectively $\mathcal{T}(\mathcal{F})$), if $f \in \mathcal{F}$ admits arity $n \geq 1$ and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ (respectively $\mathcal{T}(\mathcal{F})$) for any $1 \leq i \leq n$.

Definition 2.25. For any term t, $\#_c(t)$ denote the number of occurrences of the symbol or variable c in t, and $|t|$ denotes the total number of function symbols and variables occurring in t (obviously $|t| = \sum_{c \in \mathcal{F} \cup \mathcal{X}} \#_c(t)$).

We sometimes need to abstract from the actual form of the whole term and concentrate on parts of it. For that we use contexts. Intuitively a *context* is a term containing "holes" that can be filled with other terms. In general a context may have more than one occurrence of $\Box$. For our purposes, we only need to consider contexts with exactly one occurrence of $\Box$, so we give a more restricted definition of context.

Definition 2.26. Let $\mathcal{F}$ be a signature and $\Box$ a constant not occurring in $\mathcal{F}$. A *(linear) context* is a term over $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with exactly one occurrence of $\Box$ (the trivial context). Given a context $C[\Box]$ (also denoted by $C[\ ]$) and a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $C[t]$ denotes the term obtained by replacing the occurrence of $\Box$ by t.

We will often need to perform induction on the definition of linear contexts, i. e., if we want to prove some property for a term $C[t]$, for any linear context C and (any) term t, we prove that the property holds for (all) t and then assuming that the property holds for $D[t]$, where D is a linear context, we prove the property holds for $f(\ldots, D[t], \ldots)$, for any $f \in \mathcal{F}$ with appropriate arity. It is not difficult to see that this is equivalent to proving the property for (all) t and then prove that if the property holds for a term s then it also holds for $f(\ldots, s, \ldots)$, for any $f \in \mathcal{F}$, arity permitting. This fact will be used when performing induction on linear contexts.

Definition 2.27. We say that a term t is a *subterm* of a term s if we have $s = C[t]$, for some linear context C; s is also called a *superterm* of t. If C is not the trivial context then t is a *proper* subterm of s. Furthermore if $s = f(t_1, \ldots, t_n)$, for some $n \geq 1$, the terms t_i, with $1 \leq i \leq n$, are called the *principal* subterms of s and they are denoted by $\vec{s}$; the function symbol f is the *root* symbol of s, usually denoted by $\texttt{root}(s)$.

Definition 2.28. A *substitution* σ is a function from $\mathcal{X}$ to $\mathcal{T}(\mathcal{F},\mathcal{X})$; such a function can be extended to an endomorphism over $\mathcal{T}(\mathcal{F},\mathcal{X})$ as follows $\sigma(f(t_1,\ldots,t_n)) = f(\sigma(t_1),\ldots,\sigma(t_n))$, for any $f \in \mathcal{F}$ admitting arity $n \geq 0$, and terms $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$. A *ground* substitution is a substitution whose image lies in $\mathcal{T}(\mathcal{F})$. We usually denote $\sigma(t)$ by $t\sigma$.

We will deal with some particular relations on terms we define next.

Definition 2.29. A binary relation Θ over $\mathcal{T}(\mathcal{F},\mathcal{X})$ is said to be *closed under contexts* (*monotonic* or satisfying the *replacement property*) if whenever $s\Theta t$ then for any linear context $C[\ ]$ also $C[s]\Theta C[t]$. Equivalently, $s\Theta t \Rightarrow f(\ldots,s,\ldots)\Theta f(\ldots,t,\ldots)$, for all non-constant $f \in \mathcal{F}$. If Θ is a quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$, we say that Θ is *strictly* closed under contexts if both its equivalence and strict part are closed under contexts.

Definition 2.30. A binary relation Θ over $\mathcal{T}(\mathcal{F},\mathcal{X})$ is said to be *closed under substitutions* (*stable* or satisfying the *full invariance property*) if whenever $s\Theta t$ then for any substitution $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F},\mathcal{X})$ also $s\sigma\Theta t\sigma$. If Θ is a quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$, we say that Θ is *strictly* closed under substitutions if both its equivalence and strict part are closed under substitutions.

Definition 2.31. Let $\mathcal{F}$ be a signature. A *precedence* is a partial or quasi-order on $\mathcal{F}$ denoted respectively by $\rhd$ or $\unrhd$.

We will consider special orderings on terms which are simply the extension of a precedence to the set ot terms.

Definition 2.32. A quasi-order $\succeq$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ is said to be *precedence based* if there is some precedence $\unrhd$ on $\mathcal{F}$ such that $s \succeq t \iff \texttt{root}(s) \unrhd \texttt{root}(t)$, for any terms $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$. A partial order $\succ$ is *precedence based* if there is some precedence $\rhd$ such that $s \succ t \iff \texttt{root}(s) \rhd \texttt{root}(t)$, for any terms $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$

Definition 2.33. A *term rewriting system* (TRS) is a tuple $(\mathcal{F},\mathcal{X},R)$, where R is a subset of $(\mathcal{T}(\mathcal{F},\mathcal{X}) \setminus \mathcal{X}) \times \mathcal{T}(\mathcal{F},\mathcal{X})$. The elements of R are called the rules of the TRS and are usually denoted by $l \to r$, with l being the *lefthand-side* (lhs) of the rule, r the *righthand-side* (rhs) and they satisfy the following condition: any variable occurring in r also occurs in l.

The rules of a TRS induce a relation on terms as follows.

Definition 2.34. A TRS $(\mathcal{F},\mathcal{X},R)$ induces a *reduction relation* on $\mathcal{T}(\mathcal{F},\mathcal{X})$, denoted by $\to_R$, as follows: $s \to_R t$ if and only if $s = C[l\sigma]$ and $t = C[r\sigma]$, for some linear context C, substitution σ and rule $l \to r \in R$. We call $s \to_R t$ a *reduction* or *rewrite step* and say that t is obtained from s by *contracting* or *reducing* the *redex* $l\sigma$, i. e., replacing the redex $l\sigma$ by its *contractum* $r\sigma$. The transitive closure of $\to_R$ is denoted by $\to_R^+$ and its reflexive-transitive closure by $\to_R^*$. A *rewrite sequence* is a sequence of reduction steps $t_0 \to_R t_1 \to_R \cdots$, and may be finite or infinite.

3 A scheme for the definition of quasi-orders

In this section we present a construction that allows us to recursively define quasi-orders on terms. We start with a fixed quasi-order $\succeq$ and recursively build a path order which uses both the fixed quasi-order and some kind of lifting of quasi-orders, i. e., a function that transforms quasi-orders in quasi-orders. The main idea then is to define an appropriate CPO and weakly monotone function such that the quasi-order we have in mind is or can be derived from a fixed point of the function.

3.1 The CPO of quasi-orders

Since our aim is to define a quasi-order it seems reasonable to choose as underlying set for our construction the set of all quasi-orders. Then we still have to define an appropriate order on it such that the ordered structure will be a CPO.

Let S be a set and define $\mathcal{QO}_S$ to be the set of all quasi-orders on S, i. e., $\mathcal{QO}_S = \{\theta \subseteq S \times S : \theta \text{ is a quasi-order}\}$. We now define a relation $\sqsupset$ in $\mathcal{QO}_S$ as follows:

$$\theta \sqsupset \theta' \iff \begin{cases} \theta \supset \theta', \text{ and} \\ \mathtt{ord}(\theta) \supseteq \mathtt{ord}(\theta') \end{cases}$$

It is not difficult to see that $\sqsupset$ is indeed a partial order (irreflexivity follows from the first condition above and transitivity is a consequence of the fact that $\supset$ and $\supseteq$ are transitive). Furthermore we have:

Lemma 3.1. The poset $(\mathcal{QO}_S, \sqsupset)$ is a CPO with bottom element given by equality, i. e., the relation $\{(s,s)|\ s \in S\}$, and with the supremum of directed sets given by the union of the elements in the set.

Proof It is clear that the bottom element is equality since any reflexive relation contains equality which is itself a quasi-order. Suppose now that D is a directed set of quasi-orders and take $\bigcup D$. We have to see that $\bigcup D$ is a quasi-order and that for any element $\theta \in D$, we have $\bigcup D \sqsupseteq \theta$. The relation $\bigcup D$ is indeed reflexive since it is the union of reflexive relations. As for transitivity, suppose we have elements $s, t, u \in S$ such that $s\ (\bigcup D)\ t$ and $t\ (\bigcup D)\ u$; then there are elements $\theta_1, \theta_2 \in D$ such that $s\ \theta_1\ t$ and $t\ \theta_2\ u$. Since D is directed, there is an element $\theta_3 \in D$ such that $\theta_3 \sqsupseteq \theta_1, \theta_2$ and since θ_3 is transitive, we conclude that $s\ \theta_3\ u$ and so $s\ (\bigcup D)\ u$, as we wanted.

Now we see that $\bigcup D$ is an upper bound for D. Let θ be an arbitrary element of D. It is obvious that $\bigcup D \supseteq \theta$, but we still have to see that $\mathtt{ord}(\bigcup D) \supseteq \mathtt{ord}(\theta)$. Suppose that $(s,t) \in \mathtt{ord}(\theta)$, then $s\ \theta\ t$ and $\neg(t\ \theta\ s)$. We also have $s\ (\bigcup D)\ t$; suppose we have $t\ (\bigcup D)\ s$. Then an element $\theta' \in D$ has to exist such that $t\ \theta'\ s$. Since D is directed, there is an element $\theta'' \in D$ such that $\theta'' \sqsupseteq \theta, \theta'$; thus we have $s\ \theta''\ t$ and $t\ \theta''\ s$, which means that $(s,t) \in \mathtt{eq}(\theta'')$. But this contradicts the fact that $\theta'' \sqsupseteq \theta$ and $(s,t) \in \mathtt{ord}(\theta)$ (since $\mathtt{ord}(\theta'') \supseteq \mathtt{ord}(\theta)$). So we must have $\neg(t\ (\bigcup D)\ s)$ and

$\mathtt{ord}(\bigcup D) \ \supseteq \ \mathtt{ord}(\theta)$. We have just seen that $\bigcup D$ is an upper bound for any $\theta \in D$, we still need to see that it is the least upper bound. Let then $\Upsilon \in \mathcal{QO}_S$ be an upper bound for D; this means that

- $\Upsilon \supseteq \theta$, for all $\theta \in D$, and therefore $\Upsilon \supseteq \bigcup D$;
- $\mathtt{ord}(\Upsilon) \supseteq \mathtt{ord}(\theta)$, for all $\theta \in D$. We need to see that $\mathtt{ord}(\Upsilon) \supseteq \mathtt{ord}(\bigcup D)$. Let $(a,b) \in \mathtt{ord}(\bigcup D)$, then also $(a,b) \in \bigcup D$ and therefore $(a,b) \in \theta$, for some $\theta \in D$. If we would have $(a,b) \in \mathtt{eq}(\theta)$ then also $(b,a) \in \theta \subseteq \bigcup D$ and therefore we would have $(a,b) \in \mathtt{eq}(\bigcup D)$. So we must have $(a,b) \in \mathtt{ord}(\theta) \subseteq \mathtt{ord}(\Upsilon)$, so we conclude that $\mathtt{ord}(\Upsilon) \supseteq \mathtt{ord}(\bigcup D)$.

By the above $\bigcup D$ is indeed the least upper bound of D and the proof is complete. □

Some remarks are in order here. The CPO structure we have built so far is rather complicated. Another possibility yielding an easier to handle CPO would be the set of all relations ordered by strict inclusion. And indeed this structure works fine if we are dealing with *partial* orders mainly because in the case of partial orders inclusion of partial orders coincides with inclusion of sets. For quasi-orders and what we have in mind, that is not so. We want inclusion of quasi-orders to respect the strict and equivalent parts and therefore inclusion of quasi-orders no longer coincides with inclusion of sets. So a richer different kind of CPO has to be defined and a good candidate seemed to be the set of quasi-orders ordered by an appropriate partial order.

The last tool we need is the function allowing the construction of new quasi-orders from existing ones. Such a function is called a *status* and is defined as follows.

Definition 3.2. Let S be a set. A *status* is a function $\Lambda : \mathcal{QO}_S \to \mathcal{QO}_S$ which is weakly monotone with respect to the CPO $(\mathcal{QO}_S, \sqsupset)$, i. e., $\theta \ \sqsupset \theta' \Rightarrow \Lambda(\theta) \ \sqsupseteq \Lambda(\theta')$.

3.2 The quasi-order scheme

From now on we fix our CPO to be $(\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}, \sqsupset)$. Let $\succeq$ be a fixed quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and let Λ be a status, in the sense of definition 3.2, with domain $\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$. We define the following function:

Definition 3.3. The function $\mathcal{H} : \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})} \to \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$ is defined as follows: $s = f(s_1,\ldots,s_k) \ \mathcal{H}(\theta) \ t$, with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, $s_1,\ldots,s_k \in \mathcal{T}(\mathcal{F},\mathcal{X})$, if one of the following conditions holds:

1. $t = g(t_1,\ldots,t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and some $t_1,\ldots,t_m \in \mathcal{T}(\mathcal{F},\mathcal{X})$, and for all $1 \leq j \leq m$, we have $s \ \mathcal{H}(\theta) \ t_j$ and $\neg(t_j \ \mathcal{H}(\theta) \ s)$, and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s \ \mathtt{ord}(\Lambda(\theta)) \ t$, or

(c) $s \sim t$ and s $\texttt{eq}(\Lambda(\theta))$ t, and for all $1 \leq j \leq k$ we have that t $\mathcal{H}(\theta)$ s_j and $\neg(s_j$ $\mathcal{H}(\theta)$ $t)$; or

2. $\exists 1 \leq i \leq k :$ s_i $\mathcal{H}(\theta)$ t.

It is not difficult to see that the function $\mathcal{H}$ defines a relation on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ (all recursive references apply to terms of strictly smaller size, so the recursion ends), however in order to see that the function is well-defined, we need to prove the following lemma.

Lemma 3.4. If θ is a quasi-order in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ then $\mathcal{H}(\theta)$ is also a quasi-order in $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Proof We need to see that $\mathcal{H}(\theta)$ is both reflexive and transitive, i. e., that for all $s, t, u \in S$

a) s $\mathcal{H}(\theta)$ s, and

b) s $\mathcal{H}(\theta)$ t and t $\mathcal{H}(\theta)$ u implies s $\mathcal{H}(\theta)$ u.

