A Note on Edge Contraction

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Abstract. Contracting an edge is the operation that introduces a new vertex that is adjacent to all vertices the endpoints of the contracted edge are adjacent to, and then deletes the endpoints of this edge and all their incident edges. In this note, we give a formal approach to the notion of edge contraction and show some basic properties of it.

1 Introduction

In this note, we give a formal approach to the notion of edge contraction, and we derive some basic properties of the notion. In recent work, we used edge contractions to improve upon lower bounds for treewidth (see [2, 3]; and [1] for more information on treewidth in general). While in [2, 3], we use the notions more informally, here we establish the more precise definitions, and we give formal proofs of (mostly intuitive) results that we use without a proof in [2, 3].

Graph contraction is used in several important graph theoretic invetigations. We just mention here the much studied notion of *graph minor* – a graph that can be obtained from a graph by a series of vertex deletions, edge deletions and edge contractions. Well known is the fundamental work of Robertson and Seymour on graph minors, see e.g. [4].

Even though most of the statements proven in this note are intuitive, a formal proof can sometimes be more technical than expected. We start by formally defining single edge contractions and showing the commutativity of such single contractions. We will work towards the notions of contraction-set and contractions of a graph.

1.1 Preliminaries

Throughout the paper $G = (V, E)$ denotes a simple undirected graph. Most of our terminology is standard graph theory/algorithm terminology. The open neighbourhood $N_G(v) := \{w \in V \mid \{v, w\} \in E\}$ or simply $N(v)$ of a vertex $v \in V$ is the set of vertices adjacent to v in G. As usual, the degree in G of vertex v is $d_G(v)$ or simply $d(v)$, and we have $d(v) = |N(v)|$. $N(S)$ for $S \subseteq V$ denotes the open neighbourhood of S, i.e. $N(S) = \bigcup_{s \in S} N(s) \setminus S$.

2 Edge Contraction

It is easy to intuitively understand the meaning of contracting an edge. In Figure 1, we can see an example when we contract an edge e that belongs to a cycle of length three. This example is essential, since it shows that there are two ways of looking at edge contractions. In the first case, the result might be a multigraph, since the endpoints of the edge to be contracted might have common neighbours as edge e in Figure 1. In the second case, the result is always a simple graph, because parallel edges that might occur will always be replaced by a single edge. It is evident that in that case, contracting an edge can decrease the number of edges by more than one. These two ways of contracting an edge e in G are sometimes denoted as G/e and $G/\!\!/e$, (e.g. in [4]).

In this note, we only consider simple graphs, and we use the notation G/e for an edge contraction which results in a simple graph, because we replace parallel edges by a single edge. Therefore, contracting edge $e = \{v_i, v_j\}$ in the graph G, denoted as G/e, is the operation that introduces a new vertex a_e and new edges such that a_e is adjacent to all the vertices in $N(e)$ and delete vertices v_i and v_j and all edges incident to v_i or v_j.

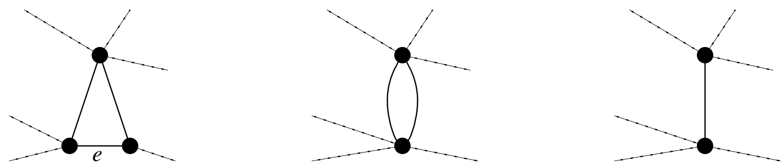

Fig. 1. Original graph; resulting multigraph after contracting edge e; resulting simple graph after contracting edge e

Definition 1. *Let be given a graph $G = (V, E)$. Let be $e \in E$, or let be $e \subseteq V$ with $|e| = 1$. Furthermore, let be $a_e \notin V$. Contracting e in G results in the graph G/e, defined as follows:*

$$\begin{aligned} G/e &:= (V/e, E/e), \text{ where} \\ V/e &= \{a_e\} \cup V \setminus e \\ E/e &= (\bigcup_{f \in E \setminus \{e\}} (f/e)) \setminus \{e\} \\ f/e &= \begin{cases} \{a_e\} \cup f \setminus e & \textit{if } f \cap e \neq \emptyset \\ f & \textit{otherwise} \end{cases} \end{aligned}$$

To be formally consistent, we included in the previous definition the case that e is 'an edge' consisting of a single vertex. This is important because of previous edge contractions if e belongs to a set of contracted edges forming a cycle, e can consist of only one vertex, (see Lemma 2).

