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On the Representation of Disk Graphs*

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Abstract. For an arbitrary graph G, we consider the problem of deciding whether G is a disk graph (DG). The problem is known to be NP-hard, but it is open whether the problem actually is in NP. The problem is related to another open problem, the Polynomial Representation Hypothesis (PRH) for disk graphs: given an n-node disk graph G, can it be embedded in the plane such that the disk centers and disk radii have a binary representation in polynomially many bits. We give several reductions of the problem, and prove that the PRH for disk graphs is equivalent to another interesting and purely geometric conjecture, the Separation Hypothesis for DGs. We give an exact, exponential algorithm for recognizing DG's that have an ϵ-separated embedding, for any given $\epsilon > 0$. Most results apply ipso facto to unit disk graphs, and will generalize to ball graphs and unit ball graphs of fixed sphericity $d > 2$.

1 Introduction

In modern information and communication technology there is a widely increasing use of ad hoc networks. In the modeling and optimization of ad hoc networks there are many questions that are best studied in graph-theoretic terms. This has given a new stimulus to the study of geometric intersection graphs such as disk graphs and their variants [11], which all model certain aspects of the interaction and possible interference of the network nodes. A graph G is a *disk graph* (DG) if it is the intersection graph of some set of closed disks in the 2-dimensional plane, say with centers c_i and radii r_i ($1 \leq i \leq n$). Any concrete set of disks in the plane that 'realizes' G is called a *representation* of G. If all r_i can be taken to be equal, then G is called an *unit disk graph* (UDG), and the representation is called an 'even disk representation'. By scaling we may as well assume that all r_i are equal to 1, in which case we speak of a 'standard (unit disk) representation'.

In this paper we study some complexity issues for disk graphs. A fundamental problem for (unit) disk graphs is the following:

DG Recognition
Given a graph G, decide whether G is a disk graph.

The DG recognition problem is algorithmically decidable ([18], Sect. 4.3) and known to be NP-hard [3] and in PSPACE [11, 13], but it is open whether the problem actually is in NP. Even the UDG recognition is not yet known to be in NP. Disk graphs are not abundant: one can show that of the n-node graphs at most $2^{O(n \log n)}$ of them can be disk graphs, using a purely algebraic argument ([18], Sect. 4.1).

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Graph Class	Recognition	Representation	Reference
planar graphs	linear	polynomial	[6]
interval graphs	linear	polynomial	[2]
unit (proper) interval graphs	linear	polynomial	[4]
circular-arc graphs	linear	polynomial	[12, 16]
unit circular-arc graphs	quadratic	polynomial	[8, 14]
tolerance graphs	$\in$ NP	polynomial	[10]
disk graphs	NP-hard	?	[3]
unit disk graphs	NP-hard	?	[3]

Fig 1. *Some geometric graph classes*

A related and equally fundamental problem is the DG construction problem: given a graph G that is known to be a disk graph, construct a representation of G as the intersection graph of a concrete set of disks in the plane. The problem can be solved algebraically (cf. [18], Sect. 4.3) but this may well lead to a representation in which some of the disk centers and radii are exponentially large or small or even irrational, which is highly undesirable from a computational viewpoint. A similar issue arises in other geometrically defined classes of graphs (Figure 1) and in the so-called *localization problem* for wireless sensor networks. It is widely conjectured that all disk graphs do have a polynomial representation. (Whenever the term 'polynomial' is used in this paper, we essentially mean a polynomial n^k for some integer $k \geq 1$.)

Polynomial Representation Hypothesis (PRH, for disk graphs)
Every n-node disk graph G has a representation by means of n disks in the plane of which the centers and radii are all integral and have a binary representation in at most $p(n)$ bits, for some fixed polynomial p.

In Section 2 we prove the easy part that every (unit) disk graph has an all-integer (even disk) representation, but the PRH itself is open. Note that, if the PRH holds, then the DG recognition problem is in NP. For unit disk graphs we implicitly assume in the PRH that all disks in the polynomial representation must have equal radius. For the case of unit disk graphs we also formulate the PRH in a modified form for the unit representation. In this case we want all radii in the representation equal to 1 and thus need a scaled version of the PRH.

'Special' Polynomial Representation Hypothesis (s-PRH, for unit disk graphs)
Every n-node unit disk graph G has a representation by means of n disks in the plane, all of radius 1 and with centers that have a (possibly fractional) binary representation in at most $p(n)$ bits, for some fixed polynomial p.

In Section 2 we show the easy part of the hypothesis, namely that all unit disk graphs have a finite fractional binary representation with radii equal to 1. In Section 3 we prove that the s-PRH is indeed a correct equivalent of the PRH in the case of unit disk graphs.

Returning to the original motivation behind disk graphs, it is also of interest to study the relative degree by which the disks in a representation overlap or are disjoint. Let $G = (V, E)$ be an (even) disk graph, and let `dist` denote the Euclidean distance measure in the plane.

Definition 1. *A representation of G by (even) disks is called* ϵ-separated*, for some* $0 < \epsilon \leq 1$*, if the following holds for all nodes* $i, j \in V$: $(i, j) \in E \Rightarrow \mathtt{dist}(c_i, c_j) \leq (1-\epsilon)(r_i+r_j)$*, and* $(i, j) \notin E \Rightarrow \mathtt{dist}(c_i, c_j) \geq (1 + \epsilon)(r_i + r_j)$.

Observe that an ϵ-separated representation is ϵ'-separated for every $0 < \epsilon' < \epsilon$. Scaling an ϵ-separated representation leaves it ϵ-separated.

Definition 2. *An (even, unit) disk graph G is said to have a* q-separated *representation, for some polynomial* $q = q(n)$*, if and only if G has a ϵ-separated representation with* $\epsilon = \frac{1}{2^{q(n)}}$.

Separation is a purely geometric notion for sets of disks, which we show to be well-defined in Section 2. From a practical viewpoint we wish to have embeddings that are at least 'well separated', and it is reasonable to believe that this can always be achieved for all (unit) disk graphs. Then the following hypothesis is realistic:

> **Separation Hypothesis** (SH, for disk graphs)
> *For some fixed polynomial q, every n-node disk graph G has a q-separated representation.*

We prove that the Polynomial Representation Hypothesis and the Separation Hypothesis are equivalent, for disk graphs as well as for unit disk graphs. In other words:

Theorem 1 (Equivalence). PRH $\Leftrightarrow$ SH.

This 'translates' the PRH into a plausible and purely geometric conjecture. The proof uses some observations from mathematical programming. For the case of unit disk graphs we also prove that **s-PRH $\Leftrightarrow$ SH**.

Corollary 1. SH $\Rightarrow$ *disk graph recognition $\in$ NP (and thus DG is NP-complete).*

A side-result of the proof is the first exact, exponential-time algorithm for recognizing DG's (UDG's), assuming they are ϵ-separated for some $\epsilon > 0$. The algorithm delivers a representation that satisfies the PRH, if the tested graph indeed has a q-separated representation for some polynomial q. In Section 5 we give some reductions of the PRH. Most results will generalize to ball graphs and unit ball graphs in higher dimensions, i.e. of fixed sphericity $d > 2$.

2 Preliminaries

In this section we list a number of simple, but useful facts for DG's and their representation. A number of these facts are analogues of similar results or observations for other classes such as tolerance graphs (cf. [9], Ch 2).

2.1 ϵ-Separation

We first show that ϵ-separation is a well-defined concept. The main tools in the observations that follow are *moving*, *scaling* and the density of the rationals in the reals.

Proposition 1. *Every (unit) disk graph has an ϵ-separated embedding, for some $\epsilon > 0$.*

Proof. Let G be a n-node (unit) disk graph and consider a representation of G with n disks, with centers c_i and radii r_i ($1 \leq i \leq n$). If G has no touching disks we are done, so consider the case in which some disks do touch. We will make all touching disks overlapping by slightly moving all disks, without creating new intersections. (This leaves the radii unchanged, which is desirable.)

Consider the given representation. For all nodes $i, j \in V$ define the values $\delta_{ij} \geq 0$ by: $(i,j) \in E \Rightarrow \texttt{dist}(c_i, c_j) = (1-\delta_{ij})(r_i+r_j)$, and $(i,j) \notin E \Rightarrow \texttt{dist}(c_i, c_j) = (1+\delta_{ij})(r_i+r_j)$. Let δ be the *smallest* of the *nonzero* δ_{ij}'s. If all δ_{ij}'s are zero, take $\delta = 1$. Let $\alpha = \frac{2}{2+\delta}$ and $\epsilon = \frac{\delta}{2+\delta}$, and note that $0 < \alpha < 1$ and $0 < \epsilon < \delta$.

Now move all disks from c_i to $c'_i = \alpha c_i$. We claim that we have again a representation of G, now without touching disks. To show this, take any nodes $i, j \in V$.