We prove a) and b) simultaneously by induction on $|s| + |t| + |u|$. Let then s, t, u be minimal terms (i. e., terms for which the sum $|s| + |t| + |u|$ is minimal) such that s $\mathcal{H}(\theta)$ t and t $\mathcal{H}(\theta)$ u and for which properties a) and b) are not yet satisfied. We first see that s $\mathcal{H}(\theta)$ s. Suppose $s = f(s_1, \ldots, s_k)$, for some $f \in \mathcal{F} \cup \mathcal{X}$, and $k \geq 0$. If there is $1 \leq i \leq k$ such that s_i $\mathcal{H}(\theta)$ s, then by condition 2 of definition 3.3, we get that s $\mathcal{H}(\theta)$ s. Otherwise we have that for all $1 \leq i \leq k$, $\neg(s_i$ $\mathcal{H}(\theta)$ $s)$ and (by induction hypothesis) s_i $\mathcal{H}(\theta)$ s_i and therefore (clause 2 of definition 3.3) s $\mathcal{H}(\theta)$ s_i, for all $1 \leq i \leq k$. Since both $\sim$ and $\texttt{eq}(\Lambda(\theta))$ are equivalence relations, we also have that $s \sim s$ and s $\texttt{eq}(\Lambda(\theta))$ s, so applying condition 1c of definition 3.3, we conclude that s $\mathcal{H}(\theta)$ s, and this proves a).

For b), we have to consider the different cases in definition 3.3 by which we can conclude that s $\mathcal{H}(\theta)$ t and t $\mathcal{H}(\theta)$ u: 16 cases in total. The cases where 2 occurs are trivially solved using the induction hypothesis: for 2 *vs.* {1a, 1b, 1c, 2 }, we have s_i $\mathcal{H}(\theta)$ t $\mathcal{H}(\theta)$ u, and induction hypothesis allows us to conclude that s_i $\mathcal{H}(\theta)$ u and therefore that s $\mathcal{H}(\theta)$ u. As for the cases { 1a, 1b, 1c } *vs.* 2, we have s $\mathcal{H}(\theta)$ t_j $\mathcal{H}(\theta)$ u and again induction hypothesis gives s $\mathcal{H}(\theta)$ u.

In the remaining cases, namely { 1a, 1b, 1c } *vs.* { 1a, 1b, 1c }, the following holds: s $\mathcal{H}(\theta)$ u_l and $\neg(u_l$ $\mathcal{H}(\theta) s)$, for all $1 \leq l \leq k$ where $s = f(s_1, \ldots, s_m), t = g(t_1, \ldots, t_n)$ and $u = h(u_1, \ldots, u_k)$, for some $f, g, h \in \mathcal{F} \cup \mathcal{X}$, and $m, n, k \geq 0$. Indeed, since s $\mathcal{H}(\theta)$ t $\mathcal{H}(\theta)$ u_l, for all $1 \leq l \leq k$, induction hypothesis immediately gives s $\mathcal{H}(\theta)$ u_l, for all such l; and if for some l it would be u_l $\mathcal{H}(\theta)$ s then since s $\mathcal{H}(\theta)$ t, induction hypothesis would also give u_l $\mathcal{H}(\theta)$ t, contradicting the fact that, for all l holds $\neg(u_l$ $\mathcal{H}(\theta)$ $t)$ (remember which cases we are considering). Therefore it must be $\neg(u_l$ $\mathcal{H}(\theta)$ $s)$ for all $1 \leq l \leq k$. If we now see that one of the cases 1a, 1b, 1c holds for s and u, we are done. For cases 1a *vs.* { 1a, 1b, 1c }, we have that $s \succ t \succeq u$, so $s \succ u$, and case 1a holds for s, u, showing that s $\mathcal{H}(\theta)$ u. We can draw the same conclusion for cases { 1b, 1c } *vs.* 1a.

As for cases 1b *vs.* { 1b, 1c }, we have that $s \sim u$ (from $s \sim t \sim u$) and that s $\texttt{ord}(\Lambda(\theta))$ u (from s $\texttt{ord}(\Lambda(\theta))$ t and t $\Lambda(\theta)$ u). In this cases, we can conclude that case 1b holds for s, u. We can draw the same conclusion for case 1c *vs.* 1b.

Finally we check the case 1c *vs.* 1c. First note that whenever $p\ \mathcal{H}(\theta)\ q$ by case 1c, then also $q\ \mathcal{H}(\theta)\ p$ again by case 1c. We now see that for all $1 \leq i \leq m$, $u\ \mathcal{H}(\theta)\ s_i$ and $\neg(s_i\ \mathcal{H}(\theta)\ u)$. We have that $\forall 1 \leq i \leq m :\ t\ \mathcal{H}(\theta)\ s_i$ and $\neg(s_i\ \mathcal{H}(\theta)\ t)$. By the remark above we also have $u\ \mathcal{H}(\theta)\ t$, so applying the induction hypothesis, we conclude that $u\ \mathcal{H}(\theta)\ s_i$, for all $1 \leq i \leq m$. If there would be an index i such that $s_i\ \mathcal{H}(\theta)\ u$, then from $u\ \mathcal{H}(\theta)\ t$ and induction hypothesis we would get $s_i\ \mathcal{H}(\theta)\ t$, deriving a contradiction. So for all $1 \leq i \leq m$, $\neg(s_i\ \mathcal{H}(\theta)\ u)$ also holds. Finally, we have that $s \sim u$ (since $s \sim t \sim u$) and that s $\texttt{eq}(\Lambda(\theta))$ u (since s $\texttt{eq}(\Lambda(\theta))$ t and t $\texttt{eq}(\Lambda(\theta))$ u); so case 1c holds for s, u. □

The following lemma will be quite useful.

Lemma 3.5. For any $\theta \in \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$, for any non-trivial context C and any term $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, we have that $\neg(s\ \mathcal{H}(\theta)\ C[s])$.

Proof Let θ be an arbitrary element of $\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$. We proceed by induction on the lexicographic product $(|s|, C)$. For terms of size 1, the result holds since for concluding that $s\ \mathcal{H}(\theta)\ f(\ldots, s, \ldots)$, for any $f \in \mathcal{F}$, with arity $\geq$1, the only case of the definition of $\mathcal{H}$ applicable is case 1. and then we must have simultaneously $s\ \mathcal{H}(\theta)\ s$ and $\neg(s\ \mathcal{H}(\theta)\ s)$, which is impossible; and if D is a context for which the result holds then $s\ \mathcal{H}(\theta)\ f(\ldots, D[s], \ldots)$ again would imply (case 1 is the only possibility) that $s\ \mathcal{H}(\theta)\ D[s]$, contradicting the induction hypothesis.

Take now a term s with $|s| = k$, for a fixed $k > 1$, for which the result is not yet verified. i. e., the result holds for all terms u and contexts D if $|u| < k$. Take $f \in \mathcal{F}$ with arity $\geq$1. If $s\ \mathcal{H}(\theta)\ f(\ldots, s, \ldots)$, case 1 of definition 3.3 is not applicable, since both $s\ \mathcal{H}(\theta)\ s$ and its negation would have to hold. So we must have $s = h(s_1, \ldots, s_m)$, for some $h \in \mathcal{F}$ with arity $m \geq 1$, and $s_i\ \mathcal{H}(\theta)\ f(\ldots, s, \ldots)$, for some $1 \leq i \leq m$. But $f(\ldots, s, \ldots)$ can be written as $D[s_i]$, for some non-trivial context D, so we have $s_i\ \mathcal{H}(\theta)\ D[s_i]$, contradicting the induction hypothesis. Suppose now that the result holds for the pair s and some non-trivial context C. Suppose also that $s\ \mathcal{H}(\theta)\ f(\ldots, C[s], \ldots)$, for some $f \in \mathcal{F}$ with arity $\geq$1. Again case 1 gives a contradiction (since we would have $s\ \mathcal{H}(\theta)\ C[s]$) and case 2 will also give a contradiction since then we conclude that $s_i\ \mathcal{H}(\theta)\ D[s_i]$, for some non-trivial context D and proper subterm s_i of s, contradicting the induction hypothesis. □

Lemma 3.6. For any quasi-order θ and terms s, t:

1. if $s\ \mathcal{H}(\theta)\ t$ by case 1c of definition 3.3, then also $t\ \mathcal{H}(\theta)\ s$, by the same case .

2. $s\ \mathcal{H}(\theta)\ t$ and $t\ \mathcal{H}(\theta)\ s$ iff case 1c of definition 3.3 is applicable to derive both $s\ \mathcal{H}(\theta)\ t$ and $t\ \mathcal{H}(\theta)\ s$.

Proof 1 is a trivial consequence of definition 3.3 and reflexivity of $\sim$ and $\texttt{eq}(\Lambda(\theta))$. As for 2, the if part is a consequence of part 1. For the only-if part, remark that if both $s\ \mathcal{H}(\theta)\ t$ and $t\ \mathcal{H}(\theta)\ s$, then given the decomposition properties of quasi-orders, lemma 3.5 and transitivity of $\mathcal{H}(\theta)$, it is not difficult to see that the only possible combination of cases from definition 3.3 is 1c *vs.* 1c. □

Lemma 3.7. The function $\mathcal{H}$ is weakly monotone.

Proof We have to see that if $\theta \sqsupseteq \theta'$ then $\mathcal{H}(\theta) \sqsupseteq \mathcal{H}(\theta')$ or equivalently that

- $s\ \mathcal{H}(\theta')\ t \Rightarrow s\ \mathcal{H}(\theta)\ t$, for all terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and
- $\texttt{ord}(\mathcal{H}(\theta)) \supseteq \texttt{ord}(\mathcal{H}(\theta'))$.

We prove, by induction on $|s| + |t|$, that if $s\ \mathcal{H}(\theta')\ t$ then $s\ \mathcal{H}(\theta)\ t$ and if additionally $\neg(t\ \mathcal{H}(\theta')\ s)$ then also $\neg(t\ \mathcal{H}(\theta)\ s)$. It is not difficult to see that the statement holds for terms s, t with $|s| + |t| = 2$. Let s, t be a minimal pair of terms such that $s\ \mathcal{H}(\theta')\ t$ and for which the implications are not yet verified, i. e., if u, v are terms such that $|u| + |v| < |s| + |t|$, then u and v satisfy the implications. We have to do some case analysis. If $s\ \mathcal{H}(\theta')\ t$ by case

1. We have $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s\ \mathcal{H}(\theta')\ t_j$ and $\neg(t_j\ \mathcal{H}(\theta')\ s)$. By induction hypothesis we also have $s\ \mathcal{H}(\theta)\ t_j$ and $\neg(t_j\ \mathcal{H}(\theta)\ s)$, for all $1 \leq j \leq m$.
   (a) If case 1a is applicable then we have $s \succ t$ and consequently also $s\ \mathcal{H}(\theta)\ t$. Suppose additionally that $\neg(t\ \mathcal{H}(\theta')\ s)$. If we would have $t\ \mathcal{H}(\theta)\ s$, then cases 1a, 1b and 1c cannot be applied since we cannot have simultaneously $s \succ t$ and $t \succ s$ or $t \sim s$; therefore we must have $t\ \mathcal{H}(\theta)\ s$ by case 2 and this means that $t_j\ \mathcal{H}(\theta)\ s$, for some $1 \leq j \leq m$, which gives a contradiction.
   (b) If case 1b is applicable then we have $s \sim t$ and $s\ \texttt{ord}(\Lambda(\theta'))\ t$. Since Λ is weakly monotone, we also have $s\ \texttt{ord}(\Lambda(\theta))\ t$ and so also $s\ \mathcal{H}(\theta)\ t$. If additionally $\neg(t\ \mathcal{H}(\theta')\ s)$ and $t\ \mathcal{H}(\theta)\ s$, then we conclude that we must have $t\ \mathcal{H}(\theta)\ s$ by case 2 (case 1a, 1b and 1c are not applicable since we cannot have both $s \sim t$ and $t \succ s$, nor $s\ \texttt{ord}(\Lambda(\theta))\ t$ and $t\ \texttt{ord}(\Lambda(\theta))\ s$ nor $t\ \texttt{eq}(\Lambda(\theta))\ s$); and case 2 leads to a contradiction as above.
   (c) If case 1c is applicable then we have $s \sim t$ and $s\ \texttt{eq}(\Lambda(\theta'))\ t$. Furthermore we also have $t\ \mathcal{H}(\theta')\ s_j$ and $\neg(s_j\ \mathcal{H}(\theta')\ t)$, for all $1 \leq j \leq k$, where $s = f(s_1, \ldots, s_k)$, for some $f \in \mathcal{F} \cup \mathcal{X}$, $k \geq 0$. By weak monotonicity of Λ, we also have $s\ \texttt{eq}(\Lambda(\theta))\ t$ and by induction hypothesis we can conclude that $t\ \mathcal{H}(\theta)\ s_j$ and $\neg(s_j\ \mathcal{H}(\theta)\ t)$, for all $1 \leq j \leq k$; so by case 1c we conclude that $s\ \mathcal{H}(\theta)\ t$. By lemma 3.6, we can have neither $\neg(t\ \mathcal{H}(\theta')\ s)$ nor $\neg(t\ \mathcal{H}(\theta)\ s)$.
2. case 2; then $s = f(s_1, \ldots, s_n)$, for some $f \in \mathcal{F}$, having arity $n \geq 1$, and $s_i\ \mathcal{H}(\theta')\ t$, for some $1 \leq i \leq n$. By induction hypothesis we conclude that $s_i\ \mathcal{H}(\theta)\ t$ and so that $s\ \mathcal{H}(\theta)\ t$. Suppose additionally that $\neg(t\ \mathcal{H}(\theta')\ s)$. If we would have

$t\ \mathcal{H}(\theta)\ s$ then transitivity of $\mathcal{H}(\theta)$ would give $s_i\ \mathcal{H}(\theta)\ s$, contradicting lemma 3.5.

We have just established that

$$\mathcal{H}(\theta)\ \supseteq \mathcal{H}(\theta')\ \text{ and }\ \mathtt{ord}(\mathcal{H}(\theta))\ \supseteq \mathtt{ord}(\mathcal{H}(\theta'))$$

Thus we have that $\mathcal{H}$ is weakly monotone. □

Since the function $\mathcal{H}$ is weakly monotone (or order-preserving), theorem 2.22 tells us that $\mathcal{H}$ has a least fixed point which we take to be the *path quasi-order*.

