Connected Rectilinear Graphs on Point Sets

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Abstract

Let V be a set of n points in $\mathbb{R}^d$. We study the question whether there exists an orientation such that V is the vertex set of a connected rectilinear graph in that orientation. A graph is rectilinear if its edges are straight line segments in d pairwise perpendicular directions. We prove that at most one such orientation can be possible, up to trivial rotations of 90° around some axis. In addition, we present an algorithm for computing this orientation (if it exists) in $O(n^2)$ time when $d = 2$.

1 Introduction

Suppose you are given a set of points in the plane, or more generally in $\mathbb{R}^d$, and you want to know whether this is the vertex set of some connected rectilinear graph. Then you have to find the orientation that this graph should have. At the Canadian Conference on Computational Geometry in 2007, Therese Biedl asked the question: can a given set of points in the plane be the vertex set of two rectilinear polygons that have *different* orientations (other than trivial rotations by multiples of 90°)?

Here we show that the answer is no, even when the points are in $\mathbb{R}^d$ and even for arbitrary connected rectilinear graphs instead of polygons. Figure 1 shows an example of two rectilinear graphs on the same point set, but note that G' is not connected. The connectedness is essential: without this restriction, it is sometimes possible to have sets of rectilinear polygons in two different orientations, as Biedl already showed with a regular octagon as example, see Figure 2. For the special case where the points are in the plane and have rational coordinates, Fekete and Woeginger already proved that at most one orientation is possible [10] (Theorem 4.7).

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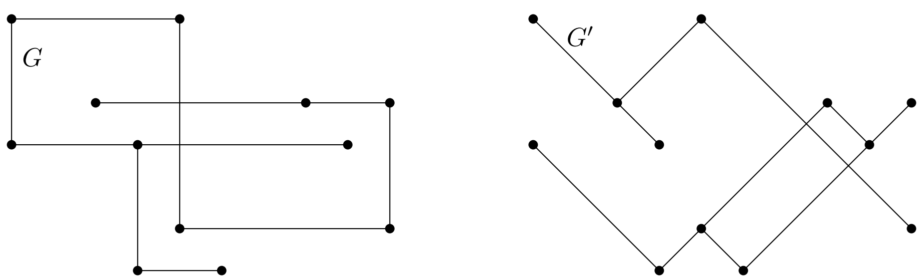

(a) (b)

Figure 1: Two rectilinear graphs in the plane with the same vertex set, but different slopes. Note that G' is not connected.

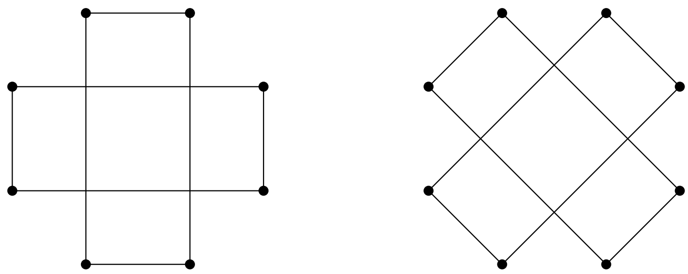

Figure 2: The vertices of a regular octagon allow a spanning set of rectilinear polygons in two different orientations.

1.1 Related Work

Problems on orthogonol polygons have been studied a lot before. O'Rourke [21] proves that there can be at most one way to connect a given point set into a rectilinear polygon that makes a 90° turn at each vertex, and gives a simple algorithm to compute it. On the other hand, if turns of 180° are allowed, Rappaport [23] shows that the problem is NP-hard. Durocher and Kirkpatrick [7] study the problem of finding a collection of rectilinear tours that use the given points as vertices, where the tours are allowed to have different orientations. They prove that this is NP-hard as well. Jansen and Woeginger [14] show that deciding if a given rectilinear geometric graph has a *planar* sub-graph is NP-complete. This is a more restricted case of the more general problem for topological graphs, which was shown NP-hard earlier [18], and for which Knauer *et al.* [17] show several hardness-of-approximation results.

Many papers exist that study how to draw the edges of a given graph where the vertices are mapped to a given point set. For example, Pach and Wenger [22] show that in order to make such a graph planar, a linear number of bends per edge may be necessary. On the other hand, if the mapping between vertices and points is not given, Kaufmann and Wiese [16] show that a planar embedding with at most 2 bends can be computed in polynomial time. Cabello [6] shows that deciding whether a planar straight line embedding exists on a given point set, is NP-complete. If the given graph is outerplanar, Bose [5] shows that a planar straight line embedding on a fixed point set can be computed in near-linear time. Efrat *et al.* [9] study the possibility of drawing a crossing-free graph with circular arcs as edges.

On the other hand, rectilinear graphs also received a lot of attention from a graph drawing perspective. Vijayan and Wigderson [25] show how to embed an abstract graph with an additional 'direction' associated to each edge can be embedded as a rectilinear graph in the plane in $O(n^2)$ time; Hoffman and Kriegel [13] improve this to $O(n)$ time. Garg and Tamassia [12] show that without such associated directions, it is NP-hard to decide if a graph has a rectilinear planar embedding. Bodlaender and Tel [4] show that all graphs with *angular resolution* of at least 90° have a rectilinear embedding. Biedl and Kant [2] give algorithms to draw graphs with rectilinear edges, that are allowed to have a few bends.

Orienting point sets is a frequent task in matching and surface reconstruction. In his classic work on rotating calipers, Toussaint [24] shows, among other problems, how to find an orientation that minimises the axis-aligned bounding box. A different way to orient a point set is principal component analysis [15]. Finding orientations that best match a point set to another point set or object is still an active research topic [3, 19].