- If $(i,j) \in E$ and disks i and j intersect but do not touch, then $\texttt{dist}(c'_i, c'_j) = \alpha \texttt{dist}(c_i, c_j) = \alpha(1-\delta_{ij})(r_i+r_j) < (1-\delta)(r_i+r_j) < (1-\epsilon)(r_i+r_j)$ and thus the disks i and j again intersect and do not touch.
- If $(i,j) \in E$ and disks i and j touch, then $\texttt{dist}(c'_i, c'_j) = \alpha \texttt{dist}(c_i, c_j) = \frac{2}{2+\delta}(r_i+r_j) = (1-\frac{\delta}{2+\delta})(r_i+r_j) \leq (1-\epsilon)(r_i+r_j)$ and the disks i and j intersect but now do not touch anymore. Note that this is the only case in which all δ_{ij}'s can be zero (in which case δ was chosen to be equal to 1).
- If $(i,j) \notin E$ and disks i and j do not intersect, then $\alpha \texttt{dist}(c_i, c_j) \geq \alpha(1+\delta)(r_i+r_j) = \frac{2}{2+\delta}(1+\delta)(r_i+r_j) = (1+\frac{\delta}{2+\delta})(r_i+r_j) \geq (1+\epsilon)(r_i+r_j)$.

This shows that we have a good representation again. From the argument it follows that the new representation of G is ϵ-separated, for the value $\epsilon > 0$ as defined. □

We will see later that no polynomial representations are missed by moving to ϵ-separated representations.

Given that all (unit) disk graphs have an ϵ-separated representation, it follows that we do not need to insist on hard boundaries of the disks in order to have a representation. Define an *open* (unit) disk graph to be any graph that is the intersection graph of a set of open (unit) disks.

Proposition 2. *The classes of open (unit) disk graphs and (unit) disk graphs coincide.*

Proof. We prove this in two parts. Let G be any (unit) disk graph. By Proposition 1, G has an ϵ-separated representation. By simply replacing the closed disks by their open version one keeps a representation of G, but now as an open (unit) disk graph.

Conversely, let G be any open (unit) disk graph and consider a representation of G by means of n open disks, with centers c_i and radii r_i $(1 \le i \le n)$. Replacing the open disks by their closed version does not work, as it would formally turn touching disks into intersecting disks. We will separate all touching disks by slightly moving all disks, without creating new touchings or intersections (and leaving radii unchanged).

Consider the representation by open disks and define the values $\delta_{ij} \ge 0$ as in the proof of Proposition 1. Let δ be the *smallest* of the values 1 and the *nonzero* δ_{ij}'s. Let $\alpha = \frac{2}{2-\delta}$ and $\epsilon = \frac{\delta}{2-\delta}$, and note that $\alpha \ge 1$ and $0 < \epsilon \le \delta$. Now move all disks from c_i to $c'_i = \alpha c_i$. We claim that we have again a representation of G, now without touching disks. To show this, take any nodes $i, j \in V$.

- If $(i,j) \in E$ and (open) disks i and j intersect but do not touch, then $\alpha\texttt{dist}(c_i, c_j) = \alpha(1-\delta_{ij})(r_i+r_j) \le \frac{2}{2-\delta}(1-\delta)(r_i+r_j) = (1-\frac{\delta}{2-\delta})(r_i+r_j) = (1-\epsilon)(r_i+r_j)$ and thus the disks i and j again intersect and do not touch.
- If $(i,j) \notin E$ but disks i and j touch, then $\texttt{dist}(c'_i, c'_j) = \alpha\texttt{dist}(c_i, c_j) = \frac{2}{2-\delta}(r_i + r_j) = (1 + \frac{\delta}{2-\delta})(r_i + r_j) \ge (1+\epsilon)(r_i + r_j)$ and the disks i and j do not touch anymore but also do not intersect.
- If $(i,j) \notin E$ and disks i and j do not intersect, then $\texttt{dist}(c'_i, c'_j) = \alpha\texttt{dist}(c_i, c_j) \ge \alpha(1+\delta)(r_i + r_j) \ge (1+\delta)(r_i + r_j) \ge (1+\epsilon)(r_i + r_j)$. (Note that this is the only case in which all δ_{ij}'s can be > 1, in which caseδ was chosen to be equal to 1).

This shows that we have a good representation again. Now replacing the open disks by their closed versions keeps a good representation of G, now as an ordinary (unit) disk graph. Note that the new representation happens to be ϵ-separated. □

We will see later that no polynomial representations are missed by restricting to closed disks, as we will do from now on.

2.2 Finite representation

With Proposition 1 we have a useful tool for proving the easy part of the Polynomial Representation Hypothesis (PRH), i.e. the part without the explicit polynomial bound.

Proposition 3. *Every (unit) disk graph has an all-integer (even disk) representation, i.e. a representation in which all disk centers have integer coordinates and all radii are integers.*

Proof. Let G be an n-node (unit) disk graph. By Proposition 1 it has an ϵ-separated representation, say consisting of disks with centers $c_i = (x_i, y_i)$ and radii r_i $(1 \le i \le n)$. We thus assume w.l.o.g. that $\epsilon < 1$. We apply two steps of scaling to obtain our result.

Let $\delta = \frac{1}{4}\sqrt{2} \cdot \epsilon$. Let r_{min} be the smallest radius, and let p be any integer such that $0 < \frac{1}{p} < \delta r_{min}$. Then any interval $(x_i - \delta r_i, x_i + \delta r_i)$ contains some integer multiple of $\frac{1}{p}$, say x'_i. Likewise any interval $(y_i - \delta r_i, y_i + \delta r_i)$ contains some integer multiple of $\frac{1}{p}$, say y'_i. The point $c'_i = (x'_i, y'_i)$ lies within the $\frac{1}{2}\epsilon r_i$-ball around the center c_i. Now move the disks from c_i to c'_i, for $1 \le i \le n$. We claim that this gives a valid representation of G again. To prove it, consider any $i, j \in V$:

- If $(i,j) \in E$ then the old disks i and j overlap but for the new positions of the disk we have
  $\mathtt{dist}(c'_i, c'_j) \leq \frac{1}{2}\epsilon r_i + \mathtt{dist}(c_i, c_j) + \frac{1}{2}\epsilon r_j \leq \frac{1}{2}\epsilon r_i + (1-\epsilon)(r_i + r_j) + \frac{1}{2}\epsilon r_j = (1 - \frac{1}{2}\epsilon)(r_i + r_j)$
  and thus they also overlap.
- If $(i,j) \notin E$ then the old disks i and j are disjoint and for the new positions of the disk we have
  $\mathtt{dist}(c'_i, c'_j) \geq -\frac{1}{2}\epsilon r_i + \mathtt{dist}(c_i, c_j) - \frac{1}{2}\epsilon r_j \geq -\frac{1}{2}\epsilon r_i + (1+\epsilon)(r_i + r_j) - \frac{1}{2}\epsilon r_j = (1 + \frac{1}{2}\epsilon)(r_i + r_j)$
  and thus they are also disjoint.

The resulting representation is still ϵ'-separated, for $\epsilon' = \frac{1}{2}\epsilon$, but the centers c'_i are now all multiples of $\frac{1}{p}$. Scale by multiplying all centers and radii by p. It keeps a valid and ϵ'-separated representation of G but now all centers have integer coordinates.

For the next step, assume that we have an ϵ-separated representation of G for some $\epsilon > 0$, with integer centers. We devise another scaling step to obtain integer radii as well. Let $\delta = \frac{\frac{1}{2}\epsilon}{1+\frac{1}{2}\epsilon}$ and note that $\frac{1+\epsilon}{1+\delta} = 1 + \frac{1}{2}\epsilon$. Let r_{min} be the smallest radius, and let q be any integer such that $0 < \frac{1}{q} < \delta r_{min}$. Then any interval $(r_i, r_i + \delta r_i)$ contains some integer multiple of $\frac{1}{q}$, say r'_i. Now change the radius of disk i from r_i to r'_i, for $1 \leq i \leq n$. This keeps a valid representation of G again. To prove it, consider any $i, j \in V$:

- If $(i,j) \in E$ then the old disks i and j overlap but for the new disks we have
  $\mathtt{dist}(c_i, c_j) \leq (1-\epsilon)(r_i + r_j) \leq (1-\epsilon)(r'_i + r'_j)$
  and thus they also overlap.
- If $(i,j) \notin E$ then the old disks i and j are disjoint but for the new disks we have
  $\mathtt{dist}(c_i, c_j) \geq (1+\epsilon)(r_i + r_j) = \frac{1+\epsilon}{1+\delta}(r_i + \delta r_i + r_j + \delta r_j) \geq (1 + \frac{1}{2}\epsilon)(r'_i + r'_j)$
  and thus they are also disjoint.

The resulting representation is again ϵ'-separated, for $\epsilon' = \frac{1}{2}\epsilon$, the centers haven't moved, but the radii r'_i are all multiples of $\frac{1}{q}$. Scale by multiplying all centers and radii by q. This results in a valid representation of G with centers and radii integral. □

In our definition of *unit* disk graphs we have only insisted on the radii being equal, not on the radii being all 1. The proof of proposition 3 gives the possibility to conclude a slightly stronger integrality result for unit disk graphs.

Corollary 2. *Let G be an unit disk graph. Then for every integer s sufficiently large, G has an all-integer (even disk) representation with radii equal to s, i.e. a representation in which all disk centers have integer coordinates and all radii are equal to s.*

Proof. Let G be a unit disk graph. Consider an ϵ-separated representation of G by means of even-sized disks, say of radius equal to r. Scale by multiplying the centers and radii by a factor of $\frac{1}{r}$. This gives an ϵ-separated representation of G, say with centers c_i but now with all radii equal to 1.