Definition 3.8. The *path quasi-order* associated with a status Λ and the quasi-order $\succeq$ is denoted by $\geq_{po}^{\Lambda,\succeq}$ and is defined as the least fixed point of function $\mathcal{H}$.

In order to ease the notation we omit, whenever possible, both the status Λ and the quasi-order $\succeq$, and write $\geq_{po}$ instead of $\geq_{po}^{\Lambda,\succeq}$.

Obviously, as a consequence of the definition of $\geq_{po}$ we have that:

Proposition 3.9. The relation $\geq_{po}$ is a quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ satisfying

$$s = f(s_1,\ldots,s_k)\ \geq_{po}\ t$$

with $f \in \mathcal{F}\cup\mathcal{X}$, having arity $k \geq 0$, if and only if one of the following conditions holds:

1. $t = g(t_1,\ldots,t_m)$, for some $g \in \mathcal{F}\cup\mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s\ \geq_{po}\ t_j$ and $\neg(t_j\ \geq_{po}\ s)$, and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s\ \mathtt{ord}(\Lambda(\geq_{po}))\ t$, or
   (c) $s \sim t$ and $s\ \mathtt{eq}(\Lambda(\geq_{po}))\ t$, and for all $1 \leq j \leq k$ we have that $t \geq_{po} s_j$ and $\neg(s_j \geq_{po} t)$; or
2. $\exists 1 \leq i \leq k:\ s_i\ \geq_{po}\ t$.

The quasi-order $\geq_{po}$ bears a striking similarity with the usual definition of *spo* (Kamin and Lévy [13]). Indeed, $\geq_{po}$ is a generalization of *spo* and *rpo* since this orders can be obtained from $\geq_{po}$ by proper instantiations of the parameters Λ and $\succeq$, as we shall later see. $\geq_{po}$ represents a class of quasi-orders on terms that share the same structure and differ only in the way "equivalent" terms are handled.

In order to simplify the notation we give now characterizations of the strict and equivalent part of the quasi-order $\geq_{po}$. Usually we would denote such parts by $\mathtt{ord}(\geq_{po})$ and $\mathtt{eq}(\geq_{po})$, respectively, but this is a bit cumbersome, so whenever possible we will use the notations $>_{po}$ instead of $\mathtt{ord}(\geq_{po})$ and $\sim_{po}$ instead of $\mathtt{eq}(\geq_{po})$. We can then state:

Proposition 3.10. The equivalence relation $\sim_{po}$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ satisfies

$$s = f(s_1,\ldots,s_k) \ \sim_{po} \ t,$$

with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, if and only if

1. $t = g(t_1,\ldots,t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s \ >_{po} \ t_j$ and

   - $s \sim t$ and $s \ \texttt{eq}(\Lambda(\geq_{po})) \ t$, and for all $1 \leq j \leq k$ holds $t \ >_{po} \ s_j$.

Proof This is a trivial consequence of lemma 3.6, part 2. □

Proposition 3.11. The partial order $>_{po}$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ satisfies

$$s = f(s_1,\ldots,s_k) \ >_{po} \ t,$$

with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, if and only if one of the following conditions holds:

1. $t = g(t_1,\ldots,t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s \ >_{po} \ t_j$ and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s \ \texttt{ord}(\Lambda(\geq_{po})) \ t$; or
2. $\exists 1 \leq i \leq k : \ s_i \ \geq_{po} \ t$.

Proof Note that $s \ >_{po} \ t$ means that $s \ \geq_{po} \ t$ and $\neg(t \ \geq_{po} \ s)$ (in other words $\neg(s \ \sim_{po} \ t)$). Using lemma 3.6 (or proposition 3.10) and proposition 3.9, we can conclude that $s \ \geq_{po} \ t$ only by application of cases 1a, 1b or 2; on the other hand if $s \ \geq_{po} \ t$ can be concluded by application of one of these cases then (using the same results) we also have $s \ \geq_{po} \ t$ and $\neg(t \ \geq_{po} \ s)$. □

3.3 Properties of the scheme

We now discuss what kind of properties does the quasi-order $\geq_{po}$ enjoy. Since $\geq_{po}$ is parameterised by the status Λ and the quasi-order $\succeq$ it is to be expected that the properties $\geq_{po}$ enjoys depend directly on the properties enjoyed by $\succeq$ and maintained by Λ. There is however one such property which is universal, i. e., does not depend on the parameters, namely the *subterm property*. As we now show, the quasi-order $\geq_{po}$ enjoys the subterm property; more precisely the strict part of it does.

Lemma 3.12. The partial order $>_{po}$ satisfies $C[s] \ >_{po} \ s$, for any term s and any non-trivial context C.

Proof (Sketch) Since reflexivity of $\geq_{po}$ ensures that $s \geq_{po} s$, case 2 of proposition 3.9 gives $f(\ldots, s, \ldots) \geq_{po} s$, for any $f \in \mathcal{F}$ having arity $n \geq 1$. That the relation is strict, i. e., that $\neg(s \geq_{po} f(\ldots, s, \ldots))$ is a consequence of lemma 3.5. □

What about other important properties as closedness under substitutions, closedness under contexts, well-foundedness and totality? In general the quasi-order $\geq_{po}$ will not enjoy these other properties. The reason why stems from the use (and definition) of status appearing in the construction of the order. For example, if the status function produces an order which is not closed under substitutions or if the parameter $\succeq$ is itself not closed under substitutions, $\geq_{po}$ will not be closed under substitutions. A similar observation applies to the other properties as well. In many cases some of these properties are not only desirable but essential - that is the case of well-foundedness for termination proofs - therefore it is important that sufficient conditions can be given in order to ensure those properties. Before doing so, we present a series of examples to illustrate the problems envolved.

Example 3.13. Suppose $\mathcal{F} = \{f/2, a/0, b/0\}$, $\mathcal{X} = \{x\}$. Let $\succeq$ be a quasi-order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ satisfying:

- $a \succ b \succ x$,
- $f(s,t) \sim f(t,s)$, for all $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ (f is a commutative symbol).

Define the status Λ by

$$s \; \Lambda(\theta) \; t \iff \begin{cases} s = t \quad \text{or} \\ s \neq t \quad \text{and} \quad s = f(s_1, s_2), t = f(t_1, t_2), \text{ and} \\ \qquad\qquad\qquad \langle s_1, s_2 \rangle \; \theta_{lex} \; \langle t_1, t_2 \rangle \end{cases}$$

where for any quasi-order θ, θ_{lex} denotes its lexicographic extension (in this case in sequences of length 2).

It is not difficult to see that $\Lambda(\theta)$ is a quasi-order and that Λ is weakly monotone, thus a well-defined status. Consider now $\geq_{po}$ associated with this particular choice of Λ and $\succeq$. We see that $\geq_{po}$ is not closed under substitutions. Indeed we have $f(b,x) \geq_{po} f(x,b)$ since $f(b,x) \geq_{po} b, x$, and neither $x \geq_{po} f(b,x)$ nor $b \geq_{po} f(b,x)$, and $f(b,x) \sim f(x,b)$ and $\langle b, x \rangle \; \texttt{ord}(\geq_{po,lex}) \; \langle x, b \rangle$ (this last inequality comes as a consequence of $b \succ x$).

Let σ be the substitution associating a to the variable x. Then $f(b,x) \geq_{po} f(x,b)$ but $f(b,x)\sigma \not\geq_{po} f(x,b)\sigma$, actually $f(x,b)\sigma = f(a,b) \geq_{po} f(b,a) = f(b,x)\sigma$.

Note that in this case the quasi-order $\succeq$ is not closed under substitutions: we have $b \succ x$ but if $\sigma(x) = f(a,b)$ then we do not have $b\sigma = b \succeq f(a,b) = \sigma(x)$. One can wonder whether closedness under substitutions of $\succeq$ is enough to obtain closedness under substitution for $\geq_{po}$. That is not so as the following example shows.

Example 3.14. Suppose $\mathcal{F} = \{g/2, f/1, a/0\}$, $\mathcal{X} = \{x\}$. Let $\succeq$ be the quasi-order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ defined by: $s \succeq t \iff \texttt{root}(s) = \texttt{root}(t)$. Note that $\succeq$ is closed under substitutions.

Define the status Λ as the function that assigns to any quasi-order the quasi-order $\sqsupseteq$ denoting the size of a term (i. e., $s \sqsupseteq t \iff |s| \geq |t|$). Note that this relation is not closed under substitutions and consequently neither will $\geq_{po}$ be. It is not difficult to see that $\sqsupseteq$ is a quasi-order and that Λ is weakly monotone, thus a well-defined status.

Let $s = g(f(a), x), t = g(x, x)$, then $s >_{po} t$: indeed $s >_{po} x$, by lemma 3.12, and further $s \sim t$ (they have equal roots) and $s \sqsupset t$ (since $|s| > |t|$), thus the statement follows from case 1b in proposition 3.11. Take the substitution σ such that $\sigma(x) = f(f(a))$, then $s\sigma = g(f(a), f(f(a)))$ and $t\sigma = g(f(f(a)), f(f(a)))$ and it is easy to see that $t\sigma \ >_{po} \ s\sigma$.

Example 3.15. Let $\mathcal{F} = \{f/1, a/0, b/0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be a quasi-order on $\mathcal{T}(\mathcal{F})$ satisfying: $a \succ b$ and $f(b) \succ f(a)$. Define the status Λ by

$$s \ \Lambda(\theta) \ t \iff \begin{cases} s = t & \text{or} \\ s \neq t & \text{and} \quad s = f(s_1), t = f(t_1), \text{ and } s_1 \ \theta \ t_1 \end{cases}$$

As in the previous examples it is not difficult to see that Λ is a well-defined status.

Consider the quasi-order $\geq_{po}$ obtained using these parameters. Then it is easy to see that $a \geq_{po} b$ (actually $a >_{po} b$) while if we place both a and b within the context $f(\Box)$, we cannot conclude that $f(a) \geq_{po} f(b)$.

Note that if we would replace $f(b) \succ f(a)$ by $f(b) \sim f(a)$, then we could conclude that $f(a) >_{po} f(b)$.

Example 3.16. Let $\mathcal{F} = \{f/1, a/0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be defined again by $s \succeq t \iff \mathtt{root}(s) = \mathtt{root}(t)$. Note that this quasi-order is well-founded since its strict part is empty.

Let the status Λ be a constant function given by the quasi-order

$$s \sqsupseteq t \iff s = f^i(a), t = f^j(a), \text{ and } 0 \leq i \leq j$$

Again it is not difficult to see that Λ is a well-defined status. Note that $\sqsupseteq$ is not well-founded, in particular we have the infinite descending chain:

$$f(a) \sqsupset f(f(a)) \sqsupset f(f(f(a))) \sqsupset \cdots$$

Consider the quasi-order $\geq_{po}$ obtained using these parameters. It is not difficult to see that $\geq_{po}$ is well-founded. Indeed, if $s, t \in \mathcal{T}(\mathcal{F})$ then we must have $s = f^i(a), t = f^j(a)$, for some $i, j \geq 0$ and if $s \ >_{po} \ t$ then we must have that t is a proper subterm of s. If $i = j$ then the terms are equal and we cannot have $s \ >_{po} \ t$; if $j > i$ then s is a proper subterm of t and by lemma 3.12 we conclude that $t \ >_{po} \ s$, so the only remaining case is $i > j$ which means that t is a proper subterm of s. So if $s \ >_{po} \ t$ we also have $|s| > |t|$ and since all terms are finite this rules out the existence of an infinite descending chain $s_0 \ >_{po} \ s_1 \ >_{po} \ s_2 \ldots$.

Example 3.17. Let $\mathcal{F} = \{c_i/1 |\ i \geq 0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be a precedence based quasi-order for the precedence $\trianglerighteq$ satisfying $c_i \triangleright c_j$ if and only if $i < j$. Note that $\succeq$ is not well-founded since we have $c_0 \succ c_1 \succ c_2 \succ \cdots$.

Let Λ be any well-defined status yielding a well-founded quasi-order. The quasi-order $\geq_{po}$ associated with these parameters is not well-founded since we also have

$$c_0 \ >_{po} \ c_1 \ >_{po} \ c_2 \ >_{po} \cdots$$

Example 3.18.

Let $\mathcal{F} = \{a/0, b/0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be equality. Note that $\succeq$ is not total in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ since a and b are not comparable. No matter what our choice for the status Λ is, a and b will remain incomparable under $\geq_{po}$.

Example 3.19. Let $\mathcal{F} = \{f/2, a/0, b/0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be any total quasi-order in $\mathcal{T}(\mathcal{F})$ satisfying $f(s,t) \sim f(t,s)$, for any $s, t \in \mathcal{T}(\mathcal{F})$.

Let the status Λ be a constant function given by equality, i. e.,

$$\forall \theta : \ s \ \Lambda(\theta) \ t \iff s = t$$

Again Λ is a well-defined status. Note that $\texttt{ord}(\Lambda(\theta)) = \emptyset$, for any quasi-order θ.

Consider the quasi-order $\geq_{po}$ obtained using these parameters. This quasi-order is not total: the terms $f(a,b)$ and $f(b,a)$ are not comparable. Due to lemma 3.12 we could never apply case 2 (in proposition 3.9) and since $f(a,b) \sim f(b,a)$ we can only try cases 1b or 1c. The first is not applicable because $\texttt{ord}(\Lambda(\theta)) = \emptyset$, and the second would require the terms to be equal.

The previous examples hint that if we want the quasi-order $\geq_{po}$ to enjoy a certain property we may need to require that both $\succeq$ and Λ enjoy the same property (or a variation thereof). That may not be necessary but will help provide sufficient conditions.

3.3.1 Closedness under substitutions

As we saw in examples 3.13, 3.14, $\geq_{po}$ need not be closed under substitutions. We now provide a sufficient condition to guarantee that that happens. First we need a definition.

Definition 3.20. We say that a status Λ is *(strictly) stable* if it satisfies the following condition: if θ is (strictly) closed under substitutions for $S \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ then $\Lambda(\theta)$ is (strictly) closed under substitutions for $\overline{S} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, where

$$\overline{S} = \{f(s_1, \ldots, s_k) |\ f/k \in \mathcal{F} \cup \mathcal{X}, k \geq 0, \text{ and } \forall 1 \leq i \leq k : \ s_i \in S\}$$

Theorem 3.21. If $\succeq$ is strictly closed under substitutions and the status Λ is strictly stable then $\geq_{po}$ is strictly closed under substitutions.