Lemma 1. *Let be given a graph $G = (V, E)$ and two distinct edges $e, f \in E$. Let be $G_1 = (V_1, E_1) = (G/e)/(f/e)$ and $G_2 = (V_2, E_2) = (G/f)/(e/f)$. Then G_1 and G_2 are isomorphic, i.e. $G_1 \cong G_2$.*

Proof. In case e and f are not adjacent, the lemma is easy to see, since the contractions of e and f do not influence each other. Thus, we assume $e = \{u, v\}$ and $f = \{v, w\}$.

Claim. $V_1 = \{a_f\} \cup V \setminus \{u, v, w\}$.

Proof. This is easy to see:

$$\begin{aligned} (V/e)/(f/e) &= (\{a_e\} \cup V \setminus \{u, v\})/(\{a_e, w\}) \\ &= \{a_f\} \cup (\{a_e\} \cup V \setminus \{u, w\}) \setminus \{a_e, w\} \\ &= \{a_f\} \cup V \setminus \{u, v, w\} \end{aligned}$$

$\diamond$

In the same way, we obtain: $V_2 = \{a_e\} \cup V \setminus \{u, v, w\}$, and therefore, we see that V_1 and V_2 only differ in the name of one element, namely the only new vertex.

We define a mapping $i : V_1 \leftrightarrow V_2$ in the following way:

$$i(v) = \begin{cases} v & \text{if } v \in V \\ a_e & \text{otherwise, i.e. if } v = a_f \end{cases}$$

Now, we will prove that the mapping i is an isomorphism between G_1 and G_2. From above, we already know that $|V_1| = |V_2|$. What remains to be shown is that two vertices x and y are joined by an edge in G_1 if and only if the corresponding vertices are joined by an edge in G_2.

The next two claims can be deduced from the Definition 1 and the definition of G_1 and G_2. First, we consider the case that the considered vertices x and y are pairwise different from the new vertex. Note that $i(x) = x$ and $i(y) = y$.

Claim. $\{x,y\} \in E_1 \wedge a_f \notin \{x,y\} \Longleftrightarrow \{x,y\} \in E_2 \wedge a_e \notin \{x,y\}$

Proof.

$$\begin{aligned}\{x,y\} \in E_1 \wedge a_f \notin \{x,y\} &\Longleftrightarrow \{x,y\} \in E/e \wedge \{a_e, w\} \cap \{x,y\} = \emptyset \\ &\Longleftrightarrow \{x,y\} \in E \wedge \{u,v,w\} \cap \{x,y\} = \emptyset \\ &\Longleftrightarrow \{x,y\} \in E/f \wedge \{u,a_f\} \cap \{x,y\} = \emptyset \\ &\Longleftrightarrow \{x,y\} \in E_2 \wedge a_e \notin \{x,y\} = \emptyset\end{aligned}$$

◇

Now, we consider the case that $a_f \in \{x,y\}$. W.l.o.g. let be $x = a_f$. Note that $i(x) = a_e$ and $i(y) = y$.

Claim. $\{a_f, y\} \in E_1 \Longleftrightarrow \{a_e, y\} \in E_2$

Proof.

$$\begin{aligned}\{a_f, y\} \in E_1 &\Longleftrightarrow \{a_e, y\} \in E/e \vee \{w,y\} \in E/e \\ &\Longleftrightarrow \{u,y\} \in E \vee \{v,y\} \in E \vee \{w,y\} \in E \\ &\Longleftrightarrow \{u,y\} \in E/f \vee \{a_f,y\} \in E/f \\ &\Longleftrightarrow \{a_e,y\} \in E_2\end{aligned}$$

◇

We can summarise this: $\{x,y\} \in E_1 \Longleftrightarrow \{i(x), i(y)\} \in E_2$. Therefore, mapping i is an isomorphism and G_1 and G_2 are isomorphic. □

Note that any two sequences of the same elements can be transformed into each other by successively swapping the positions of neighbouring elements. Therefore, with Lemma 1, we can conclude the following corollary.