Consider the first part of the proof of Proposition 3, and apply it to this representation of G. Following the proof, the centers of the disks are moved slightly (without changing the radii), and an integer p is determined such that scaling by a factor p leads to a valid representation with purely integral centers, and in this case all radii equal to p. But this part of the proof can be carried out with any integer $s \geq p$ instead of p, thus leading to a valid representation with purely integral centers and all radii equal to s. □

Another observation is the following, now for the special case of *unit* disk representations where we have radii equal to 1. The result is the basis for the s-PRH hypothesis.

Corollary 3. *Every unit disk graph has a representation in which the centers have a finite (possibly fractional) binary representation and all radii are equal to* 1*.*

Proof. Let G be a unit disk graph. Consider an ϵ-separated representation of G by means of even-size disks, say of radius equal to r. Scale it again to obtain a representation with centers $c_i = (x_i, y_i)$ and all radii equal to 1.

Consider the first part of the proof of Proposition 3 again, and only look at the step in which new centers $c'_i = (x'_i, y'_i)$ are determined. We keep a valid representation of G for *any* choice of x'_i and y'_i, as long as they are chosen from the intervals $(x_i - \delta r_i, x_i + \delta r_i)$ and $(y_i - \delta r_i, y_i + \delta r_i)$ respectively, with $\delta > 0$ as chosen in the proof. Like all non-empty intervals, these intervals certainly contain numbers that have a finite binary representation, possibly in a fractional form. This proves the result. □

Corollary 3 clearly implies that every unit disk graph has a standard representation in which the centers have *rational coordinates* (and all radii are equal to 1).

3 On Polynomial Representation

The relevant question is: *are all (unit) disk graphs polynomial objects*. This question was phrased more accurately in the Polynomial Representation Hypothesis (PRH). In this section we give some first observations. We are especially interested in the relationship between polynomial representations and q-separated representations.

3.1 Implications for q-separation

An important observation in relation to the formulated hypotheses is the following.

Lemma 1. PRH $\Rightarrow$ SH, *in other words: if all (unit) disk graphs have a polynomial representation, then all (unit) disk graphs have a q-separated representation.*

Proof. Assume that all (unit) disk graph have an integral representation in which the center-coordinates and radii are all $p(n)$-bit integers, for some (fixed) polynomial $p(n)$. In the proof we use that for $0 < x < 1$, $\sqrt{1-x} < 1 - \frac{1}{4}x$ and $\sqrt{1+x} > 1 + \frac{1}{4}x$. (These estimates are not best possible but suitable for our purposes.)

Let G be an arbitrary (unit) disk graph, and consider some representation of G with centers $c_i = (x_i, y_i)$ and radii r_i all given by $p(n)$-bit integers. Let $\delta = \frac{1}{2^{2p(n)+4}}$. We claim that any two disks in the representation are δ-separated, unless they touch. To see this, consider any two nodes $i, j \in V$.

- Suppose that disks i and j overlap but do not touch. Because $\texttt{dist}(c_i, c_j)^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ and $(r_i + r_j)^2$ are both integral, they must differ by at least 1. Thus

  $\texttt{dist}(c_i, c_j) \leq \sqrt{(r_i + r_j)^2 - 1} = \left(\sqrt{1 - \frac{1}{(r_i+r_j)^2}}\right)(r_i + r_j) \leq (1 - \frac{1}{4\cdot(r_i+r_j)^2})(r_i + r_j) \leq (1 - \frac{1}{4\cdot 2^{2p(n)+2}})(r_i + r_j) = (1 - \delta)(r_i + r_j)$.
- Next suppose that disks i and j do not overlap. Then by a similar argument we have

  $\texttt{dist}(c_i, c_j) \geq \sqrt{(r_i + r_j)^2 + 1} = \left(\sqrt{1 + \frac{1}{(r_i+r_j)^2}}\right)(r_i + r_j) \geq (1 + \frac{1}{4\cdot(r_i+r_j)^2})(r_i + r_j) \geq (1 + \frac{1}{4\cdot 2^{2p(n)+2}})(r_i + r_j) = (1 + \delta)(r_i + r_j)$.

All we need to do now is to make the touching disks overlap. Let $\alpha = \frac{2}{2+\delta}$ and $\epsilon = \frac{\delta}{2+\delta}$. We can now use the exact same argument as in the proof of Proposition 1. It means that by moving the disks from c_i to $c_i' = \alpha c_i$, we obtain a fully ϵ-separated representation. Because $\epsilon \geq \frac{\delta}{4} = \frac{1}{2^{2p(n)+6}}$, it follows at once that all (unit) disk graphs have a q-separated representation with $q = q(n) = 2p(n) + 6$. □

Corollary 4. *If all (unit) disk graphs have a polynomial representation, then all (unit) disk graphs have an q-separated 'all-integer' polynomial (even disk) representation.*

Proof. We continue the argument in the preceding proof. The result after moving all disks to c_i' $(1 \leq i \leq n)$ is a q-separated representation in which the $p(n)$-bit integer radii r_i of the disks haven't changed but the centers now have coordinates equal to $\frac{2}{2+\delta}\, x_i$ and $\frac{2}{2+\delta}\, y_i$. Scale the representation by multiplying all coordinates and radii by a factor $2^{2p(n)+5} + 1$. It gives a q-separated representation, now with disk centers $c_i'' = (2^{2p(n)+5} x_i, 2^{2p(n)+5} y_i)$ and radii equal to $(2^{2p(n)+5} + 1)\, r_i$. Thus this is a q-separated and at the same time a polynomial representation in which all relevant parameters are at most $(3p(n) + 7)$-bit integers. □

3.2 Implications for unit disk graphs

For unit disk graphs we show that any polynomial representation with even disks can be transformed into a 'standard' representation with unit disks of radius 1, with all centers having a fractional binary representation in polynomially many bits.

Proposition 4. *For unit disk graphs:* **PRH** $\Leftrightarrow$ **s-PRH**, *in other words: the s-PRH is a correct equivalent of the PRH in the case of unit disk graphs.*

Proof. Clearly **s-PRH** $\Rightarrow$ **PRH** for unit disk graphs, by scaling the polynomial even disk representation with a factor of $2^{p(n)}$.

For the converse, assume that every unit disk graph has a representation with even disks of which the centers (u_i, v_i) and radius r are all $p(n)$-bit integers. By Corollary 4 we may assume w.l.o.g that the graphs are also q-separated, for some polynomial $q(n)$. Scale the representation, to obtain one with centers $c_i = (x_i, y_i)$ and radii equal to 1, where

$x_i = \frac{u_i}{r}$ and $y_i = \frac{v_i}{r}$. Note that $\lfloor x_i \rfloor$ and $\lfloor y_i \rfloor$ are $p(n)$-bit integers, and the representation is still q-separated. Assume w.l.o.g. that $q(n) \geq p(n)$.

Consider the argument in the first part of the proof of Proposition 3 and apply it to the representation we have. In this case $r_i = 1$ and $\delta = \frac{1}{4}\sqrt{2}\frac{1}{2^{q(n)}}$. Observe that $\frac{1}{2^{q(n)+2}} < \frac{1}{4}\sqrt{2}\frac{1}{2^{q(n)}} r_{min}$, where clearly $r_{min} = 1$. Then any interval $(x_i - \delta r_i, x_i + \delta r_i)$ contains some integer multiple of $\frac{1}{2^{q(n)+2}}$, say x'_i, and any interval $(y_i - \delta r_i, y_i + \delta r_i)$ contains some integer multiple of $\frac{1}{2^{q(n)+2}}$, say y'_i. Let $c'_i = (x'_i, y'_i)$. The proof of Proposition 3 shows that moving the disks from c_i to c'_i, for $1 \leq i \leq n$, gives a valid (unit disk) representation of the original graph again.

But note that for every $1 \leq i \leq n$, x'_i and y'_i lie within distance 1 from $\lfloor x_i \rfloor$ and $\lfloor y_i \rfloor$, respectively, which are both $p(n)$- thus $q(n)$-bit integers. Thus x'_i and y'_i both have a finite binary representation, with integer and fractional parts both of at most $q(n) + 2$ bits. This proves the s-PRH. □

Corollary 5. PRH $\Rightarrow$ *for some polynomial $q = q(n)$ every unit disk graph has a (standard) representation in which the centers have a (possibly fractional) binary representation in $q(n)$ bits, all radii are equal to* 1 *and the representation is q-separated.*

In Lemma 1 we established that **PRH** $\Rightarrow$ **SH**. In the Corollary this is strengthened for unit disk graphs. In this case we can show the converse.

Proposition 5. *For all unit disk graphs one has: '***SH** *in at least one standard representation'* $\Rightarrow$ **PRH***, in other words: if for some fixed polynomial q all unit disk graphs have at least one q-separated standard (unit disk) representation, then they have a polynomial representation.*

Proof. Assume that for a certain polynomial $q = q(n)$, every unit disk graph has a standard representation that is q-separated, i.e. with all disks of radius 1 and ϵ-separated with $\epsilon = \frac{1}{2^{q(n)}} \leq \frac{1}{2}$. We first show that the representation can be boxed in by a small box.