Proof We have to prove both that if $s \geq_{po} t$ and σ is any arbitrary substitution we have $s\sigma \geq_{po} t\sigma$ and if $s >_{po} t$ we also have $s\sigma >_{po} t\sigma$. We prove both assertions simultaneously and proceed by induction on $|s| + |t|$. Suppose we have two minimal terms s, t with $s \geq_{po} t$ and for which the property is not yet verified, i. e., if s', t' are terms such that $|s'| + |t'| < |s| + |t|$ then s', t' satisfy the property. We now proceed by case analysis.

- If $s \geq_{po} t$ holds by cases 1a, 1b or 1c of proposition 3.9, then $s = f(s_1, \ldots, s_k)$ for some $k \geq 0$, and $f \in \mathcal{F} \cup \mathcal{X}$, and $t = g(t_1, \ldots, t_m)$ for some $m \geq 0$ and $g \in \mathcal{F} \cup \mathcal{X}$. Furthermore $s >_{po} t_j$ for all $1 \leq j \leq m$, so by induction hypothesis we get $s\sigma >_{po} t_j\sigma$ for all $1 \leq j \leq m$.

  If case 1a is applicable we have $s \succ t$ and therefore $s\sigma \succ t\sigma$; by the same case we conclude that $s\sigma \geq_{po} t\sigma$. Since $s \geq_{po} t$ by case 1a, proposition 3.11 ensures that actually we have $s >_{po} t$. We have concluded that $s\sigma \geq_{po} t\sigma$ also by case 1a and using again proposition 3.11, we can conclude that $s\sigma >_{po} t\sigma$, as we wanted.

  If case 1b is applicable, we have $s \sim t$ and so also $s\sigma \sim t\sigma$. We also have $s \; \texttt{ord}(\Lambda(\geq_{po})) \; t$. By induction hypothesis $\geq_{po}$ is strictly closed under substitutions in $S = \{s_1, \ldots, s_k, t_1, \ldots, t_m\}$, and since Λ is strictly stable we get that $\Lambda(\geq_{po})$ is strictly closed under substitutions in $\overline{S}$. Consequently (and because $s, t \in \overline{S}$), we have $s\sigma \; \texttt{ord}(\Lambda(\geq_{po})) \; t\sigma$. By the same case we conclude that $s\sigma \geq_{po} t\sigma$. As in the previous case, proposition 3.11 ensures that actually we have $s >_{po} t$, and since $s\sigma \geq_{po} t\sigma$ was derived using the same case, proposition 3.11 gives us $s\sigma >_{po} t\sigma$.

  If case 1c is applicable, we again have $s \sim t$ and therefore $s\sigma \sim t\sigma$. We also have $s \; \texttt{eq}(\Lambda(\geq_{po})) \; t$. By a similar argument as in the previous case we conclude that $s\sigma \; \texttt{eq}(\Lambda(\geq_{po})) \; t\sigma$. By the same case (1c) we conclude that $s\sigma \geq_{po} t\sigma$. Due to proposition 3.11, we do not have neither $s >_{po} t$ nor $s\sigma >_{po} t\sigma$.

- If $s \geq_{po} t$ holds by case 2, of proposition 3.9, then $s = f(s_1, \ldots, s_k)$ for some $k \geq 1$ and $f \in \mathcal{F}$ and $s_i \geq_{po} t$, for some $1 \leq i \leq k$. By induction hypothesis we get that $s_i\sigma \geq_{po} t\sigma$. Furthermore, if $s_i >_{po} t$, again by induction hypothesis we conclude that $s_i\sigma >_{po} t\sigma$.

□

3.3.2 Closedness under contexts

As we saw in example 3.15, $\geq_{po}$ need not be closed under contexts. Note that while most existing path orders are indeed closed under substitutions, that does not hold for closedness under contexts; in particular *spo* is not closed under contexts in general.

We can try to provide a sufficient condition to guarantee closedness under contexts in a similar way we did it for closedness under substitutions, i. e., requiring that the parameter

$\succeq$ is closed under contexts and that the status Λ enjoys some kind of closure property akin to stability. However that is not possible as the following example will show.

Example 3.22.
Let $\mathcal{F} = \{f/1, g/1\}$, $\mathcal{X} = \{x\}$. Consider the following rewrite system:

$$\begin{array}{rcl} f(f(x)) & \to & f(g(f(x))) \\ f(f(x)) & \to & g(f(x)) \end{array}$$

This rewrite system is terminating: we sketch a proof of termination. It is well-known that the system consisting only of the first rule is terminating (see for example [9] for a proof of termination); thus we can say that there is some algebra $(A, >_A)$, with $>_A$ being a well-founded partial order, where terms are interpreted and such that if $s \to t$ (using that rule) then $\phi(s) >_A \phi(t)$, being ϕ the interpretation function. Consider now another measure denoted by $\#_f(s)$, that given a term s counts the number of occurrences of the symbol f in the term. It is not difficult to see that for the rewrite system above, for every reduction $s \to t$ then either $\#_f(s) > \#_f(t)$, if the rule used was the second one, or $\#_f(s) = \#_f(t)$, if the rule used was the first one, and in this case we can use the measure $>_A$ to deduce that $\phi(s) >_A \phi(t)$. Combining lexicografically $\#_f$ with $>_A$ (in this order) we obtain a well-founded order, say $\sqsupset$, such that if $s \to t$ then $s \sqsupset t$, giving us termination of the system.

We now take $\succeq$ as follows: $\sim$ is just equality and $\succ$ is exactly the transitive closure of the reduction relation above, i. e., $\succ = \to^+$. Note that $\succeq$ is strictly closed under contexts (and substitutions).

Let the status Λ be a constant function given by equality. This status fulfils any condition akin to context stability that we may think of since for any input the quasi-order produced is strictly closed under contexts!

Consider the quasi-order $\geq_{po}$ obtained using these parameters. This quasi-order is not closed under contexts: we have that $g(f(x)) >_{po} f(x)$ due to subterm compatibility (lemma 3.12) but we do not have $f(g(f(x))) \geq_{po} f(f(x))$. In fact we have the reverse relation: $f(f(x)) >_{po} f(x)$ and since $f(f(x)) \succ g(f(x))$, we may conclude that $f(f(x)) >_{po} g(f(x))$. But since also $f(f(x)) \succ f(g(f(x)))$, we conclude that $f(f(x)) >_{po} f(g(f(x)))$.

If we restrict the parameter $\succeq$ to be a precedence on $\mathcal{F}$ and we impose a trivial condition on the status Λ, we can come up with a sufficient condition for closedness under contexts. It is no coincidence that the resulting $\geq_{po}$ is similar to *rpo*, known to be closed under contexts.

Definition 3.23. We say that a status Λ is *context stable* if it satisfies the following condition: if $s \; \theta \; t$ then $f(\ldots, s, \ldots) \; \Lambda(\theta) \; f(\ldots, t, \ldots)$, for any $f \in \mathcal{F}$ and terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. A status Λ is *strictly context stable* if the previous condition holds for both the strict parts of θ and $\Lambda(\theta)$, and the equivalent parts of θ and $\Lambda(\theta)$, respectively.

Theorem 3.24. If $\succeq$ is precedence based and the status Λ is strictly context stable then $\geq_{po}$ is strictly closed under contexts.

Proof We have to prove both that

- if $s \ \geq_{po}\ t$ then $f(s_1,\ldots,s,\ldots,s_k) \ \geq_{po}\ f(s_1,\ldots,t,\ldots,s_k)$, and
- if $s \ >_{po}\ t$ we also have $f(s_1,\ldots,s,\ldots,s_k) \ >_{po}\ f(s_1,\ldots,t,\ldots,s_k)$,

for any arbitrary function symbol $f \in \mathcal{F}$ with arity $k \geq 1$, and arbitrary terms s_j, $1 \leq j \leq k, j \neq i$, with s and t occurring at position i. We prove both assertions simultaneously and proceed by induction on the context. For the the trivial context the result obviously holds. Suppose then that $s \ \geq_{po}\ t$ and let $f \in \mathcal{F}$ have arity $k \geq 1$. We have either $s \ >_{po}\ t$ and in this case we also have $f(\ldots,s,\ldots)\ \mathtt{ord}(\Lambda(\geq_{po}))\ f(\ldots,t,\ldots)$, or $s \ \sim_{po}\ t$ and in this case $f(\ldots,s,\ldots)\ \mathtt{eq}(\Lambda(\geq_{po}))\ f(\ldots,t,\ldots)$ (this is due to the hypothesis that Λ is strictly context stable). Since $\succeq$ is precedence based, we have that $f(\ldots,s,\ldots) \sim f(\ldots,t,\ldots)$, so we only need to see that $f(s_1,\ldots,s,\ldots,s_k) \ >_{po}\ s_i$, for $i \neq j$ and $f(s_1,\ldots,s,\ldots,s_k) \ >_{po}\ t$. The first part follows directly from lemma 3.12, as for the second, the same lemma and the hypothesis allows us to write:

$$f(s_1,\ldots,s,\ldots,s_k) \ >_{po}\ s \ \geq_{po}\ t$$

and the result follows from transitivity and the fact that for any quasi-order holds $\sim \circ \succ \circ \sim \subseteq \succ$.

Now if $s \ \geq_{po}\ t$, by the exposed above and using case 1c or 1b in proposition 3.9, we conclude that $f(\ldots,s,\ldots) \ \geq_{po}\ f(\ldots,t,\ldots)$, and if $s \ >_{po}\ t$, again from by the exposed above and using case 1b of proposition 3.11, we conclude that $f(\ldots,s,\ldots) \ >_{po}\ f(\ldots,t,\ldots)$. □

3.3.3 Totality

Totality is a property which is usually relevant in the restricted context of ground terms. This is so because totality and closedness under substitutions are incompatible properties: if one has a set of variables containing more than one variable, in order to have closedness under substitutions one needs to have different variables to be incomparable, thus no totality is possible in the set of open terms. But totality on ground terms can be achieved even combined with closedness under substitutions. We now present a sufficient condition for obtaining a total $\geq_{po}$ on ground terms. It is easy to transpose this condition to the open terms case. Nevertheless, because we believe that in practice totality on ground terms is more relevant, we restrict our presentation to the ground case.

Definition 3.25. We say that a status Λ is *totality stable* if it satisfies the following condition: for any $S \in \mathcal{T}(\mathcal{F})$, if θ is total in S then $\Lambda(\theta)$, is total in $\widehat{S}$, where

$$\widehat{S} = \{f(s_1,\ldots,s_k)|\ f/k \in \mathcal{F}, k \geq 0, \text{ and } \forall 1 \leq i \leq k:\ s_i \in S\}$$

Theorem 3.26. If $\succeq$ is total on $\mathcal{T}(\mathcal{F})$ and the status Λ is totality stable then $\geq_{po}$ is total on $\mathcal{T}(\mathcal{F})$.

Proof We need to see that for any $s, t \in \mathcal{T}(\mathcal{F})$ either $s \geq_{po} t$ or $t \geq_{po} s$. We proceed by induction on $|s| + |t|$.

If s, t are constants then either $s \succ t$ or $t \succ s$ or $s \sim t$. In the first two cases we obtain respectively $s \geq_{po} t$ and $t \geq_{po} s$. In the last case since $\geq_{po}$ is total in $\emptyset$ and Λ is totality stable, we have that $\Lambda(\geq_{po})$ is total in $\mathcal{F}_0$, the set of constants, and therefore either $s\ \texttt{ord}(\Lambda(\geq_{po}))\ t$, or $t\ \texttt{ord}(\Lambda(\geq_{po}))\ s$ or $s\ \texttt{eq}(\Lambda(\geq_{po}))\ t$. In all cases we can state that either $s \geq_{po} t$ or $t \geq_{po} s$.

Suppose now that $t = g(t_1, \ldots, t_m)$. Either there is an j such that $t_j \geq_{po} s$ and then $t \geq_{po} s$ or, by induction hypothesis, for all $1 \leq j \leq m$, $s >_{po} t_j$. Also if $s = f(s_1, \ldots, s_k)$, then either there is an i such that $s_i \geq_{po} t$ and then $s \geq_{po} t$ or, by induction hypothesis, for all $1 \leq i \leq k$, $t >_{po} s_i$. Suppose then that both $s >_{po} t_j$, for all $1 \leq j \leq m$, and $t >_{po} s_i$, for all $1 \leq i \leq k$. If $s \succ t$ we can conclude that $s \geq_{po} t$ (by case 1a in proposition 3.9), if $t \succ s$, we conclude the reverse by the same case. If $s \sim t$, since by induction hypothesis $\geq_{po}$ is total in $S = \{s_1, \ldots, s_k, t_1, \ldots, t_m\}$ and Λ is totality stable, then $\Lambda(\geq_{po})$ is total in $\widehat{S}$ and since $s, t \in \widehat{S}$, we have either $s\ \texttt{ord}(\Lambda(\geq_{po}))\ t$, or $t\ \texttt{ord}(\Lambda(\geq_{po}))\ s$, or $s\ \texttt{eq}(\Lambda(\geq_{po}))\ t$. In all cases we can state that either $s \geq_{po} t$ or $t \geq_{po} s$. $\square$

3.3.4 Well-foundedness

An important and extensive use of orders on terms is in termination proofs, but for an order to be used for such purpose it is essential that the order is well-founded.

As we saw in examples 3.16, 3.17, in general $\geq_{po}$ will not be well-founded, but we can impose conditions both on the quasi-order $\succeq$ and on the status Λ in order to obtain well-founded quasi-orders.

A generalized way of proving well-foundedness of terms orders is through Kruskal's theorem ([17, 19, 11]). Roughly this theorem implies that any simplification ordering (an order closed under substitutions and contexts and satisfying the subterm property) is well-founded; clearly it cannot be applied to non-simplification orderings, so Kruskal's theorem cannot help us prove well-foundedness of our orders, which are not, in general, simplification orderings. In [10, 9] the problem of proving well-foundedness of (quasi) orders was studied and simple sufficient conditions are given; these results apply also to orders which are not closed under contexts and/or substitutions.

The following definition and theorem are taken from [10, 9], slightly modified in order to fit the quasi-order framework.