Corollary 1. *Contracting edges in a graph is commutative.*

Hence, it is sensible to define the contraction of a set of edges. Hereafter, we use the following shorthand:

$$e_1/e_2/e_3/.../e_p := (e_1/e_2)/(e_3/e_2)/.../(((e_p/e_2)/(e_3/e_2))...)$$

Definition 2. *Given a graph G and a set of edges $E' = \{e_1, ..., e_p\}$, we define:*

$$G/E' := G/e_1/.../e_p$$

When contracting a set of edges, the edges to be contracted might be modified due to earlier contractions. The next lemma makes a statement about the edge which is contracted last, if a set of edges is contracted that forms a cycle in the graph.

Lemma 2. *Let $C = (v_1, ..., v_p)$ be a cycle of length $p \geq 3$, and let E' be the set of edges in C, i.e. $E' = \{e_1 = \{v_1, v_2\}, ..., e_p = \{v_p, v_1\}\}$. W.l.o.g. we contract the edges in the following order: $e_1, ..., e_p$. Then e_p will degenerate to a single vertex due to the contractions of $e_1, ..., e_{p-1}$, i.e.:*

$$e_p/e_1/.../e_{p-1} = \{a_{e_{p-1}}\}$$

for a new vertex $a_{e_{p-1}}$.

Proof. We prove this by induction on p. For $p = 3$, we have: $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, and $e_3 = \{v_3, v_1\}$.

$$\begin{aligned} e_3/e_1/e_2 &= (e_3/e_1)/(e_2/e_1) \\ &= (\{v_3, v_1\}/\{v_1, v_2\})/(\{v_2, v_3\}/\{v_1, v_2\}) \\ &= \{a_{e_1}, v_3\}/\{a_{e_1}, v_3\} \\ &= \{a_{e_2}\} \end{aligned}$$

Hence, we assume the lemma holds for cycles of length p, and we will show it also holds for cycles of length $p+1$. Let be given a cycle $e_1, ..., e_{p+1}$ of length $p+1$. Contracting e_1 introduces a new vertex a_{e_1} and we see that $e_2/e_1 = \{a_{e_1}, v_3\}$ and $e_{p+1}/e_1 = \{a_{e_1}, v_{p+1}\}$. All other edges are not influenced by contracting e_1. Hence, this results in a cycle of length p, for which we know that the lemma holds. □

Definition 1 defines edge contractions. For technical reasons, it also defines the contraction of 'an edge' consisting of a single vertex. However, such a contraction in G results in a graph isomorphic to G.

Lemma 3. *Let be given a graph $G = (V, E)$ and $x \in V$. Furthermore, let be $G' = (V', E') = G/x$. Then we have: $G' \cong G$.*

Proof. Looking at Definition 1, we see that $V' = \{a_x\} \cup V \setminus x$. We do not delete any edges from E to obtain E', but we update all single edges which contained x to contain a_x. Therefore, G' is isomorphic to G, since we only changed the name of vertex x into a_x. □

Lemma 4. *Let be given a graph $G = (V, E)$ and a set of edges $E_1 = \{h_1, ..., h_q\}$ and a set of edges $E_2 = \{e_1, ..., e_p\}$, $E_1 \cap E_2 = \emptyset$. Let E_2 form a cycle $e_1, ..., e_p$ with $3 \leq p$. Let be $E' = \{h_1, ..., h_q, e_1, ..., e_p\}$ and $E'' = \{h_1, ..., h_q, e_1, ..., e_{p-1}\}$. Then G/E' is isomorphic to G/E'', i.e. $G/E' \cong G/E''$.*

Proof. From Corollary 1, we know edge contractions are commutative. Therefore, we choose to contract edges in the following order: $e_1, ..., e_p, h_1, ..., h_q$. Then we have:

$$\begin{aligned} G/E' &= G/e_1/.../e_{p-1}/e_p/h_1/.../h_q \\ &\quad \text{(from Lemma 2 follows:)} \\ &= G/e_1/.../e_{p-1}/\{a_p\}/h_1/.../h_q \\ &\quad \text{(from Lemma 3 follows:)} \\ &\cong G/e_1/.../e_{p-1}/h_1/.../h_q \\ &\cong G/E'' \end{aligned}$$

□

From the last lemma, we easily conclude that we can delete an arbitrary edge in a cycle in a set of edges $E' \subseteq E$ to be contracted, for a graph $G = (V, E)$. We can repeat this until there are no cycles left . The result will be a maximal spanning forest E'' of $G[E'] := (\bigcup_{e \in E'} e, E')$, and we have G/E' is isomorphic to G/E'', i.e. $G/E' \cong G/E''$. Since we can restrict ourself to edge sets without cycles, we use the term *contraction-set* to refer to such sets.