To show this, consider the given standard representation consisting of the n unit disks. Move the representation into the first quadrant so the leftmost and lowest disks touch the y- and x-axis respectively. Projecting the unit disks on the x-axis, gives a set of one or more disjoint segments of size ≥ 2 on the line. If two consecutive segments are more than 2ϵ apart, slide all disks on the right horizontally leftward so the two segments become precisely 2ϵ apart. Note that this leaves the representation (valid and) ϵ-separated, because all centers of non-intersecting disks remain at least $2 + 2\epsilon = (1 + \epsilon) \cdot 2$ apart as in the original representation. Thus we can assume that every two consecutive segments on the x-axis are 2ϵ apart. It follows that the horizontal projection of the representation is at most $2n + (n - 1)2\epsilon \leq 3n$. By the same argument and without affecting this bound, the representation can be shifted vertically so it is valid and the projection on the y-axis is bounded by $3n$. It follows that we may assume that every unit disk graph has a standard representation that is q-separated and fits in a $3n$ x $3n$-square, fitted against the origin in the first quadrant of the plane. (This bound can be improved but is good enough for our purposes.)

Thus we may assume that every unit disk graph has a standard representation that is q-separated, and has (x_i, y_i) such that $0 \leq x_i, y_i \leq 3n$, and thus that $\lfloor x_i \rfloor$ and $\lfloor y_i \rfloor$ are $\log 3n$-bit integers. But now we have very much the same situation as in the second part of the proof of Proposition 4. The proof shows that the representation can be altered to obtain a polynomial-size fractional binary standard representation. By scaling this gives a (not necessarily standard) polynomial representation as required in the PRH. □

With the results at hand we can conclude the basic equivalence between polynomial representation and q-separation for unit disk graphs, in which the q-separation condition is limited to standard representations by unit disks of radius 1. (Without this constraint, there would not be an á priori polynomial bound on the radii in the q-separated representation.)

Corollary 6. *For all unit disk graphs one has:* **PRH** $\Leftrightarrow$ '**SH** *in at least one standard representation'.*

Proof. Consider unit disk graphs. The proof of Lemma 1 shows that PRH $\Rightarrow$ SH, where it is implicit that all disks have equal radius $r_i = r$. By scaling (dividing) by a factor equal to r, we obtain a standard representation that continues to be q-separated. (Corollary 5 gives a stronger implication but this is not needed here.) Thus PRH $\Rightarrow$ SH in at least one standard representation. The converse follows by Proposition 5. □

4 Polynomial Representation versus Separation in General

In Lemma 1 we showed **PRH** $\Rightarrow$ **SH**. This section is devoted to a proof of the converse. The observations in Section 3 showed what is essential for the converse: in order to prove that **SH** $\Rightarrow$ **PRH** one needs to show that *under the given condition* every (general) disk graph has a q-separated representation that fits within a $2^{s(n)}$ x $2^{s(n)}$ box, for some polynomial $s = s(n)$ and has disk radii of (possibly fractional) polynomial size only, which is important for preventing extremely small radii. For unit disk graphs this could be done easily. We now show it for general disk graphs. In the sequel $q = q(n)$ need not be a polynomial but can be any other suitable function e.g. $q(n) = \log n$, as long as $\epsilon = \frac{1}{2^q}$.

4.1 A model

Assume the Separation Hypothesis. We write $\epsilon = \frac{1}{2^q}$ where $q = q(n)$ throughout. We assume w.l.o.g. that ϵ is smaller than a suitable constant e.g. $\frac{1}{256}$, i.e. that q is larger than a suitable constant e.g. 8 or 10. Let $G = (V, E)$ be an arbitrary disk graph. Clearly any q-separated representation of G is completely determined by the following quadratic model (where we use that every representation can be moved entirely into the first quadrant):

$$\begin{array}{ll} (x_i - x_j)^2 + (y_i - y_j)^2 \leq (1-\epsilon)^2 (r_i + r_j)^2 & \text{if } (i,j) \in E \\ (x_i - x_j)^2 + (y_i - y_j)^2 \geq (1+\epsilon)^2 (r_i + r_j)^2 & \text{if } (i,j) \notin E \\ r_i > 0 & \text{for } 1 \leq i \leq n \\ x_i, y_i \geq 0 & \text{for } 1 \leq i \leq n. \end{array}$$

Because the model is scalable by positive factors, we may as well replace the third set of inequalities by $r_i \geq 1$ without loss of generality. Thus we consider the model

$$\begin{array}{ll} (x_i - x_j)^2 + (y_i - y_j)^2 \leq (1-\epsilon)^2 (r_i + r_j)^2 & \text{if } (i,j) \in E \\ (x_i - x_j)^2 + (y_i - y_j)^2 \geq (1+\epsilon)^2 (r_i + r_j)^2 & \text{if } (i,j) \notin E \\ r_i \geq 1 & \text{for } 1 \leq i \leq n \\ x_i, y_i \geq 0 & \text{for } 1 \leq i \leq n. \end{array}$$

Let C_r denote the circle with its center at the origin and radius r. The model can then be restated in the following terms (model $\mathcal{Q}$).

$$\begin{array}{ll} (x_i - x_j, y_i - y_j) \text{ lies } \textit{inside} \text{ (or on) } C_{(1-\epsilon)(r_i+r_j)} & \text{if } (i,j) \in E \\ (x_i - x_j, y_i - y_j) \text{ lies } \textit{outside} \text{ (or on) } C_{(1+\epsilon)(r_i+r_j)} & \text{if } (i,j) \notin E \\ r_i \geq 1 & \text{for } 1 \leq i \leq n \\ x_i, y_i \geq 0 & \text{for } 1 \leq i \leq n. \end{array}$$

We proceed to *approximate* this system of $\frac{1}{2}n(n-1)$ quadratic constraints by a system of linear constraints in such a way that feasibility of solutions is preserved. Divide every circle C_r by *sectors* of width 2α, where $\alpha > 0$ will be specified later. (We will be choosing α in the order of $\sqrt{\epsilon}$ or less.) Let a given sector be bounded by the rays at angles $\varphi - \alpha$ and $\varphi + \alpha$. Let the ray at angle φ intersect the circle of radius $(1-\epsilon)r$ in A and let the rays at angles $\varphi - \alpha$ and $\varphi + \alpha$ intersect the circle of radius $(1+\epsilon)r$ in B and C. We now *linearize* the constraints as follows:

– A constraint of the form '$(x_i - x_j, y_i - y_j)$ lies *inside* (or on) $C_{(1-\epsilon)r}$' will be replaced by the constraint '$(x_i - x_j, y_i - y_j)$ lies *inside* (or on) the *triangle* formed by the rays at the angles $\varphi - \alpha$ and $\varphi + \alpha$ and the tangent to $C_{(1-\epsilon)r}$ in the point A' (outer linearization).
– A constraint of the form '$(x_i - x_j, y_i - y_j)$ lies *outside* (or on) $C_{(1+\epsilon)r}$' will be replaced by the constraint '$(x_i - x_j, y_i - y_j)$ lies *outside* (or on) the *open quadrilateral* formed by the rays at angles $\varphi - \alpha$ and $\varphi + \alpha$ and the chord BC of the circle $C_{(1+\epsilon)r}$' (inner linearization).

Note that for a given sector at angles $\varphi \pm \alpha$, the tangent at A and the chord BC are parallel lines, irrespective of the choice of circles.

With some elementary geometry, this leads to the following model in terms of linear inequalities (model $\mathcal{Q}_L$). For the sake of argument we only specify the model for angles φ_{ij} with $0 < \varphi_{ij} < \frac{1}{2}\pi$ and omit the obvious case specifications for sectors in other quadrants.

If $(i,j) \in E$, there exists an angle φ_{ij} such that for $(x,y) = (x_i - x_j, y_i - y_j)$ and $r = r_i + r_j$:

$$\begin{array}{l} \sin(\varphi_{ij} - \alpha) \cdot x - \cos(\varphi_{ij} - \alpha) \cdot y \leq 0 \\ \sin(\varphi_{ij} + \alpha) \cdot x - \cos(\varphi_{ij} + \alpha) \cdot y \geq 0 \\ \cos\varphi_{ij} \cdot x + \sin\varphi_{ij} \cdot y \leq (1-\epsilon) \cdot r \end{array}$$

If $(i,j) \notin E$, there exists an angle φ_{ij} such that for $(x,y) = (x_i - x_j, y_i - y_j)$ and $r = r_i + r_j$:

$$\begin{array}{l} \sin(\varphi_{ij} - \alpha) \cdot x - \cos(\varphi_{ij} - \alpha) \cdot y \leq 0 \\ \sin(\varphi_{ij} + \alpha) \cdot x - \cos(\varphi_{ij} + \alpha) \cdot y \geq 0 \\ \cos\varphi_{ij} \cdot x + \sin\varphi_{ij} \cdot y \geq (1+\epsilon)\cos\alpha \cdot r \end{array}$$

$$\begin{array}{ll} r_i \geq 1 & \text{for } 1 \leq i \leq n \\ x_i, y_i \geq 0 & \text{for } 1 \leq i \leq n. \end{array}$$

The linearized model has some useful and important properties. Recall that model $\mathcal{Q}$ precisely captures all ϵ-separated representations of G.

Proposition 6. *(a) (*Consistency*) Every solution of model $\mathcal{Q}$ is a solution of model $\mathcal{Q}_L$.*
*(b) (*Scalability*) If x_i, y_i, r_i $(1 \leq i \leq n)$ is a solution of model $\mathcal{Q}_L$, then $k \cdot x_i, k \cdot y_i, k \cdot r_i$ $(1 \leq i \leq n)$ is a solution of model $\mathcal{Q}_L$ also, for every $k \geq 1$.*
(c) If α satisfies $\tan \alpha \leq \frac{1}{2}\sqrt{\epsilon}$ then every solution of model $\mathcal{Q}_L$ is a $\frac{1}{2}\epsilon$-separated representation of G.