Definition 3.27. A status Λ is said to be *well-foundedness stable*[2] if it satisfies the following condition: if θ is a quasi-order in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ well-founded in $A \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ then $\Lambda(\theta)$ is well-founded in $\overline{A} = \{f(t_1, \ldots, t_k) : f/k \in \mathcal{F} \cup \mathcal{X}, k \geq 0, \text{ and } t_i \in A \text{ for all } 1 \leq i \leq k\}$.

[2] *Term status* in the terminology of [10, 9].

Theorem 3.28. Let $\geq$ be a quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and let Υ be a well-foundedness stable status. Suppose $>$ has the subterm property and satisfies the following condition:

- $\forall f/m, g/n \in \mathcal{F}\cup\mathcal{X},\ s_1,\ldots,s_m,t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$:
  if $s = f(s_1,\ldots,s_m) > g(t_1,\ldots,t_n) = t$ then either
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1,\ldots,t_n)$, or
  - $s\ \texttt{ord}(\Upsilon(\geq))\ t$

Then $\geq$ is well-founded on $\mathcal{T}(\mathcal{F},\mathcal{X})$.

We can now give a suficient condition for $\geq_{po}$ to be well-founded.

Theorem 3.29. If $\succeq$ is a well-founded quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and Λ is a well-foundedness stable status then $\geq_{po}$ is well-founded.

Proof We define the following function Υ on quasi-orders on $\mathcal{T}(\mathcal{F},\mathcal{X})$.

$$s\ \Upsilon(\theta)\ t \iff \begin{cases} s \succ t & \text{or} \\ s \sim t & \text{and} \quad s\ \Lambda(\theta)\ t \end{cases}$$

It is not difficult to see that whenever Λ is a well-foundedness stable status then Υ is also a well-foundedness stable status. Furthermore we have

$$s\ \texttt{ord}(\Upsilon(\theta))\ t \iff \begin{cases} s \succ t & \text{or} \\ s \sim t & \text{and} \quad s\ \texttt{ord}(\Lambda(\theta))\ t \end{cases}$$

Due to the definition of Υ, the statement above and proposition 3.11, we have indeed that whenever $s = f(s_1,\ldots,s_m) >_{po} g(t_1,\ldots,t_n) = t$, for some $f/m, g/n \in \mathcal{F}\cup\mathcal{X},\ s_1,\ldots,s_m,t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$ then either

- $\exists 1 \leq i \leq m : s_i \geq_{po} g(t_1,\ldots,t_n)$, or
- $s\ \texttt{ord}(\Upsilon(\geq_{po}))\ t$

Since by lemma 3.12 $>_{po}$ has the subterm property, all conditions of theorem 3.28 are met so we can apply it to conclude that $\geq_{po}$ is well-founded. □

3.4 Semantic and Recursive path orders

We now show how the quasi-order versions of *spo* and *rpo* can be obtained as instances of the scheme presented. We will restrict ourselves to the "pure" versions of these orders in which the extension associated to the orders is either the multiset extension or the lexicographic extension. In the last years other extensions have been proposed (see eg. [24, 25]), notably mixing the two before mentioned extensions. The idea is that the extension used is associated with function symbols, so for example if we compare $f(\ldots)$ with $g(\ldots)$

and f and g are "equal" then their subterms are compared in a way which depends on the symbols f and g. It is not difficult to see that these variations are still within our framework (see [9, 8]); for the sake of simplicity and because we believe that dealing with the more complicated cases would not add anything to the understanding of the mechanism, we choose not to treat those cases here.

3.4.1 *rpo*

The following definition of *rpo* is due to Dershowitz [4].

Definition 3.30. (rpo) Let $\trianglerighteq$ be a quasi-order in the set $\mathcal{F}$. The *recursive path order* denoted by $\geq_{rpo}$ on the set $\mathcal{T}(\mathcal{F})$ is defined as follows:

$$s = f(s_1, \ldots, s_k) \geq_{rpo} g(t_1, \ldots, t_m) = t$$

if one of the following conditions holds:

1. $s_i \geq_{rpo} t$, for some $i = 1, \ldots, k$; or
2. $f \triangleright g$ and $s >_{rpo} t_j$, for all $j = 1, \ldots, m$; or
3. $f \sim g$ and $\{\{s_1, \ldots, s_k\}\} \geq_{rpo,mul} \{\{t_1, \ldots, t_m\}\}$.

To see how $\geq_{rpo}$ can be obtained from our construction we define the parameters $\succeq$ and Λ. For $\succeq$ we take the precedence based order for the precedence $\trianglerighteq$, extending it to cope with the variables in $\mathcal{X}$:

$$f(s_1, \ldots, s_k) \succeq g(t_1, \ldots, t_m) \iff \begin{cases} f \trianglerighteq g & \text{if } f, g \in \mathcal{F} \\ f = g & \text{if } f, g \in \mathcal{X} \end{cases}$$

As for Λ we define it as follows:

$$s = f(s_1, \ldots, s_k)\ \Lambda(\theta)\ g(t_1, \ldots, t_m) \iff \{\{s_1, \ldots, s_k\}\}\ \theta_{mul}\ \{\{t_1, \ldots, t_m\}\}$$

For any quasi-order its multiset extension is also a quasi-order; furthermore if $\theta \sqsupseteq \theta'$ then also $\theta_{mul} \sqsupseteq \theta'_{mul}$. Indeed it is well-known that $\theta_{mul} \supseteq \theta'_{mul}$; suppose that $X = \{\{x_1, \ldots, x_k\}\}\ \texttt{ord}(\theta'_{mul})\ \{\{y_1, \ldots, y_m\}\} = Y$ then for each y_j there is an element x_i such that $x_i\ \texttt{ord}(\theta')\ y_j$ and then also $x_i\ \texttt{ord}(\theta)\ y_j$ implying $X\ \texttt{ord}(\theta_{mul})\ Y$. So Λ is a well-defined status.

The quasi-order $\geq_{po}$ obtained using these parameters can be writen as (see proposition 3.9):
$s = f(s_1, \ldots, s_k)\ \geq_{po}\ t$, with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, if and only if one of the following conditions holds:

1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s\ >_{po}\ t_j$, and either

(a) $f \rhd g$, or

(b) $f \sim g$ and $\{\{s_1, \ldots, s_k\}\}$ $\texttt{ord}(\geq_{po,mul})$ $\{\{t_1, \ldots, t_m\}\}$, or

(c) $f \sim g$ and $\{\{s_1, \ldots, s_k\}\}$ $\texttt{eq}(\geq_{po,mul})$ $\{\{t_1, \ldots, t_m\}\}$, and for all $1 \leq j \leq k$ we have that $t >_{po} s_j$; or

2. $\exists 1 \leq i \leq k : \; s_i \; \geq_{po} \; t$.

Apparently the two definition do not coincide: in case 1c extra conditions are imposed on the terms which do not occur in definition 3.30. However it is not difficult to see that both definitions are the same.

Indeed in this particular case the status Λ satisfies the following:

$$f(s_1, \ldots, s_k) \; \texttt{eq}(\Lambda(\geq_{po})) \; g(t_1, \ldots, t_m) \Rightarrow g(t_1, \ldots, t_m) >_{po} s_i, \forall 1 \leq i \leq k$$

Due to the definition of Λ, $f(s_1, \ldots, s_k)$ $\texttt{eq}(\Lambda(\geq_{po}))$ $g(t_1, \ldots, t_m)$ is equivalent to $\{\{s_1, \ldots, s_k\}\}$ $\texttt{eq}(\geq_{po,mul})$ $\{\{t_1, \ldots, t_m\}\}$ and then for each term s_i there is a term t_j such that s_i $\texttt{eq}(\geq_{po})$ t_j. From $g(t_1, \ldots, t_m) >_{po} t_j$ $\texttt{eq}(\geq_{po})$ s_i we conclude that $g(t_1, \ldots, t_m) >_{po} s_i$. So we can eliminate the condition " $t >_{po} s_j$, for all $1 \leq j \leq k$", in clause 1c, and then clauses 1b and 1c simply correspond to clause 3 in definition 3.30, with the strict and equivalent parts discriminated.

Thus $\geq_{po}$ coincides with $\geq_{rpo}$ in the set $\mathcal{T}(\mathcal{F})$. Note that it is not essential to define $\geq_{rpo}$ in this set; in definition 3.30 we could have used the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Then the orders coincide in the whole set of terms.

Since $\geq_{rpo}$ is a particular instance of our scheme, we can apply all the derived results to it. So we immediately conclude that (lemma 3.12) $>_{rpo}$ enjoys the subterm property. The other usual properties of $\geq_{rpo}$ can also be derived from the results presented, as we now see.

Note that the status Λ is strictly stable. Suppose θ is a quasi-order strictly closed under substitutions in $S \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$. Let $X, Y \in \mathcal{M}(S)$ be such that $X \; \theta_{mul} \; Y$. Then for every element $y \in Y$ there is an element $x \in X$ such that $x \; \theta \; y$; since both $x, y \in S$ we can conclude that $x\sigma \; \theta \; y\sigma$, for any substitution σ and this implies that $X\sigma \; \theta_{mul} \; Y\sigma$ (where $Z\sigma$ denotes the multiset obtained from Z by applying the substitution σ to each one of its elements). If the relation is strict, i. e., if X $\texttt{ord}(\theta_{mul})$ Y, in a similar way we can also see that $X\sigma$ $\texttt{ord}(\theta_{mul})$ $Y\sigma$.

Since $\succeq$ is also strictly closed under substitutions, we can apply theorem 3.21 to conclude that $\geq_{rpo}$ is strictly closed under substitutions.

The status Λ is also strictly context stable. If $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $s \; \theta \; t$ then $X \sqcup \{\!\{s\}\!\} \; \theta_{mul} \; X \sqcup \{\!\{t\}\!\}$, for any multiset of terms X, implying $f(\ldots, s, \ldots) \; \Lambda(\theta) \; f(\ldots, t, \ldots)$. Note that the relation is strict if the relation between s and t is itself strict. Application of theorem 3.24 yields that $\geq_{rpo}$ is strictly closed under contexts.

As for well-foundedness, it is well-known that the multiset extension of a quasi-order is well-founded if and only if the quasi-order is well-founded. Therefore Λ is a well-foundedness stable status and we can state (cf. theorem 3.29) that if $\unrhd$ is well-founded on $\mathcal{F}$ then $\geq_{rpo}$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. The reverse statement also holds.

3.4.2 *spo*

The definition of *spo* is due to Kamin and Lévy [13]. The reader should note the similarity between $\geq_{po}$ and $\geq_{spo}$: $\geq_{po}$ is actually a general formulation of $\geq_{spo}$, being this generalization a consequence of the freer definition of status used.

The following definition of *spo* extends the original one [13] to quasi-orders and can be found in [4].

Definition 3.31. (**spo**) Let $\geq$ be a well-founded quasi-order on $\mathcal{T}(\mathcal{F})$. The *semantic path order* $>_{spo}$ is defined on $\mathcal{T}(\mathcal{F})$ as follows: $s = f(s_1, \ldots, s_m) \geq_{spo} t$ if either

1. $t = g(t_1, \ldots, t_n)$, for some $g \in \mathcal{F}$ having arity $n \geq 0$, and $s >_{spo} t_i$, for all $1 \leq i \leq n$, and either:
   (a) $s > t$, or
   (b) $s \sim t$ and $\{\!\{s_1 \ldots, s_m\}\!\} \geq_{spo,mul} \{\!\{t_1, \ldots, t_n\}\!\}$, or
2. $\exists i \in \{1, \ldots, m\} : s_i \geq_{spo} t$.

It can be seen [13] that $>_{spo}$ satisfies the subterm property and is in general not monotonic though closed under substitutions.

As with $\geq_{rpo}$, we can define $\geq_{spo}$ over the set of open terms: the easiest way may be to take $\geq$ as defined over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and not only over $\mathcal{T}(\mathcal{F})$.

The way $\geq_{spo}$ can be obtained from our our scheme is again by appropriate instantiation of the parameters $\succeq$ and Λ. For $\succeq$ we take $\geq$ and for Λ we take the same status we did for $\geq_{rpo}$, namely

$$s = f(s_1, \ldots, s_k)\ \Lambda(\theta)\ g(t_1, \ldots, t_m) \iff \{\{s_1, \ldots, s_k\}\}\ \theta_{mul}\ \{\{t_1, \ldots, t_m\}\}$$

Note that just like in the case of $\geq_{rpo}$, $\geq_{spo}$ and the path order $\geq_{po}$ obtained using these parameters do not coincide at first sight: again in clause 1c in proposition 3.9 some extra conditions seem to be required. But just like in the previous case (and for the same reason) the extra conditions are always satisfied for this choice of status and thus can be omitted.

As we saw before, the status Λ is strictly stable, strictly context stable and well-foundedness stable. This allows us to state that:

- whenever $\geq$ is strictly closed under substitutions, $\geq_{spo}$ also is strictly closed under substitutions (theorem 3.21);
- whenever $\geq$ is strictly closed under contexts, $\geq_{spo}$ also is strictly closed under contexts (theorem 3.24);
- whenever $\geq$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, $\geq_{spo}$ also is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ (theorem 3.29).

As for the subterm property, lemma 3.12 ensures that $>_{spo}$ enjoys this property.

4 A scheme for definition of partial orders

We now turn to the problem of defining partial orders instead of quasi-orders. The procedure is similar but because we are interested in objects having different properties, we need to work with a different CPO and a different notion of status.

4.1 The CPO of partial orders

Since our aim is to define partial orders, we choose as underlying set for our construction the set of all partial orders. Let S be a set and let $\mathcal{PO}_S$ be the set of all partial orders over S. We consider the structure $(\mathcal{PO}_S, \supset)$, where $\supset$ is set inclusion. It is not difficult to see that $(\mathcal{PO}_S, \supset)$ is a partially ordered set. Furthermore:

Lemma 4.1. The poset $(\mathcal{PO}_S, \supset)$ is a CPO with the bottom element being the empty set and with the supremum of directed sets given by the union of the elements in the set.

Proof Clearly $\emptyset$ is a partial order over any set S, and for any other partial order $\theta \in \mathcal{PO}_S$, $\theta \supseteq \emptyset$, so $\emptyset$ is the least element of $\mathcal{PO}_S$.