Definition 3. *A* contraction-set *E' in $G = (V, E)$ is a set of edges $E' \subseteq E$, such that $G[E']$ is a forest. A* contraction *H of G is a graph such that there exists a contraction-set E' with: $H = G/E'$.*

After the previous observations, we develop another view on contracting a set E' of edges. The graph $G[E']$ is composed of connected components $O_1, ..., O_z$, with $O_i = (V_i, E_i)$. When contracting E' in G, then every connected component O_i will be replaced by a new single vertex a_i. This vertex a_i will be made adjacent to every vertex in $N_G(V_i)$, i.e. all vertices that are neighbours in G of a vertex in O_i, but that do not belong to V_i. It becomes evident from the previous lemmas that this definition of edge contractions is equivalent to Definition 1.

Note once again that after each single edge-contraction the names of the vertices are updated in the graph. Hence, for two adjacent edges $e = \{u, v\}$ and $f = \{v, w\}$, edge f will be different after contracting edge e, namely in G/e we have $f = \{a_e, w\}$. However, it might be convenient to use f to represents the same edge in G and in G/e. The same applies also to vertices.

The next lemma tells us that an edge contraction might decrease the degree of a vertex, but it can never decrease it by more than one.

Lemma 5. *Let be given a graph $G = (V, E)$, $v \in V$ and $e \in E$.*

$$v \notin e \Longrightarrow d_{G/e}(v) \geq d_G(v) - 1$$
$$v \in e \Longrightarrow d_{G/e}(a_e) \geq d_G(v) - 1$$

Proof. We prove this by considering an exhaustive case distinction.

Case 1: $e = \{u, v\} \wedge v \in e$. Clearly, we have:

$$N_{G/e}[v] = N_G[v] \cup N_G[u] \cup \{a_e\} \setminus \{u, v\}$$

And therefore it holds:

$$\begin{aligned} d_{G/e}(v) &= |N_{G/e}(v)| = |N_{G/e}[v]| - 1 \\ &= |N_G[v] \cup N_G[u] \cup \{a_e\} \setminus \{u, v\}| - 1 \\ &\geq |N_G[v] \cup \{a_e\} \setminus \{u, v\}| - 1 \\ &\geq |N_G[v]| + 1 - 2 - 1 = |N_G[v]| - 2 = |N_G(v)| - 1 = d_G(v) - 1 \end{aligned}$$

Case 2: $e = \{u, w\} \wedge |e \cap N(v)| = 2$. We have:

$$N_{G/e}(v) = N_G(v) \cup \{a_e\} \setminus \{u, w\}$$

and thus:

$$\begin{aligned} d_{G/e}(v) &= |N_{G/e}(v)| = |N_G(v) \cup \{a_e\} \setminus \{u, w\}| \\ &= |N_G(v)| + 1 - 2 = |N_G(v)| - 1 = d_G(v) - 1 \end{aligned}$$

Case 3: $e = \{u, w\} \wedge |e \cap N(v)| \leq 1$. In this case, the neighbourhood of v is not affected, apart from a possibe change of the name of one vertex in $N(v)$. Therefore, $d_{G/e}(v) = d_G(v)$. □

References

1. H. L. Bodlaender. A tourist guide through treewidth. *Acta Cybernetica*, 11:1–23, 1993.
2. H. L. Bodlaender, A. M. C. A. Koster, and T. Wolle. Contraction and treewidth lower bounds. In S. Albers and T. Radzik, editors, *Proceedings 12th Annual European Symposium on Algorithms, ESA2004*, pages 628–639. Springer, Lecture Notes in Computer Science, vol. 3221, 2004.
3. H. L. Bodlaender, A. M. C. A. Koster, and T. Wolle. Degree-based treewidth lower bounds. Technical Report To appear., Institute for Information and Computing Sciences, Utrecht University, Utrecht, The Netherlands, 2004.
4. B. Mohar and C. Thomassen. *Graphs on Surfaces.* The Johns Hopkins University Press, Baltimore, 2001.