Proof. Consistency follows by construction, and it is easily seen that multiplication of solutions by an arbitrary factor $k \geq 1$ leaves all inequalities in the model satisfied. For *(c)* we distinguish two cases as in the model. The cases prove that any solution of model $\mathcal{Q}_L$ is a $\frac{1}{2}\epsilon$-separated representation of G.

Consider the case $(i, j) \in E$. With the earlier notation, let the tangent to $C_{(1-\epsilon)r}$ at A intersect the ray $\varphi + \alpha$ in D. Let $z = |AD|$. Considering the right triangle $\triangle OAD$, the base line of the triangle and thus the triangle itself lies entirely within the circle $C_{(1-\frac{1}{2}\epsilon)r}$ if and only if

$$(1-\epsilon)^2 r^2 + z^2 \leq (1 - \frac{1}{2}\epsilon)^2 r^2,$$

or: $z^2 \leq (\epsilon - \frac{3}{4}\epsilon^2) r^2$. If $\tan \alpha \leq \frac{1}{2}\sqrt{\epsilon}$, this is indeed satisfied because:

$$\tan \alpha = \frac{|AD|}{|OD|} = \frac{z}{(1-\epsilon)r} \leq \frac{1}{2}\sqrt{\epsilon} \Rightarrow z \leq \frac{1}{2}\sqrt{\epsilon}(1-\epsilon)r \leq \frac{1}{2}\sqrt{\epsilon}\, r$$

and thus $z^2 \leq \frac{1}{4}\epsilon r^2 \leq (\epsilon - \frac{3}{4}\epsilon^2) r^2$.

Next consider the case $(i, j) \notin E$. With the earlier notation, let the edge BC of the open quadrilateral intersect the ray at angle φ in E. Let $z = |EB|$. Consider the right triangle $\triangle OEB$, where $|OB| = (1+\epsilon)r$. If $\tan \alpha \leq \frac{1}{2}\sqrt{\epsilon}$, we also have $\sin \alpha = \frac{|EB|}{|OB|} = \frac{z}{(1+\epsilon)r} \leq \tan \alpha \leq \frac{1}{2}\sqrt{\epsilon}$ and hence $z \leq \frac{1}{2}(1+\epsilon)\sqrt{\epsilon}\, r$. It follows that

$$|OE|^2 = (1+\epsilon)^2 r^2 - z^2 \geq (1+\epsilon)^2 r^2 - \frac{1}{4}\epsilon(1+\epsilon)^2 r^2 \geq (1+\epsilon)^2 (1 - \frac{1}{4}\epsilon)\, r^2 \geq (1 + \frac{1}{2}\epsilon)^2 r^2$$

as an easy calculation shows. Thus $|OE| \geq (1 + \frac{1}{2}\epsilon)r$ and (x, y) indeed lies outside $C_{(1+\frac{1}{2}\epsilon)r}$.
□

Note that the condition $\tan \alpha \leq \frac{1}{2}\sqrt{\epsilon}$ is satisfied for all angles α small enough e.g. $\alpha \leq \frac{1}{4}\sqrt{\epsilon}$. We assume this bound on α from now on, to keep feasible solutions $\frac{1}{2}\epsilon$-separated.

4.2 Bounding the model

Recall that $\epsilon = \frac{1}{2^q}$ where $q = q(n)$ is some polynomial. We assumed that G is an arbitrary disk graph on n nodes and that it has some ϵ-separated representation, i.e. that model $\mathcal{Q}$ has a solution. We have shown in Proposition 6 that in this case it must also be the solution of a linear system of constraints (model $\mathcal{Q}_L$) and that this system is even a good approximation of model $\mathcal{Q}$ for any α small enough.

However, model $\mathcal{Q}_L$ does not tell us very much yet, as its coefficients can be 'very irrational' (and even arbitrarily small). In this subsection we will relax model $\mathcal{Q}_L$ a little further to obtain a model that has only integer coefficients, while preserving the nice properties of Proposition 6. The coefficients a will all satisfy $0 \le |a| \le c \cdot 2^q$, for some constant c.

The model we aim at is the following, now in terms of linear inequalities entirely with integer coefficients (model $\mathcal{Q}_N$). Again we only specify the model with triangles and open quadrilaterals in the first quadrant and omit the case specifications for other quadrants.

If $(i,j) \in E$, there exist an angle φ_{ij} and integer coefficients $0 \le a_{ij}, b_{ij}, c_{ij}, d_{ij} \le 2^{q+2}$ (with $a_{ij} > c_{ij}$ and $b_{ij} < d_{ij}$) such that

line $b_{ij} \cdot x - a_{ij} \cdot y = 0$ lies between rays $\varphi_{ij} - \alpha$ and $\varphi_{ij} - \frac{1}{2}\alpha$,
line $d_{ij} \cdot x - c_{ij} \cdot y = 0$ lies between rays $\varphi_{ij} + \frac{1}{2}\alpha$ and $\varphi_{ij} + \alpha$,
the segment from $(\frac{c_{ij}}{2^{q+2}}, \frac{d_{ij}}{2^{q+2}})$ to $(\frac{a_{ij}}{2^{q+2}}, \frac{b_{ij}}{2^{q+2}})$ lies between $C_{1-\epsilon}$ and $C_{1-\frac{1}{2}\epsilon}$

and for $(x,y) = (x_i - x_j, y_i - y_j)$ and $r = r_i + r_j$:

$b_{ij} \cdot x - a_{ij} \cdot y \le 0,$
$d_{ij} \cdot x - c_{ij} \cdot y \ge 0,$
$(d_{ij} - b_{ij})2^{q+2} \cdot x + (a_{ij} - c_{ij})2^{q+2} \cdot y \le (a_{ij}d_{ij} - b_{ij}c_{ij}) \cdot r$

If $(i,j) \notin E$, there exist an angle φ_{ij} and integer coefficients $0 \le a_{ij}, b_{ij}, c_{ij}, d_{ij} \le 2^{q+2}$ (with $a_{ij} > c_{ij}$ and $b_{ij} < d_{ij}$) such that

line $b_{ij} \cdot x - a_{ij} \cdot y = 0$ lies between rays $\varphi_{ij} - \alpha$ and $\varphi_{ij} - \frac{1}{2}\alpha$,
line $d_{ij} \cdot x - c_{ij} \cdot y = 0$ lies between rays $\varphi_{ij} + \frac{1}{2}\alpha$ and $\varphi_{ij} + \alpha$,
the segment from $(\frac{c_{ij}}{2^{q+2}}, \frac{d_{ij}}{2^{q+2}})$ to $(\frac{a_{ij}}{2^{q+2}}, \frac{b_{ij}}{2^{q+2}})$ lies between $C_{1+\frac{1}{2}\epsilon}$ and $C_{1+\epsilon}$

and for $(x,y) = (x_i - x_j, y_i - y_j)$ and $r = r_i + r_j$:

$b_{ij} \cdot x - a_{ij} \cdot y \le 0,$
$d_{ij} \cdot x - c_{ij} \cdot y \ge 0,$
$(d_{ij} - b_{ij})2^{q+2} \cdot x + (a_{ij} - c_{ij})2^{q+2} \cdot y \ge (a_{ij}d_{ij} - b_{ij}c_{ij}) \cdot r.$

$r_i \ge 1$ for $1 \le i \le n.$
$x_i, y_i \ge 0$ for $1 \le i \le n.$

We first show how one can approximate the (inner) 'triangles' defined by the constraints of model $\mathcal{Q}_L$ in case $(i,j) \in E$. Next we do it for the (outer) 'open quadrilaterals' defined by the constraints in case $(i,j) \notin E$.

4.2.1 The case $(i,j) \in E$

Consider the case $(i,j) \in E$ (see Figure 1) and assume without loss of generality that $0 < \varphi < \frac{1}{2}\pi$. Our aim will be to show that points $(\frac{c_{ij}}{2^{q+2}}, \frac{d_{ij}}{2^{q+2}})$ and $(\frac{a_{ij}}{2^{q+2}}, \frac{b_{ij}}{2^{q+2}})$ can be found with the property stated in model $\mathcal{Q}_N$. To give the argument in a general way we do it by locating points $(\frac{c_{ij}}{2^{q+2}}\, r, \frac{d_{ij}}{2^{q+2}}\, r)$ and $(\frac{a_{ij}}{2^{q+2}}\, r, \frac{b_{ij}}{2^{q+2}}\, r)$ with the desired property, in between the circles $C_{(1-\epsilon)r}$ and $C_{(1-\frac{1}{2}\epsilon)r}$. However, the argument is independent of the (model variable) r: just carry out the construction for $r = 1$ and scale the points and the line that connects by a factor r. This gives the same result.