Let D be a directed set of partial orders over S. We see that $\bigcup D$ is irreflexive and transitive (and thus an element of $\mathcal{PO}_S$). Suppose $\bigcup D$ is not irreflexive, then there is an element $s \in S$ such that $(s, s) \in \bigcup D$, which means there is an element $\theta \in D$ such that $(s, s) \in \theta$, contradicting irreflexivity of θ. Transitivity of $\bigcup D$ is proven just like in lemma 3.1. Finally we have to prove that $\bigcup D$ is the least upper bound for D. Obviously, for any $\theta \in D$, $\bigcup D \supseteq \theta$. Suppose Υ is another upper bound for D; then $\Upsilon \supseteq \theta$, for every $\theta \in D$, and therefore $\Upsilon \supseteq \bigcup D$. [3] □

We need to adapt the notion of status (cf. definition 3.2) to this new setting. In the quasi-order case, a status is a function that given a quasi-order delivers also a quasi-order. As it should be expected, now a status will be a function that given a *partial* order will deliver a partial order. To keep the notation used minimal, we will overload our terminology: the same names and notations will be used for similar concepts both in the quasi-order as in the partial order setting. It should be clear from context which one is meant at any given time.

Definition 4.2. Let $(S, >)$ be a a partially ordered set. A status is a function $\Lambda : \mathcal{PO}_S \to \mathcal{PO}_S$ which is weakly monotone with respect to the CPO $(\mathcal{PO}_S, \supset)$.

4.2 The partial order scheme

We now fix our CPO to be $(\mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}, \supset)$. Let $\succeq$ be a fixed quasi-order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and let Λ be a status with domain $\mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$.

[3] Actually this structure is far richer that we needed it to be, $(\mathcal{PO}_S, \supset)$ is not only a CPO but a complete lattice.

Definition 4.3. The function $\mathcal{G} : \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})} \to \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$ is given by:

$$s = f(s_1, \ldots, s_k)\ \mathcal{G}(\theta)\ t$$

with $f \in \mathcal{F} \cup \mathcal{X}$ having arity $k \geq 0$, if one of the following conditions holds:

1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$ having arity $m \geq 0$, and $s\ \mathcal{G}(\theta)\ t_j$, for all $1 \leq j \leq m$, and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s\ \Lambda(\theta)\ t$, or
2. $\exists 1 \leq i \leq k :\ s_i\ \mathcal{G}(\theta)\ t$ or $s_i = t$.

The first thing that needs to be checked is that the function $\mathcal{G}$ defines a relation on $\mathcal{T}(\mathcal{F},\mathcal{X})$, or in other words that the recursion in the definition of $\mathcal{G}$ stops. This can easily be done by induction on the sum of the size of the terms: in every recursive "call" of $\mathcal{G}$, the size of the terms involved decreases. We also need to see that the result of applying $\mathcal{G}$ to a partial order on terms results in a partial order on terms. This is what the next lemma is about.

Lemma 4.4. If θ is a partial order in $\mathcal{T}(\mathcal{F},\mathcal{X})$ then $\mathcal{G}(\theta)$ is a partial order in $\mathcal{T}(\mathcal{F},\mathcal{X})$.

Proof We need to see that for any $\theta \in \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$, $\mathcal{G}(\theta) \in \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$, i. e., that $\mathcal{G}(\theta)$ is irreflexive and transitive. Fix then $\theta \in \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$. We first prove transitivity: if $s\ \mathcal{G}(\theta)\ t$ and $t\ \mathcal{G}(\theta)\ u$ then $s\ \mathcal{G}(\theta)\ u$, by induction on $|s| + |t| + |u|$. Let then $s, t, u \in \mathcal{T}(\mathcal{F},\mathcal{X})$ be minimal terms satisfying $s\ \mathcal{G}(\theta)\ t$ and $t\ \mathcal{G}(\theta)\ u$ and for which we still have to see that $s\ \mathcal{G}(\theta)\ u$ holds. As in the proof of transitivity in lemma 3.4, we have to consider the possible cases of definition 4.3 by which we can conclude that $s\ \mathcal{G}(\theta)\ t$ and $t\ \mathcal{G}(\theta)\ u$: 9 cases in total. The case analysis is quite similar to the one done in the proof of lemma 3.4, though substantially simpler, so we skip most of it and present only case 1b *vs.* 1b, the most interesting. We have $s = f(s_1, \ldots, s_k)$, $t = g(t_1, \ldots, t_m)$ and $u = h(u_1, \ldots, u_n)$, for some $f, g, h \in \mathcal{F} \cup \mathcal{X}$, with arities respectively $k, m, n \geq 0$. Since $t\ \mathcal{G}(\theta)\ u$ by case 1b, we have that $t\ \mathcal{G}(\theta)\ u_l$, for all $1 \leq l \leq n$, and combining this with $s\ \mathcal{G}(\theta)\ t$, the induction hypothesis gives us $s\ \mathcal{G}(\theta)\ u_l$, for all $1 \leq l \leq n$. Also $s \sim t \sim u$ and since $\sim$ is transitive we obtain $s \sim u$. Finally $s\ \Lambda(\theta)\ t$ and $t\ \Lambda(\theta)\ u$ and since $\Lambda(\theta)$ is transitive, we also have $s\ \Lambda(\theta)\ u$; applying case 1b we can conclude that $s\ \mathcal{G}(\theta)\ u$.

We now prove irreflexivity, i. e., that for any $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$ we don't have $s\ \mathcal{G}(\theta)\ s$, by induction over $|s|$ and using the already proven fact that $\mathcal{G}(\theta)$ is transitive. Suppose then that $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is a minimal term such that $s\ \mathcal{G}(\theta)\ s$, i. e., if $u \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is such that $|u| < |s|$ then $\neg(u\ \mathcal{G}(\theta)\ u)$. If $s\ \mathcal{G}(\theta)\ s$ holds by case 1a of definition 4.3 then we must have $s \succ s$, contradicting irreflexivity of $\succ$. If $s\ \mathcal{G}(\theta)\ s$ holds by case 1b of definition 4.3 then we must have $s \sim s$ and $s\ \Lambda(\theta)\ s$, contradicting irreflexivity of $\Lambda(\theta)$. And finally if $s\ \mathcal{G}(\theta)\ s$ holds by case 2 of definition 4.3 then we must have either $s_i = s$ or $s_i\ \mathcal{G}(\theta)\ s$, for some principal subterm s_i of s; clearly the first case is impossible and in the second, since $s\ \mathcal{G}(\theta)\ s_i$, by transitivity we conclude that $s_i\ \mathcal{G}(\theta)\ s_i$, contradicting the minimality of s. So we derive that $\mathcal{G}(\theta)$ is also irreflexive, concluding our proof. □

We want to use the function $\mathcal{G}$ to obtain a definition of a path order. For that order to be uniquely determined it is enough to ensure that $\mathcal{G}$ has a least fixed point; we then take the order to be that least fixed point. From theorem 2.22 we know that if $\mathcal{G}$ is weakly monotone then it has a least fixed point. Our next task is to prove weak monotonicity of $\mathcal{G}$.

Lemma 4.5. The function $\mathcal{G}$ is weakly monotone with respect to the CPO $(\mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}, \supset)$.

Proof We need to prove that if $\theta, \theta' \in \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$ and $\theta \supset \theta'$ then $\mathcal{G}(\theta) \supseteq \mathcal{G}(\theta')$. This is achieved by showing that for any terms s, t if $s\ \mathcal{G}(\theta')\ t$ then also $s\ \mathcal{G}(\theta)\ t$. We proceed by induction on $|s| + |t|$. Let s, t be two minimal terms (i. e., terms for which $|s| + |t|$ is minimal) such that $s\ \mathcal{G}(\theta')\ t$. Our induction hypothesis states that $(u\ \mathcal{G}(\theta')\ v) \Rightarrow (u\ \mathcal{G}(\theta)\ v)$ for any terms u, v with $|u| + |v| < |s| + |t|$. We consider the cases of definition 4.3 by which we can derive $s\ \mathcal{G}(\theta')\ t$. If $s\ \mathcal{G}(\theta')\ t$ holds by cases 1a or 1b then $s = f(s_1, \ldots, s_k), t = g(t_1, \ldots, t_m)$, for some $f, g \in \mathcal{F} \cup \mathcal{X}$ with arities respectively $k, m \geq 0$, and $s\ \mathcal{G}(\theta')\ t_j$ for all $1 \leq j \leq m$. By induction hypothesis we conclude that also $s\ \mathcal{G}(\theta)\ t_j$ for all $1 \leq j \leq m$. If

- case 1a holds then $s \succ t$ and by the same case we can conclude that $s\ \mathcal{G}(\theta)\ t$;
- case 1b holds then $s \sim t$ and $s\ \Lambda(\theta')\ t$. Since Λ is weakly monotone, we also have that $s\ \Lambda(\theta)\ t$ and we can use the same case to conclude that $s\ \mathcal{G}(\theta)\ t$;

If $s\ \mathcal{G}(\theta')\ t$ holds by case 2 then $s = f(s_1, \ldots, s_k)$, for some $f \in \mathcal{F}$ with arity $k \geq 1$, and for some $1 \leq i \leq k$, either $s_i\ \mathcal{G}(\theta')\ t$, and by induction hypothesis we conclude that $s_i\ \mathcal{G}(\theta)\ t$, or $s_i = t$; in both cases we can apply case 2 of definition 4.3 to conclude that $s\ \mathcal{G}(\theta)\ t$. □

Since the function $\mathcal{G}$ is weakly monotone (or order-preserving), theorem 2.22 tells us that $\mathcal{G}$ has a least fixed point which we take to be the *path order*.

Definition 4.6. The *path order* associated with a status Λ and the quasi-order $\succeq$ is denoted by $\gg_{po}^{\Lambda,\succeq}$ and is defined as the least fixed point of the function $\mathcal{G}$.

As we did in the quasi-order case, we omit, whenever possible, both the status Λ and the quasi-order $\succeq$, and write $\gg_{po}$ instead of $\gg_{po}^{\Lambda,\succeq}$.

As a consequence of the definition of $\gg_{po}$ we have that:

Proposition 4.7. The relation $\gg_{po}$ is a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ satisfying

$$s = f(s_1, \ldots, s_k) \gg_{po} t$$

with $f \in \mathcal{F} \cup \mathcal{X}$ having arity $k \geq 0$, if and only if one of the following conditions holds:

1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$ having arity $m \geq 0$, and $s \gg_{po} t_j$, for all $1 \leq j \leq m$, and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s\ \Lambda(\gg_{po})\ t$; or
2. $\exists 1 \leq i \leq k :\ s_i \gg_{po} t$ or $s_i = t$.

Example 4.8. Let $\mathcal{F} = \{f/3, s/1\}$ and let $\mathcal{X} = \{x, y, z\}$. Suppose $\succeq$ is defined by $s \succeq t \iff \texttt{root}(s) \trianglerighteq \texttt{root}(t)$, where $\trianglerighteq$ is defined by $f \triangleright s$ and $p \sim q \iff p = q$. Let the status Λ be defined as:

$$s\ \Lambda(\theta)\ t \iff \begin{cases} s = f(s_1, s_2, s_3), t = f(t_1, t_2, t_3) \text{ and} \\ \{\!\{s_1, s_2\}\!\}\ \theta_{mul}\ \{\!\{t_1, t_2\}\!\} \text{ or } (\{\!\{s_1, s_2\}\!\} = \{\!\{t_1, t_2\}\!\} \text{ and } s_3\ \theta\ t_3) \end{cases}$$

It is not difficult to see that for any partial order θ, $\Lambda(\theta)$ is also a partial order. Furthermore since multiset and lexicographic extension of partial orders are monotone, we can also easily derive that Λ is monotone, thus a proper status according to definition 4.2.

Using the $\gg_{po}$ associated with these parameters we can conclude that

$$f(s(x), s(y), z) \gg_{po} f(y, x, s(z))$$

From the subterm property, we can conclude that $f(s(x), s(y), z) \gg_{po} x, y, z$. Also since $f \triangleright s$, we have $f(s(x), s(y), z) \gg_{po} s(z)$. Finally $f(s(x), s(y), z) \sim f(y, x, s(z))$ (they have the same root) and (with $\gg_{po,mul}$ denoting the multiset extension of $\gg_{po}$) $\{\!\{s(x), s(y)\}\!\} \gg_{po,mul} \{\!\{y, x\}\!\}$, giving the result.

4.3 Properties of the partial order scheme

We discuss what properties are enjoyed by the partial order $\gg_{po}$. Not surprisingly this section follows closely section 3.3.

As remarked in section 3.3, since $\gg_{po}$ is parameterised by a quasi-order $\succeq$ and a status Λ, the properties enjoyed by $\gg_{po}$ depend directly on which properties are enjoyed by $\succeq$ and maintained by Λ. The subterm property however is once again universal, i. e., we have the following lemma (akin to lemma 3.12).

Lemma 4.9. The partial order $\gg_{po}$ satisfies $C[s] \gg_{po} s$, for any term s and any non-trivial context C.

Proof Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be an arbitrary term. We proceed by induction on the context. If $C[s] = f(\ldots, s, \ldots)$ for some $f \in \mathcal{F}$ then case 2 in proposition 4.7 allows us to conclude that $f(\ldots, s, \ldots) \gg_{po} s$. Suppose $D[\,]$ is a context for which the property is valid and let $C[s] = f(\ldots, D[s], \ldots)$, for some $f \in \mathcal{F}$. Again by case 2 in proposition 4.7 we conclude that $f(\ldots, D[s], \ldots) \gg_{po} D[s]$ and by induction hypothesis $D[s] \gg_{po} s$; from transitivity follows $f(\ldots, D[s], \ldots) \gg_{po} s$. $\square$

Just as with $\geq_{po}$, the other relevant properties (closedness under substitutions and/or contexts, well-foundedness and totality) are in general not enjoyed by $\gg_{po}$. We will also present sufficient conditions to ensure these properties, similar to the ones presented for $\geq_{po}$. Before doing so we give some examples.

Example 4.10. Suppose $\mathcal{F} = \{f/1\}$, $\mathcal{X} = \{x, y, z\}$. Let $\succeq$ be given by:

- $x \succ y \succ z$, and $s \succ t$ iff it can be derived by the previous inequalities using transitivity, and
- $s \sim t \iff s = t$.

Define the status Λ by $\Lambda(\theta) = \emptyset$, for all θ.

The path order $\gg_{po}$ associated with these parameters is not closed under substitutions nor contexts: we have $x \gg_{po} y$ while $\neg(f(x) \gg_{po} f(y))$ and $\sigma(x) = f(z) \not\gg_{po} f(f(z)) = \sigma(y)$, for example.

Example 4.11. Suppose $\mathcal{F} = \{f/2, a/0, b/0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be the size of a term. Note that $\succeq$ is total in $\mathcal{T}(\mathcal{F})$. Define the status Λ by

$$s\ \Lambda(\theta)\ t \iff \vec{s}\ \theta_{mul}\ \vec{t}$$

It is not difficult to check that Λ is a well-defined status (cf. definition 4.2). Using the path order $\gg_{po}$ associated with these parameters, the terms $f(a, b)$ and $f(b, a)$ are incomparable. That is so because $\{\!\{a, b\}\!\} = \{\!\{b, a\}\!\}$.

Example 4.12. Let $\mathcal{F} = \{g/1, f/1, c/0\}$, $\mathcal{X} = \emptyset$. Let $\succeq$ be a precedence based quasi-order for the precedence $\trianglerighteq$ satisfying $g \triangleright f, c$. Note that $\succeq$ is well-founded.

Let the status Λ be defined as follows. For any partial order θ in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, $\Lambda(\theta)$ is the partial order $\sqsupset$ given by:

$$s \sqsupset t \iff s = g(f^i(c)), t = g(f^j(c)) \text{ and } 0 \leq i < j$$

Note that $\sqsupset$ is not well-founded since we have $g(c) \sqsupset g(f(c)) \sqsupset g(f(f(c))) \sqsupset \cdots$. It is not difficult to see that taking $\gg_{po}$ associated with these parameters, we also have $g(c) \gg_{po} g(f(c)) \gg_{po} g(f(f(c))) \gg_{po} \cdots$, so $\gg_{po}$ is not well-founded.

4.3.1 Closedness under substitutions

We present a sufficient condition for obtaining closedness under substitutions for the partial order $\gg_{po}$.

Definition 4.13. We say that a status Λ is *stable* if it satisfies the following condition: if θ is closed under substitutions in $S \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ then $\Lambda(\theta)$ is closed under substitutions in $\overline{S} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, where

$$\overline{S} = \{f(s_1, \ldots, s_k)|\ f/k \in \mathcal{F} \cup \mathcal{X}, k \geq 0, \text{ and } \forall 1 \leq i \leq k:\ s_i \in S\}$$

Theorem 4.14. If $\succeq$ is strictly closed under substitutions and the status Λ is stable then $\gg_{po}$ is closed under substitutions.

Proof We have to prove that if $s \gg_{po} t$ and σ is any arbitrary substitution we have $s\sigma \gg_{po} t\sigma$. We prove this by induction on $|s| + |t|$. Suppose we have two minimal terms s, t with $s \gg_{po} t$ and for which the property is not yet verified, i. e., if s', t' are terms such that $|s'| + |t'| < |s| + |t|$ then s', t' satisfy the property. We now proceed by case analysis.

- If $s \gg_{po} t$ holds by cases 1a, 1b of proposition 4.7, then $s = f(s_1, \ldots, s_k)$ for some $k \geq 0$, $f \in \mathcal{F} \cup \mathcal{X}$, and $t = g(t_1, \ldots, t_m)$ for some $m \geq 0$, $g \in \mathcal{F} \cup \mathcal{X}$. Furthermore $s \gg_{po} t_j$ for all $1 \leq j \leq m$, so by induction hypothesis we get $s\sigma \gg_{po} t_j\sigma$ for all $1 \leq j \leq m$.

  If case 1a is applicable we have $s \succ t$ and therefore $s\sigma \succ t\sigma$; by the same case we conclude that $s\sigma \gg_{po} t\sigma$.

  If case 1b is applicable, we have $s \sim t$ and so also $s\sigma \sim t\sigma$. We also have $s\ \Lambda(\gg_{po})\ t$. By induction hypothesis $\gg_{po}$ is closed under substitutions in $S = \{s_1, \ldots, s_k, t_1, \ldots, t_m\}$, and since Λ is stable we get that $\Lambda(\gg_{po})$ is closed under substitutions in $\overline{S}$. Consequently (and because $s, t \in \overline{S}$), we have $s\sigma\ \Lambda(\gg_{po})\ t\sigma$. By the same case we conclude that $s\sigma \gg_{po} t\sigma$.
- If $s \gg_{po} t$ holds by case 2, of proposition 4.7, then $s = f(s_1, \ldots, s_k)$ for some $k \geq 1$, $f \in \mathcal{F}$, and $s_i \gg_{po} t$ or $s_i = t$, for some $1 \leq i \leq k$. In the first case by induction hypothesis we get that $s_i\sigma \gg_{po} t\sigma$, and in the second case obviously $s_i\sigma = t\sigma$. Using proposition 4.7 we conclude that $s_i\sigma \gg_{po} t\sigma$.

□

4.3.2 Closedness under contexts

As we did in section 3.3.2, we provide here a sufficient condition for closedness under contexts of $\gg_{po}$.

Definition 4.15. We say that a status Λ is *context stable* if it satisfies the following condition: if $s\ \theta\ t$ then $f(\ldots, s, \ldots)\ \Lambda(\theta)\ f(\ldots, t, \ldots)$, for any $f \in \mathcal{F}$, $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.

Theorem 4.16. If $\succeq$ is precedence based and the status Λ is context stable then $\gg_{po}$ is closed under contexts.

Proof We have to prove that if $s \gg_{po} t$ then $f(s_1, \ldots, s, \ldots, s_k) \gg_{po} f(s_1, \ldots, t, \ldots, s_k)$, for any arbitrary function symbol $f \in \mathcal{F}$ with arity $k \geq 1$, and arbitrary terms s_j, $1 \leq j \leq k, j \neq i$, with s and t occurring at position i. We proceed by induction on the context. For the the trivial context the result obviously holds. Suppose that $s \gg_{po} t$ and let $f \in \mathcal{F}$ have arity $k \geq 1$. We have $f(\ldots, s, \ldots)\ \Lambda(\geq_{po})\ f(\ldots, t, \ldots)$, (this is due to the hypothesis that Λ is context stable). Since $\succeq$ is precedence based, we have that $f(\ldots, s, \ldots) \sim f(\ldots, t, \ldots)$, so we only need to see that $f(s_1, \ldots, s, \ldots, s_k) \gg_{po} s_i$, for $i \neq j$ and $f(s_1, \ldots, s, \ldots, s_k) \gg_{po} t$. The first part follows directly from lemma 4.9, as for the second, the same lemma and the hypothesis allows us to write:

$$f(s_1, \ldots, s, \ldots, s_k) \gg_{po} s \gg_{po} t$$

and the result follows from transitivity. □

4.3.3 Totality

Again we restrict the study of totality to the set of ground terms. It is not difficult to modify the notions given for achieving totality in the set of open terms. We do not do it because it seems not natural and of no practical use to require totality on open terms.

Definition 4.17. We say that a status Λ is *totality stable* if it satisfies the following condition: for any $S \in \mathcal{T}(\mathcal{F})$, if θ is total in S then $\Lambda(\theta)$, is total in $\widehat{S}$, where

$$\widehat{S} = \{f(s_1, \ldots, s_k)|\ f/k \in \mathcal{F}, k \geq 0, \text{ and } \forall 1 \leq i \leq k:\ s_i \in S\}$$

Theorem 4.18. If $\succeq$ is total on $\mathcal{T}(\mathcal{F})$ and the status Λ is totality stable then $\gg_{po}$ is total on $\mathcal{T}(\mathcal{F})$.

Proof We need to see that for any $s, t \in \mathcal{T}(\mathcal{F})$, if $s \neq t$ then either $s \gg_{po} t$ or $t \gg_{po} s$. We proceed by induction on $|s| + |t|$.

Suppose $s \neq t$. If s, t are constants then either $s \succ t$ or $t \succ s$ or $s \sim t$. In the first two cases we obtain respectively $s \gg_{po} t$ and $t \gg_{po} s$. In the last case since $\gg_{po}$ is total in $\emptyset$ and Λ is totality stable, we have that $\Lambda(\geq_{po})$ is total in $\mathcal{F}_0$, the set of constants, and therefore either $s\ \Lambda(\geq_{po})\ t$, or $t\ \Lambda(\geq_{po})\ s$. In both cases we can state that either $s \gg_{po} t$ or $t \gg_{po} s$. Suppose now that $t = g(t_1, \ldots, t_m)$. Either there is an j such that $t_j \gg_{po} s$ or $t_j = s$, and then $t \gg_{po} s$ or, by induction hypothesis, for all $1 \leq j \leq m$, $s \gg_{po} t_j$. Also if $s = f(s_1, \ldots, s_k)$, then either there is an i such that $s_i \gg_{po} t$ or $s_i = t$, and then $s \gg_{po} t$ or, by induction hypothesis, for all $1 \leq i \leq k$, $t \gg_{po} s_i$. Suppose then that both $s \gg_{po} t_j$, for all $1 \leq j \leq m$, and $t \gg_{po} s_i$, for all $1 \leq i \leq k$. If $s \succ t$ we can conclude that $s \gg_{po} t$ (by case 1a in proposition 4.7), if $t \succ s$, we conclude the reverse by the same case. If $s \sim t$, since by induction hypothesis $\gg_{po}$ is total in $S = \{s_1, \ldots, s_k, t_1, \ldots, t_m\}$ and Λ is totality stable, then $\Lambda(\gg_{po})$ is total in $\widehat{S}$ and since $s, t \in \widehat{S}$, we have either $s\ \Lambda(\gg_{po})\ t$, or $t\ \Lambda(\gg_{po})\ s$. In both cases we can state that either $s \gg_{po} t$ or $t \gg_{po} s$. □

4.3.4 Well-foundedness

Just like its quasi-order counterpart, $\gg_{po}$ will not be well-founded in general. Again Kruskal's theorem is not enough to help prove well-foundedness of the order obtained via the partial order scheme since the resulting order may not be a simplification ordering. As we did in section 3.3.4, we use results from [10, 9] to provide a sufficient condition for ensuring well-foundedness of $\gg_{po}$.

The following definition and theorem are taken from [10, 9].

Definition 4.19. A status Λ is said to be *well-foundedness stable*[4] if it satisfies the following condition: if θ is a partial order in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ well-founded in $A \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ then $\Lambda(\theta)$ is well-founded in $\overline{A} = \{f(t_1, \ldots, t_k) : \ f/k \in \mathcal{F} \cup \mathcal{X}, k \geq 0, \text{ and } t_i \in A \text{ for all } 1 \leq i \leq k\}$.

Theorem 4.20. Let $>$ be a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and let Υ be a well-foundedness stable status. Suppose $>$ has the subterm property and satisfies the following condition:

- $\forall f/m, g/n \in \mathcal{F} \cup \mathcal{X},\ s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$:
  if $s = f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n) = t$ then either
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1, \ldots, t_n)$, or
  - $s\ \Upsilon(\geq)\ t$

Then $>$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Now it is easy to give a sufficient condition for well-foundedness of $\gg_{po}$.

Theorem 4.21. If $\succeq$ is a well-founded quasi-order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and Λ is a well-foundedness stable status then $\gg_{po}$ is well-founded.

Proof (Sketch). We define $\Upsilon : \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})} \to \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$ as follows:

$$s\ \Upsilon(\theta)\ t \iff \begin{cases} s \succ t & \text{or} \\ s \sim t & \text{and} \quad s\ \Lambda(\theta)\ t \end{cases}$$

The rest of the proof is similar to the proof of theorem 3.29 replacing the results on quasi-orders by the corresponding results for partial orders. □

4.4 Lexicographic path order and Knuth-Bendix order

Here we show how the *lexicographic path order*, abbreviatedly *lpo*, and the *Knuth-Bendix order*, abbreviatedly *kbo*, can be obtained as instances of the scheme presented.

[4] *Term status* in the terminology of [10, 9].

4.4.1 *lpo*

The *lexicographic path order* was originally proposed by Kamin and Lévy [13] and is very similar to *recursive path order*: instead of comparing multisets of principal subterms, a lexicographic comparison is performed.

Definition 4.22. (lpo) Let $\trianglerighteq$ be a quasi-order in the set $\mathcal{F}$. The *lexicographic path order* denoted by $>_{lpo}$ is defined, in the set $\mathcal{T}(\mathcal{F}, \mathcal{X})$, as follows: $s = f(s_1, \ldots, s_k) >_{lpo} t$ if either:

1. $t = g(t_1, \ldots, t_m)$ and $s >_{lpo} t_j$, for all $1 \leq j \leq m$, and either:
   (a) $f \triangleright g$; or
   (b) $f \sim g$ and $\langle s_1, \ldots, s_k \rangle >_{lpo,lex} \langle t_1, \ldots, t_m \rangle$; or
2. $s_i >_{lpo} t$, or $s_i = t$, for some $i = 1, \ldots, k$.

To see how $>_{lpo}$ can be obtained from our construction we define the parameters $\succeq$ and Λ. For $\succeq$ we take the precedence based order for the precedence $\trianglerighteq$, i. e.,

$$f(s_1, \ldots, s_k) \succeq g(t_1, \ldots, t_m) \iff f \trianglerighteq g$$

As for Λ we define it as follows:

$$s = f(s_1, \ldots, s_k) \ \Lambda(\theta) \ g(t_1, \ldots, t_m) \iff \langle s_1, \ldots, s_k \rangle \ \theta_{lex} \ \langle t_1, \ldots, t_m \rangle$$

For any partial order its lexicographic extension is also a partial order; furthermore it is well-known that lexicographic lifting is monotone with respect to the order lifted, i. e., if $\theta \supset \theta'$ then also $\theta_{lex} \supset \theta'_{lex}$. So Λ is a well-defined status.

The partial order $\gg_{po}$ obtained using these parameters can be writen as (see proposition 4.7):

Proposition 4.23. The partial order $\gg_{po}$ satisfies: $s = f(s_1, \ldots, s_k) \gg_{po} t$, with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, if and only if one of the following conditions holds:

1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s \gg_{po} t_j$, and either
   (a) $f \triangleright g$, or
   (b) $f \sim g$ and $\langle s_1, \ldots, s_k \rangle \geq_{po,lex} \langle t_1, \ldots, t_m \rangle$; or
2. $\exists 1 \leq i \leq k : s_i \gg_{po} t$ or $s_i = t$.