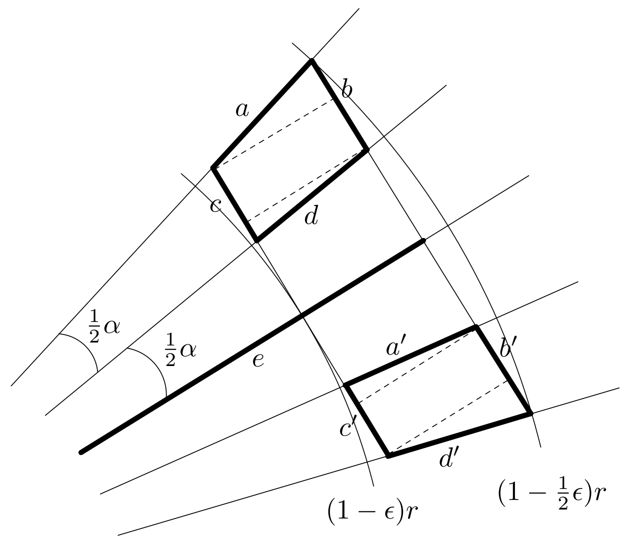

Fig. 1. The case $(i, j) \in E$.

Consider the circles $C_{(1-\epsilon)r}$ and $C_{(1-\frac{1}{2}\epsilon)r}$. Look at quadrilateral *abcd* in Figure 1. (Note that the line $c - c'$ is tangent to the circle, line $b - b'$ is parallel to $c - c'$ and both are perpendicular to e.) We show that the quadrilateral can be guaranteed to be 'sufficiently fat'. To this end we choose $\alpha = \frac{1}{8}\sqrt{\epsilon}$.

We now determine bounds on the 'width' and the 'height' of *abcd*. The width is determined by $|c|$. From Figure 1, we observe for $|c|$ that

$$\begin{aligned}
(1-\epsilon)r\tan\alpha - |c| &= (1-\epsilon)r\tan\frac{1}{2}\alpha \\
(1-\epsilon)r\frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} - |c| &= (1-\epsilon)r\tan\frac{1}{2}\alpha \\
(1-\epsilon)r\frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} - |c| &\leq (1-\epsilon)r\frac{\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} \\
|c| &\geq (1-\epsilon)r\frac{\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} \\
|c| &\geq (1-\epsilon)r\tan\frac{1}{2}\alpha
\end{aligned}$$

By the tangent inequality $\tan x \geq x$ and our choice of α it follows that

$$|b| \geq |c| \geq (1-\epsilon)r\tan\frac{1}{2}\alpha \geq \frac{1-\epsilon}{16}\sqrt{\epsilon}\ r.$$

and thus $|b| \geq |c| \geq \frac{1}{32}\sqrt{\epsilon}\ r$.

Let the height of $abcd$ be h. Note that $abcd$ contains a rectangle of size $|c| - h \tan \frac{1}{2}\alpha$ by h. For the height h we have $h = |e| - (1-\epsilon)r$. To estimate h, let f be the extension of line segment b to the intersection with line e. From Figure 1, $\tan\alpha = \frac{|f|}{|e|}$. Using the tangent inequality and given our choice of α,

$$\frac{1}{8}|e|\sqrt{\epsilon} \leq |e| \tan\alpha = |f| = \frac{\sin\alpha}{\cos\alpha}\,|e| \leq 2\sin\alpha\,|e| \leq \frac{1}{4}\sqrt{\epsilon}\,|e|.$$

Using the Pythagorean Theorem,

$$|f|^2 + |e|^2 = \left(1 - \frac{1}{2}\epsilon\right)^2 r^2$$

$$\frac{1}{64}\epsilon|e|^2 + |e|^2 \leq \left(1 - \frac{1}{2}\epsilon\right)^2 r^2 \leq \frac{1}{16}\epsilon|e|^2 + |e|^2$$

$$\frac{(1 - \frac{1}{2}\epsilon)}{\sqrt{1 + \frac{1}{16}\epsilon}}\; r \leq |e| \leq \frac{(1 - \frac{1}{2}\epsilon)}{\sqrt{1 + \frac{1}{64}\epsilon}}\; r.$$

Observing that

$$\frac{(1 - \frac{1}{2}\epsilon)}{\sqrt{1 + \frac{1}{16}\epsilon}} \geq \left(1 - \frac{3}{4}\epsilon\right)$$

for all the values of ϵ we are looking at, we have

$$(1 - \frac{3}{4}\epsilon)r \leq |e| \leq (1 - \frac{1}{2}\epsilon)r$$

and thus

$$\frac{1}{4}\epsilon r \leq h \leq \frac{1}{2}\epsilon r.$$

Using the tangent inequality again it follows that

$$|c| - h\tan\frac{1}{2}\alpha \geq (1-\epsilon)r\tan\frac{1}{2}\alpha - \frac{1}{2}\epsilon r \tan\frac{1}{2}\alpha \geq (1 - \frac{3}{2}\epsilon)r\frac{1}{2}\alpha \geq \frac{1}{32}\sqrt{\epsilon}\; r \geq \frac{1}{4}\epsilon r.$$

We conclude that $abcd$ actually contains a square of size at least $\frac{1}{4}\epsilon r$ by $\frac{1}{4}\epsilon r$, thus a circle of diameter at least $\frac{1}{4}\epsilon r$. Hence choosing a grid with point distance $\frac{1}{4}\epsilon r$ ensures that quadrilateral $abcd$ contains a point $(\frac{c_{ij}}{2^{q+2}}r, \frac{d_{ij}}{2^{q+2}}r)$ with c_{ij}, d_{ij} integral and bounded by 2^{q+2} as desired. By symmetry, the quadrilateral $a'b'c'd'$ contains a point $(\frac{a_{ij}}{2^{q+2}}r, \frac{b_{ij}}{2^{q+2}}r)$ with a_{ij}, b_{ij} as desired. The triangle formed by the origin and these points gives the lines we claimed in model $\mathcal{Q}_N$ (easily adapted in degenerate cases).

4.2.2 The case $(i,j) \notin E$

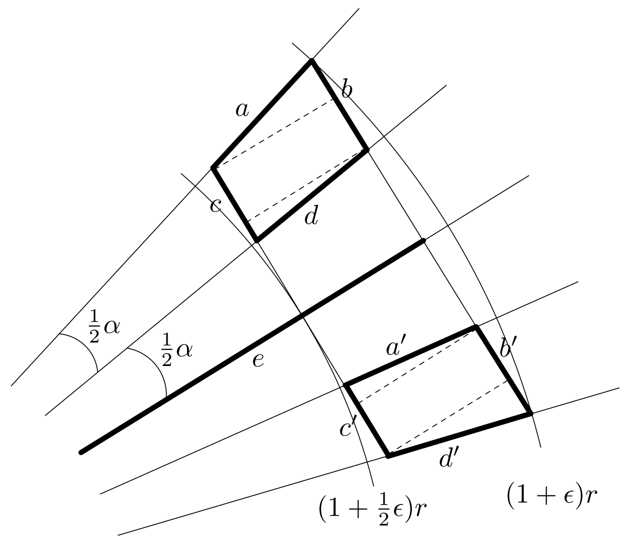

Fig. 2. The case $(i,j) \notin E$.

Now consider the case $(i,j) \notin E$ (see Figure 2) and assume without loss of generality that $0 < \varphi < \frac{1}{2}\pi$. We show again that points $(\frac{c_{ij}}{2^{q+2}}, \frac{d_{ij}}{2^{q+2}})$ and $(\frac{a_{ij}}{2^{q+2}}, \frac{b_{ij}}{2^{q+2}})$ can be found with the desired property. We do it again by appropriately locating points $(\frac{c_{ij}}{2^{q+2}}\, r, \frac{d_{ij}}{2^{q+2}}\, r)$ and $(\frac{a_{ij}}{2^{q+2}}\, r, \frac{b_{ij}}{2^{q+2}}\, r)$, now in between the circles $C_{(1+\frac{1}{2}\epsilon)r}$ and $C_{(1+\epsilon)r}$. Again, the argument is independent of the model variable r: just carry out the construction for $r = 1$ and scale the result by a factor of r. (The details are similar to Subsection 4.2.1 and only given for reasons of completeness.)

Consider the circles $C_{(1+\frac{1}{2}\epsilon)r}$ and $C_{(1+\epsilon)r}$. Look at quadrilateral *abcd* in Figure 2. (Note that the line $c - c'$ is tangent to the circle, line $b - b'$ is parallel to $c - c'$ and both are perpendicular to e.) We show that the quadrilateral can be guaranteed to be 'sufficiently fat'. To this end we can choose $\alpha = \frac{1}{4}\sqrt{\epsilon}$ this time.

We now determine bounds on the 'width' and the 'height' of *abcd*. The width is determined by $|c|$. From Figure 2, we observe for $|c|$ that

$$\begin{aligned}
\left(1+\frac{1}{2}\epsilon\right) r \tan\alpha - |c| &= \left(1+\frac{1}{2}\epsilon\right) r \tan\frac{1}{2}\alpha \\
\left(1+\frac{1}{2}\epsilon\right) r \frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} - |c| &= \left(1+\frac{1}{2}\epsilon\right) r \tan\frac{1}{2}\alpha \\
\left(1+\frac{1}{2}\epsilon\right) r \frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} - |c| &\le \left(1+\frac{1}{2}\epsilon\right) r \frac{\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} \\
|c| &\ge \left(1+\frac{1}{2}\epsilon\right) r \frac{\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha}
\end{aligned}$$

$$|c| \geq \left(1 + \frac{1}{2}\epsilon\right) r \tan \frac{1}{2}\alpha$$

By the tangent inequality and our choice of α,

$$|b| \geq |c| \geq \left(1 + \frac{1}{2}\epsilon\right) r \tan \frac{1}{2}\alpha \geq \frac{1 + \frac{1}{2}\epsilon}{8}\sqrt{\epsilon}\ r$$

and thus $|b| \geq |c| \geq \frac{1}{8}\sqrt{\epsilon}\ r$.