It is trivial to see that the order so obtained coincides with the one of definition 4.22.

Since $>_{lpo}$ is a particular instance of our scheme, we can apply all the derived results to it. By lemma 4.9 we conclude that $>_{lpo}$ enjoys the subterm property. The other usual properties of $>_{lpo}$ can also be derived from the results presented. It is not difficult to see that the status Λ is stable and context stable. Furthermore $\succeq$ is precedence based and strictly closed under substitutions and contexts so we can apply theorem 4.14 and 4.16 to conclude that $>_{lpo}$ is both closed under substitutions and contexts.

Another well-known property of $>_{lpo}$ is totality on ground terms whenever $\trianglerighteq$ is total on $\mathcal{F}$. It is not difficult to see that Λ is totality stable (cf. definition 4.17) so totality of $>_{lpo}$ can be concluded by theorem 4.18, provided $\trianglerighteq$ is total.

As for well-foundedness it is well-known that a simple lexicographic extension on sequences of any size is not well-founded even if the extended order is well-founded. If we only consider sequences of size bounded by some naturtal number, then the lexicographic extension is well-founded iff the extended order is well-founded. In the particular case of $>_{lpo}$ there are several ways of getting around this problem; one of them is to restrict equivalent function symbols to be equal, thus only sequences of the same size are compared lexicographically. Another slightly more general possibility is to require that there is a bound on the arities of equivalent function symbols. Yet another possibility is to use a definition of lexicographic extension in which lengths of sequences are first inspected and the lexicographic comparison is effectively used only on sequences of the same size. Whatever solution one choses, it is routine work to verify that well-foundedness of $>_{lpo}$ in this cases can be obtained via theorem 4.21.

4.4.2 *kbo*

The Knuth-Bendix order was originally proposed by Knuth and Bendix [16]. It is a path order of a different kind since not only is the syntactical structure of terms used for the comparison but also a “semantical” component which associates weights to terms. In its simplest form (see [26] for extensions) a weight is a natural number which is associated to each function or variable symbol and which is then extended uniquely to terms. We will consider a definition of weight and of the *kbo* similar to the one presented in [7], however we extend the precedence to be a quasi-order in $\mathcal{F}$ and allow for more than one maximum element in $\mathcal{F}$ with arity one and weight zero. Other more general possibilities for weight functions do exist. In [27] a general weight function is given using an interpretation of terms in a weakly monotone algebra.

In the following we assume lexicographic extension as defined in 2.11.

Definition 4.24.

A *weight function* $\phi : \mathcal{F} \cup \mathcal{X} \to \mathbb{N}$ is a function satisfying:

$$\phi(f) \text{ is } \begin{cases} = \phi_0 & \text{if } f \in \mathcal{X} \\ \geq \phi_0 & \text{if } \texttt{arity}(f) = 0 \\ > 0 & \text{if } \texttt{arity}(f) = 1 \text{ and } \exists g \in \mathcal{F} :\ f \not\trianglerighteq g \end{cases}$$

where $\trianglerighteq$ is a precedence in $\mathcal{F}$ and $\phi_0 \in \mathbb{N}$ is a fixed natural greater than zero. ϕ is extended to terms as follows: $\phi(f(s_1, \ldots, s_m)) = \phi(f) + \sum_{i=1}^{m} \phi(s_i)$.

Note that the last condition on ϕ in definition 4.24 means that if $f \in \mathcal{F}$, has arity 1 and weight 0, then it must satisfy $f \trianglerighteq g$ for all function symbols $g \in \mathcal{F}$. In other words, f has to be a maximum, but not necessarily unique.

We introduce the Knuth-Bendix order.

Definition 4.25. **(kbo)** We say that $s >_{kbo} t$ iff $\forall x \in \mathcal{X} : \#_x(s) \geq \#_x(t)$ and either

1. $\phi(s) > \phi(t)$, or
2. $\phi(s) = \phi(t)$, and either
   (a) $t \in \mathcal{X}$ and $\exists k > 0 : \ s = f_0^k(t)$, where f_0 is an element of $\mathcal{F}$ having weight 0, arity one and being a maximum in the precedence,
   (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$ and
      - $f \triangleright g$, or
      - $f \sim g$ and $\langle s_1, \ldots, s_m \rangle >_{kbo,lex} \langle t_1, \ldots, t_n \rangle$

where $\#_x(t)$ denotes the number of occurrences of variable x in term t.

Suppose we have a fixed precedence $\trianglerighteq$ in $\mathcal{F}$ and let ϕ be a weight function with respect to some fixed positive natural ϕ_0; let also f_0 denote any (possibly non-existent) function symbol in $\mathcal{F}$ with arity one and being a maximum in the precedence.

We define the quasi-order $\succeq$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: its strict part is empty and its equivalence part is the whole set of terms; in other words $\succ = \emptyset$ and $\sim = \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$.

We define the status Λ as follows. For any partial order $\theta \in \mathcal{PO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$:

$$s \ \Lambda(\theta) \ t \iff \begin{cases} \forall x \in \mathcal{X} : \ \#_x(s) \geq \#_x(t) \text{ and either} \\ i) \ \phi(s) > \phi(t), \text{ or} \\ ii) \ \phi(s) = \phi(t), \text{ and either} \\ \quad a) \ t \in \mathcal{X} \text{ and } \exists k > 0 : \ s = f_0^k(t), \text{ or} \\ \quad b) \ s = f(\vec{s}), t = g(\vec{t}) \text{ and } \begin{cases} f \triangleright g & \text{or} \\ f \simeq g & \text{and} \quad \langle \vec{s} \rangle \ \theta_{lex} \ \langle \vec{t} \rangle \end{cases} \end{cases}$$

It is not difficult to see that if θ is a partial order, so is $\Lambda(\theta)$ and that Λ is weakly monotone, thus a well-defined status (cf. definition 4.2).

According to proposition 4.7 and unfolding the parameters $\succeq$ and Λ we can write the associated path order $\gg_{po}$ as follows:

Proposition 4.26. The partial order $\gg_{po}$ satisfies: $s = f(s_1, \ldots, s_k) \gg_{po} t$, with $f \in \mathcal{F} \cup \mathcal{X}$ having arity $k \geq 0$, if and only if one of the following conditions holds:

1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$ having arity $m \geq 0$, and $s \gg_{po} t_j$, for all $1 \leq j \leq m$, and either
   (a) $(\forall x \in \mathcal{X} : \ \#_x(s) \geq \#_x(t))$ and either

i. $\phi(s) > \phi(t)$, or

ii. $\phi(s) = \phi(t)$ and either

- $t \in \mathcal{X}$ and $\exists k > 0 :\ s = f_0^k(t)$, or
- $s = f(\vec{s}), t = g(\vec{t})$ and $\begin{cases} f \rhd g & \text{or} \\ f \simeq g & \text{and} \quad \langle \vec{s} \rangle \ \geq_{po,lex} \ \langle \vec{t} \rangle \end{cases}$

2. $\exists 1 \leq i \leq k :\ s_i \gg_{po} t$ or $s_i = t$.

Apparently the order $\gg_{po}$ and $>_{kbo}$ are not the same object, but they actually are two different representations of the same thing. Before we prove so, we need some auxiliary results.

Lemma 4.27. Suppose $s = C[t]$, for some non-trivial context C. If $\phi(s) = \phi(t)$ then $s = f_{i_1}(\ldots f_{i_k}(t)\ldots)$, with each f_{i_j} $(1 \leq j \leq k,\ k \geq 1)$ having arity one, weight zero and satisfying $f_{i_j} \unrhd g$, for all $g \in \mathcal{F}$ (i. e., being a maximum in the precedence).

Proof If C contains any symbol with weight ≥ 0 then we would have $\phi(s) > \phi(t)$. So all symbols occurring in C have weight zero and this excludes constants and variables. If C would contain a function symbol with arity ≥ 2 it would forcibly contain a variable or a constant, so that cannot happen either. Therefore C contains only function symbols of arity one and with weight zero, and given the restrictions on the weight function, they have to be maximums in the precedence, i. e., $s = f_{i_1}(\ldots f_{i_k}(t)\ldots)$, with each f_{i_j} $(1 \leq j \leq k,\ k \geq 1)$ having arity one, weight zero and satisfying $f_{i_j} \unrhd g$, for all $g \in \mathcal{F}$, as we wanted to prove. □

Lemma 4.28. $C[t] >_{kbo} t$, for any non-trivial context C.

Proof The proof is by induction on the size of $s = C[t]$. Clearly we have $\phi(C[t]) \geq \phi(t)$ for any context C. If the inequality is strict, since $\#_x(C[t]) \geq \#_x(t)$, for all $x \in \mathcal{X}$, we have $C[t] >_{kbo} t$. Suppose then that $\phi(C[t]) = \phi(t)$. By lemma 4.27 we can write $s = C[t] = f_{i_1}(\ldots f_{i_k}(t)\ldots)$, with each f_{i_j} $(1 \leq j \leq k,\ k \geq 1)$ having arity one, weight zero and satisfying $f_{i_j} \unrhd g$, for all $g \in \mathcal{F}$. If $k > 1$ then by induction hypothesis we can state that $f_{i_k}(t) >_{kbo} t$ and since

$$\forall x \in \mathcal{X} :\ \#_x(f_{i_1}(\ldots f_{i_k}(t)\ldots)) \geq \#_x(f_{i_2}(\ldots f_{i_k}(t)\ldots)) \geq \ldots \geq \#_x(f_{i_k}(t)) \geq \#_x(t)$$

and $f_{i_1} \simeq \ldots \simeq f_{i_k}$, and all the terms $f_{i_1}(\ldots f_{i_k}(t)\ldots)$, ..., $f_{i_k}(t)$ have the same weight, we can conclude that $f_{i_1}(\ldots f_{i_k}(t)\ldots) >_{kbo} t$.

If $k = 1$ then we consider the structure of t. Note that $\#_x(f(t)) \geq \#_x(t)$, for all $x \in \mathcal{X}$. If $t \in \mathcal{X}$ then we have $s = f_0(t)$ and by definition of $>_{kbo}$ we conclude that $s >_{kbo} t$. If $t = g(t_1, \ldots, t_m)$ then either $f \rhd g$ and we are done, or $f_0 \simeq g$ and then by induction hypothesis $g(t_1, \ldots, t_k) >_{kbo} t_i$, for each $1 \leq i \leq m$ and consequently $\langle g(t_1, \ldots, t_k) \rangle >_{kbo,lex} \langle t_1, \ldots, t_m \rangle$ and so $s = f_0(t) >_{kbo} t$. □

Lemma 4.29. For any terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ $s \gg_{po} t$ iff $s >_{kbo} t$.

Proof We first see that for any $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ if $s \gg_{po} t$ then $s >_{kbo} t$, by induction on $|s| + |t|$. Suppose then s, t are minimal terms such that $s \gg_{po} t$; our induction hypothesis states that if u, v are terms such that $|u| < |t|$ then $u \gg_{po} t \Rightarrow u >_{kbo} t$. If $s \gg_{po} t$ by case 1a it is obvious that also $s >_{kbo} t$. If $s \gg_{po} t$ by case 2 then either $s_i \gg_{po} t$ or $s_i = t$, with s_i being a principal subterm of s. In the first case we conclude that $s_i >_{kbo} t$ by induction hypothesis and by lemma 4.28 we have $s >_{kbo} s_i$ and thus $s >_{kbo} t$. In the last case t is a principal subterm of s and by the same lemma we conclude that $s >_{kbo} t$.

For the opposite inclusion we proceed similarly, i. e., by proving that $s >_{kbo} t$ implies $s \gg_{po} t$, by induction on $|s| + |t|$. Note that if $s >_{kbo} t$ then we can conclude that $s \gg_{po} t$ by case 1a provided we can show that if $t = g(t_1, \ldots, t_m)$ then $s \gg_{po} t_j$ for all $1 \leq j \leq m$. But if $s >_{kbo} t$ and t has that form, by lemma 4.28, we also have that $t >_{kbo} t_j$, for all $1 \leq j \leq m$. Using transitivity we conclude that $s >_{kbo} t_j$, for all $1 \leq j \leq m$ and by induction hypothesis we have $s \gg_{po} t_j$, for all $1 \leq j \leq m$, as we wanted. $\square$

It is not difficult though cumbersome to see that the status Λ is stable, context stable and well-foundedness stable if $\trianglerighteq$ is well-founded on $\mathcal{F}$ and the lexicographic extension satisfies appropriate requirements for well-foundedness, therefore we can derive that $>_{kbo}$ is closed under substitutions and contexts and well-founded (provided $\trianglerighteq$ is well-founded), using the results presented in section 4.3.1, 4.3.2, and 4.3.4.

5 Conclusions

We provided a characterization of recursively defined path partial and quasi-orders on terms. Though many such orders are known, proofs of their well-definedness are, as far as we know, quite often not to be found in the literature. With the characterizations given, we can provide such a proof since any relation on terms fitting our schemes will be a partial or quasi-order. So by simply checking the relation falls within our schemes we ensure its well-definedness and eventually other properties characterizing the ordering; we did so for the *semantic path order*, the *recursive path order*, the *lexicographic path order*, and the *Knuth-Bendix order*. We chose these orderings for historical reasons and also because though many variations have been proposed since the advents of these orderings, they still remain as the best known representatives of path orders.

Our approach has several major advantages: proving properties like well-definedness, transitivity, (ir)reflexivity etc. has to be done only once, namely for the schemes presented. Then for any particular relation we wish to establish as a partial or quasi-order, we only need to see that the relation fits in our scheme, and that is usually much simpler than providing an independent proof. Another important aspect of our approach is that it abstracts from the form of the actual existing orders and enables us to concentrate on the properties and the ingredients needed for the definition of partial or quasi-orders. As a consequence a better understanding of the mechanisms at play is obtained. And last but not least, it is possible to use the schemes for the definition of new orderings on terms; this may be especially relevant in practical situations where orderings tailored for a particular problem may be needed.

In the future it would be interesting to investigate whether a similar scheme can be obtained for orders in an equational setting, i. e., when we want the orders to be compatible with some equational theory. The definition of path orders compatible with equational theories has proved to be a hard problem, especially when other properties like totality are required.

Acknowledgements

This work was partially done while the author was visiting the Computer Science Department of Utrecht University. I would like to thank the Department and in particular Hans Zantema for making the visit possible.

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