Let the height of *abcd* be h. Again *abcd* contains a rectangle of size $|c| - h \tan \frac{1}{2}\alpha$ by h. For the height h we have $h = |e| - (1 + \frac{1}{2}\epsilon)r$. To estimate h, let f be the extension of line segment b to the intersection with line e. From Figure 2, $\tan \alpha = \frac{|f|}{|e|}$. Using the tangent inequality and given our choice of α,

$$\frac{1}{4}|e|\sqrt{\epsilon} \leq |e| \tan \alpha = |f| = \frac{\sin \alpha}{\cos \alpha}\,|e| \leq 2 \sin \alpha\,|e| \leq \frac{1}{2}\sqrt{\epsilon}\,|e|.$$

Using the Pythagorean Theorem,

$$|f|^2 + |e|^2 = (1 + \epsilon)^2\, r^2$$

$$\frac{1}{16}\epsilon|e|^2 + |e|^2 \leq (1 + \epsilon)^2\, r^2 \leq \frac{1}{4}\epsilon|e|^2 + |e|^2$$

$$\frac{(1 + \epsilon)}{\sqrt{1 + \frac{1}{4}\epsilon}}\, r \leq |e| \leq \frac{(1 + \epsilon)}{\sqrt{1 + \frac{1}{16}\epsilon}}\, r.$$

Observing that

$$\frac{(1 + \epsilon)}{\sqrt{1 + \frac{1}{4}\epsilon}} \geq \left(1 + \frac{3}{4}\epsilon\right)$$

for all the values of ϵ we are looking at, we have

$$(1 + \frac{3}{4}\epsilon)r \leq |e| \leq (1 + \epsilon)r$$

and thus

$$\frac{1}{4}\epsilon r \leq h \leq \frac{1}{2}\epsilon r.$$

Using the tangent inequality again it follows that

$$|c| - h \tan \frac{1}{2}\alpha \geq (1 + \frac{1}{2}\epsilon) r \tan \frac{1}{2}\alpha - \frac{1}{2}\epsilon r \tan \frac{1}{2}\alpha \geq r \frac{1}{2}\alpha \geq \frac{1}{8}\sqrt{\epsilon}\ r \geq \frac{1}{4}\epsilon\ r.$$

We conclude that *abcd* again contains a square of size at least $\frac{1}{4}\epsilon r$ by $\frac{1}{4}\epsilon r$, thus a circle of diameter at least $\frac{1}{4}\epsilon r$. Hence, again a grid with point distance $\frac{1}{4}\epsilon r$ ensures that quadrilateral

$abcd$ contains a point $(\frac{c_{ij}}{2^{q+2}}r, \frac{d_{ij}}{2^{q+2}}r)$ with c_{ij}, d_{ij} integral and bounded by 2^{q+2} as desired. By symmetry, the quadrilateral $a'b'c'd'$ contains a point $(\frac{a_{ij}}{2^{q+2}}r, \frac{b_{ij}}{2^{q+2}}r)$ with a_{ij}, b_{ij} as desired. The open quadrilateral formed by the rays from the origin through these points and their connecting line segment, gives the lines we claimed in model $\mathcal{Q}_N$ (again easily adapted in degenerate cases).

4.2.3 Properties of the construction

This completes the construction of the model. Note that model $\mathcal{Q}_N$ is essentially equivalent to model $\mathcal{Q}_L$, except that the corner points of the triangles and of the open quadrilaterals are very carefully chosen as points of a suitable grid (with coefficients independent of r). The equivalence is expressed in the exact analogue of Proposition 6, where we note that α is bounded as required by Proposition 6.

Proposition 7. *(a) (*Consistency*) Every solution of model $\mathcal{Q}$ is a solution of model $\mathcal{Q}_N$.*
*(b) (*Scalability*) If x_i, y_i, r_i $(1 \leq i \leq n)$ is a solution of model $\mathcal{Q}_N$, then $k \cdot x_i, k \cdot y_i, k \cdot r_i$ $(1 \leq i \leq n)$ is a solution of model $\mathcal{Q}_N$ also, for every $k \geq 1$.*
(c) Every solution of model $\mathcal{Q}_N$ is a $\frac{1}{2}\epsilon$-separated representation of G.

Proof. By construction. □

4.3 Bounding a solution

The new model $\mathcal{Q}_N$ is important for our further, final step. Recall that *by assumption*, disk graph G has an ϵ-separated representation and thus model $\mathcal{Q}_N$ admits a feasible solution. We now proceed to show that model $\mathcal{Q}_N$ in fact must admit a feasible solution fully in integers, all of polynomially bounded size.

Consider model $\mathcal{Q}_N$ again. It consists precisely of $3 \cdot \frac{1}{2}n(n-1)$ inequalities, of the form

$$\begin{aligned}
&b_{ij} \cdot (x_i - x_j) - a_{ij} \cdot (y_i - y_j) \ \leq / \geq \ 0 \\
&d_{ij} \cdot (x_i - x_j) - c_{ij} \cdot (y_i - y_j) \ \geq / \leq 0 \\
&f_{ij} \cdot (x_i - x_j) + e_{ij} \cdot (y_i - y_j) \ \leq / \geq \ g_{ij} \cdot (r_i + r_j)
\end{aligned}$$

and the constraints

$$\begin{aligned}
&r_i \geq 1 && \text{for } 1 \leq i \leq n \\
&x_i, y_i \geq 0 && \text{for } 1 \leq i \leq n.
\end{aligned}$$

We bring the linear system into standard form by introducing non-negative variables s_i defined as $s_i = r_i - 1$ and slack variables $t_{ij}^{\alpha}, t_{ij}^{\beta}, t_{ij}^{\gamma}$ that turn the constraints into equalities:

$$\begin{aligned}
&b_{ij} \cdot (x_i - x_j) - a_{ij} \cdot (y_i - y_j) \pm t_{ij}^{\alpha} = 0 \\
&d_{ij} \cdot (x_i - x_j) - c_{ij} \cdot (y_i - y_j) \mp t_{ij}^{\beta} = 0 \\
&f_{ij} \cdot (x_i - x_j) + e_{ij} \cdot (y_i - y_j) \pm t_{ij}^{\gamma} - g_{ij} \cdot (s_i + s_j) = 2g_{ij}
\end{aligned}$$

now simply with the standard constraints

$x_i, y_i, s_i, t_{ij}^{\alpha}, t_{ij}^{\beta}, t_{ij}^{\gamma} \geq 0$ for all relevant i, j with $1 \leq i, j \leq n$

where we note that $|a_{ij}|, |b_{ij}|, |c_{ij}|, |d_{ij}| \leq 2^{q+2}$ and $|e_{ij}|, |f_{ij}|, |g_{ij}| \leq 2^{2q+4}$ according to the details of model $\mathcal{Q}_N$. The system of linear equalities can be written in matrix-vector form: $\mathbf{A}\mathbf{x} = \mathbf{b}$ with the constraint $\mathbf{x} \geq 0$, where

$\mathbf{A}$ is a $\frac{3}{2}n(n-1)$ by $3n + \frac{3}{2}n(n-1)$ all-integer matrix

all entries a of $\mathbf{A}$ satisfy $|a| \leq 2^{2q+4}$

$\frac{3}{2}n(n-1)$ columns of $\mathbf{A}$ are unit vectors, namely the columns corresponding to the variables $t_{ij}^{\alpha}, t_{ij}^{\beta}, t_{ij}^{\gamma}$ $(1 \leq i \leq j \leq n)$

$\mathbf{x} = (\cdots, x_i, \cdots, y_i, \cdots, s_i, \cdots, t_{ij}^{\alpha}, t_{ij}^{\beta}, t_{ij}^{\gamma}, \cdots)^T$

$\mathbf{b} = (\cdots 0, 0, 2g_{ij}, \cdots)^T$, thus with all entries g of $\mathbf{b}$ satisfying $|g| \leq 2^{2q+5}$.

It follows that $\text{rank}(\mathbf{A}) = \frac{3}{2}n(n-1)$. We can now show the following, using some common facts from the theory of linear programming (cf. [17]).

Proposition 8. *Model $\mathcal{Q}_N$ has an all-integer solution with* $0 \leq x_i, y_i, r_i \leq 2^{4n(2q+6+\log n)+1}$.

Proof. Because $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \geq 0$ has a feasible solution, it also has a *basic* feasible solution. As $\text{rank}(\mathbf{A}) = \frac{3}{2}n(n-1)$, this basic feasible solution has (at least) $3n$ of the coordinates of $\mathbf{x}$ equal to 0, whereas the $\frac{3}{2}n(n-1)$ by $\frac{3}{2}n(n-1)$ submatrix $\mathbf{A}'$ consisting of the columns corresponding to the other coordinates is invertible and satisfies $\mathbf{A}'\mathbf{x}' = \mathbf{b}$ (with $\mathbf{x}' \geq 0$), where $\mathbf{x}'$ is the subvector of $\mathbf{x}$ consisting of these other coordinates. Hence, using *Cramer's rule* [17], it follows that

$$(\mathbf{x}')_i = \frac{\det \mathbf{A}'_{\mathbf{i}}}{\det \mathbf{A}'},$$

where $\mathbf{A}'_{\mathbf{i}}$ is the matrix formed by replacing the i-th column of $\mathbf{A}'$ by the column vector $\mathbf{b}$. From this we obtain a solution x_i, y_i, r_i of model $\mathcal{Q}_N$ that satisfies the following, noting that $r_i = s_i + 1$:

$x_i = 0$ or $x_i = \frac{\det \mathbf{A}'_{\mathbf{i_1}}}{\det \mathbf{A}'}$

$y_i = 0$ or $y_i = \frac{\det \mathbf{A}'_{\mathbf{i_2}}}{\det \mathbf{A}'}$

$r_i = 1$ or $r_i = \frac{\det \mathbf{A}'_{\mathbf{i_3}}}{\det \mathbf{A}'} + 1$

for suitable indices i_1, i_2, i_3 for every i. Note that, because $\mathbf{A}'$ and $\mathbf{A}'_{\mathbf{i}}$ are integer matrices, their determinants are integer as well. Assume without loss of generality that $\det \mathbf{A}' > 0$, which implies that $\det \mathbf{A}' \geq 1$.

Now multiply the solution by $|\det \mathbf{A}'|$. By Proposition 7 this can be done while keeping a solution to model $\mathcal{Q}_N$. Writing it as x_i, y_i, r_i again, this solution satisfies:

$x_i = 0$ or $x_i = \det \mathbf{A}'_{\mathbf{i_1}}$

$y_i = 0$ or $y_i = \det \mathbf{A}'_{\mathbf{i_2}}$

$r_i = \det \mathbf{A}'$ or $r_i = \det \mathbf{A}'_{\mathbf{i_3}} + \det \mathbf{A}'$

To estimate the values of x_i, y_i, r_i we use that for any square matrix $\mathbf{U} = (\mathbf{u_1} \cdots \mathbf{u_N})$ one has

$$|\det \mathbf{U}| \leq \| \mathbf{u_1} \| \cdots \| \mathbf{u_N} \| .$$

If we apply this to any $\det \mathbf{A}'_{\mathbf{i}}$ or to $\det \mathbf{A}'$, note that the matrices $\mathbf{A}'_{\mathbf{i}}$ and $\mathbf{A}'$ have at least $\frac{3}{2}n(n-1) - 3n - 1$ columns that are unit vectors and thus at most $3n+1$ non-trivial columns with entries bounded by 2^{2q+5} in the worst case. Thus:

$$1 \leq |\det \mathbf{A}'_{\mathbf{i}}|, |\det \mathbf{A}'| \leq (\sqrt{3/2 \cdot n^2\, 2^{2q+5} 2^{2q+5}})^{3n+1} \leq (2^{2q+6}\, n)^{3n+1} \leq 2^{4n(2q+6+\log n)},$$

by using a crude estimate. This leads to the bounds for the solution stated in the Proposition. □

Lemma 2. SH ⇒ PRH, *in other words: if all (unit) disk graphs have a q-separated representation, then all (unit) disk graphs have a polynomial representation.*

Proof. This follows immediately from Proposition 8. Because the integers x_i, y_i, r_i in the solution are all bounded between 0 and $2^{4n(2q+6+\log n)+1}$, they can all be represented in $4n(2q+6+\log n)+1$ bits, i.e. as polynomial-size integers. (By improving the estimates, this bound may be improved but it is sufficient for our purposes.) □

The construction as presented has another interesting and useful consequence from a representational viewpoint. It is the most concrete result from the model approximation in this section. Let $\epsilon = \frac{1}{2^q}$ as before and assume again that ϵ is sufficiently small e.g. less than $\frac{1}{256}$.

Corollary 7. *If G has an ϵ-separated representation, then it has an all-integer $\frac{1}{2}\epsilon$-separated representation with all centers and radii represented within $4n(2q+6+\log n)+1$ bits.*

Lemmas 1 and 2 together prove the main result as stated in Theorem 1, the Equivalence Theorem. The bounds in Corollary 7 also hold if we look only for *unit* disk representations, by fixing the r_i to 1 throughout.

5 Further Remarks

We have shown that all (unit) disk graphs have integer representations and we have studied the question whether (unit) disk graphs can in fact be guaranteed to have integer representations in which all centers and radii are polynomially size-bounded. We have reduced it to an equivalent, purely geometric problem, the Separation Hypothesis: **PRH** $\Leftrightarrow$ **SH**. The Separation Hypothesis is plausible and adds considerable intuition to the Polynomial Representation Hypothesis for (unit) disk graphs. Does this make the representation problem easier, and what does it mean for the recognition problem for (unit) disk graphs?

5.1 Weakening the Separation Hypothesis

Can the Separation Hypothesis be weakened? We give one possible weakening that may be useful. Let $G = (V, E)$ be an (even) disk graph, and let $\mathtt{dist}$ denote the Euclidean distance measure in the plane again. Define the concept of separation again, but now with emphasis on the *non*-overlapping disks only. (Similar results hold if we emphasize the overlapping disks only.)

Definition 3. *A representation of G by (even) disks is called* ϵ-outer separated*, for some $0 < \epsilon \leq 1$, if the following holds for all nodes $i, j \in V$: $(i, j) \in E \Rightarrow \mathtt{dist}(c_i, c_j) \leq (r_i + r_j)$, and $(i, j) \notin E \Rightarrow \mathtt{dist}(c_i, c_j) \geq (1 + \epsilon)(r_i + r_j)$.*

Clearly every ϵ-separated representation is ϵ-outer separated, but there is also a converse.

Proposition 9. *Let G have an ϵ-outer separated representation, for some $0 < \epsilon \leq 1$. Then G also has a $\frac{1}{3}\epsilon$-separated representation.*

Proof. Let G be a n-node (unit) disk graph and assume it has an ϵ-outer representation. Consider a representation of G with n disks, and centers c_i and radii r_i ($1 \leq i \leq n$) that satisfy the corresponding requirement. Let $\alpha = 1 - \frac{1}{3}\epsilon$. Move all disks from c_i to $c'_i = \alpha c_i$. We claim that this gives a valid representation of G again, that is (at least) $\frac{1}{3}\epsilon$-separated. To show this, take any nodes $i, j \in V$.

- If $(i, j) \in E$, then disks i and j intersect and we have $\mathtt{dist}(c'_i, c'_j) = \alpha \mathtt{dist}(c_i, c_j) \leq \alpha(r_i + r_j) = (1 - \frac{1}{3}\epsilon)(r_i + r_j)$.
- If $(i, j) \notin E$, then disks i and j do not intersect and $\mathtt{dist}(c'_i, c'_j) = \alpha \mathtt{dist}(c_i, c_j) \geq \alpha(1+\epsilon)(r_i+r_j) = (1-\frac{1}{3}\epsilon)(1+\epsilon)(r_i+r_j) \geq (1+\frac{1}{3}\epsilon)(r_i+r_j)$.

This shows that we have a good representation again and that it is $\frac{1}{3}\epsilon$-separated. The size of the radii is unaffected in the move. $\square$

From this observation it follows that w.l.o.g. the Separation Hypothesis can be restricted to q-outer separated representations, i.e. ϵ-outer separated representations for some $\epsilon = \frac{1}{2^q}$ with q as before.

Corollary 8 (Equivalence - extended). *Every (even) n-node disk graph G has an all-integer representation with centers and radii that are $p(n)$-size bounded for some fixed polynomial p, $\Leftrightarrow$ every (even) n-node disk graph G has a q-outer separated representation, for some fixed polynomial $q = q(n)$.*

5.2 Recognition of (unit) disk graphs

The recognition of (unit) disk graphs is a known NP-hard problem. However, the notion of ϵ-separation give an interesting measure because the recognition problem for (unit) disk graphs really is the recognition problem for ϵ-separated (unit) disk graphs for a suitably small ϵ. The latter can be tackled with the models from Section 4.

One particular consequence is the following. Let a nondeterministic recognizer $\mathcal{R}_\epsilon$ for ϵ-separated (unit) disk graphs, for some fixed $\epsilon > 0$, be called an *NP-approximate recognizer* if the following holds, for some constant $c = c_\epsilon > 0$:

if G is ϵ-separated, then $\mathcal{R}_\epsilon$ outputs *YES*.
if $\mathcal{R}_\epsilon$ outputs *YES*, then G is $c\ \epsilon$-separated.
every (nondeterministic) 'run' of $\mathcal{R}_\epsilon$ is polynomial-time bounded.

Proposition 10. *For every $\epsilon > 0$, the class of ϵ-separated (unit) disk graphs has an NP-approximate recognizer.*

Proof. Without loss of generality we may assume that ϵ is sufficiently small as required by the results in Section 4, by using the scaling factor c. The simplest approach is to use the result of Corollary 7. Assume again w.l.o.g. that $\epsilon = \frac{1}{2^q}$. Guess a representation of the centers and radii in $4n(2q+6+\log n)+1$ bits each and verify, all in polynomial time, whether a $\frac{1}{2}\epsilon$-separated representation is obtained. This is easily seen to be an NP-approximate recognizer. □

The exact simulation of the NP-approximate recognizer takes time in the order of $2^{\mathcal{O}(n^2 \log \frac{1}{\epsilon})}$. Consequently, the following bound can be given if the Separation Hypothesis holds, say w.l.o.g. for all $q \geq 10$.

Corollary 9. SH $\Rightarrow$ *(unit) disk graphs can be recognized in $2^{\mathcal{O}(n^2 q)}$ time.*

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