The rest of the paper is organised as follows. First we discuss some existing terminology in Section 2. Then, in Section 3, we show that a point set allows for a connected rectilinear graph in at most one orientation. In Section 4, we discuss a related algorithmic question of finding such an orientation for a point set in the plane. We conclude in Section 5.

2 Preliminaries

In this section, we review several concepts and terminology from graph theory, geometry, and algebra, which will be used throughout the rest of the text.

In this paper, we consider a *graph* G to be an embedded graph (in $\mathbb{R}^d$) with straight line edges. We say that G is *d-dimensional* if it is not contained in any lower-dimensional subspace of $\mathbb{R}^d$. We call G *rectilinear* if each pair of edges of G is either parallel or perpendicular. We call G *axis-aligned* if each edge of G is parallel to one of the axes of the normal axis system. If B is an orthogonal basis of $\mathbb{R}^d$, we call G *B-aligned* if each edge of G is parallel to one of the basis vectors in B.

Observation 1 *If G is a d-dimensional connected rectilinear graph, there is a unique orthonormal basis B of $\mathbb{R}^d$, up to rearranging the vectors, such that G is B-aligned.*

Proof: We take the largest linearly independent set of vectors from the set of edges of G. Since G is connected and d-dimensional, it contains d vectors and since G is rectilinear it is orthogonal. ⊠

We also call this basis the *orientation* of G. In the special case where $d = 2$, B can be fully desribed by a *slope* s, and will consist of the vectors $(1, s)$ and $(-\frac{1}{s}, 1)$ (normalised). In this case we also say that G has slope s. If V is a set of points in $\mathbb{R}^d$, and B is an orthogonal basis, then we define the *maximal rectilinear graph* on V with orientation B as the graph with V as vertex set that has an edge between two vertices whenever that edge is parallel to some vector in B. If the maximal rectilinear graph is connected, we also say that B *allows* for a connected rectilinear graph on V.

If $F \subset \mathbb{R}$ is a field, and $x \in \mathbb{R}$, we denote by $F(x)$ the *extended field* generated by adjoining x to F, or in other words, the smallest subfield of $\mathbb{R}$ that contains both $\mathbb{Q}$ and x. We denote by $[F(x) : F]$ the *degree* of the extension.

If $X \subset \mathbb{R}$ is a finite subset of $\mathbb{R}$, we denote by $F\langle X\rangle$ the *vector space* that contains all sums of elements of the set $\{fx \mid f \in F, x \in X\}$. Let $k \leq |X|$ be the dimension of this vector space, and let $E = (e_1, \ldots, e_k)$ be a basis for it. Then $F\langle X\rangle = F\langle E\rangle$, and each element $x \in F\langle X\rangle$ can be written as $x = \sum_j x_j e_j$, where $x_j \in F$.

An *algebraic integer* is an element $a \in \mathbb{C}$ that is the root of a monic polynomial with integer coefficients, that is, a polynomial $p = \sum_{i=0}^{n} c_i x^i$ where $c_i \in \mathbb{N}$ and $c_n = 1$. We denote by O_F the *ring of integers* of F, which consists of all algebraic integers that are in F. Intuitively, the ring of integers O_F behaves towards the field F as $\mathbb{Z}$ behaves towards $\mathbb{Q}$. Every element $f \in F$ can be written as $f = a/b$, where $a, b \in O_F$. An element $a \in O_F$ is *irreducible* if it cannot be written as $a = b \cdot c$ with $b, c \in O_F$. Every element $a \in O_F$ can be written as the product of a finite sequence $a = \prod_i a_i$, where $a_i \in O_F$ is irreducible, though this factorisation is not necessarily unique. Two elements $a, b \in O_F$ are said to have a *common divisor* d if there exist factorisations of both a and b that include a term d.

We denote by $O_F\langle X\rangle$ the *O_F-module* that contains all sums of elements of the set $\{ax \mid a \in O_F, x \in X\}$, and again $O_F\langle X\rangle = O_F\langle E\rangle$.

3 Existence of orientations

In this section, we prove by contradiction that a point set cannot be the vertex set of two rectilinear graphs of different orientations. We first study the situation in the plane, which falls apart into two different cases, and then show how to extend the result to any dimension.

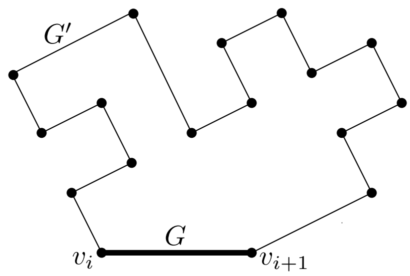

Figure 3: For any edge $\overline{v_i v_{i+1}}$ of G, there is also a path in G' connecting v_i to v_{i+1}.

Let V be a set of points in the plane, and let X and Y be the sets of all x-coordinates and y-coordinates of the points in V respectively. Assume w.l.o.g. that $\min(X) = \min(Y) = 0$. Let G be a connected axis-aligned rectilinear graph that has V as its vertex set. Let $s \in \mathbb{R}$ be any real positive real number. We will suppose that there exists another connected rectilinear graph G' that has V as vertex set and has slope s, and show that this leads to a contradiction.

Let $\mathbb{Q}(s)$ be the field generated by adjoining s to $\mathbb{Q}$, and consider the vector space $\mathbb{Q}(s)\langle X \cup Y\rangle$. Let $E = (e_1, \ldots, e_k)$ be a basis for this vector space. We now have $X, Y \subset \mathbb{Q}(s)\langle E\rangle$ so we can write $x_i = \sum_j x_{ij} e_j$ for all $x_i \in X$, and $y_i = \sum_j y_{ij} e_j$ for all $y_i \in Y$, where $x_{ij}, y_{ij} \in \mathbb{Q}(s)$.

We consider the following cases:

1. $[\mathbb{Q}(s) : \mathbb{Q}] < \infty$, s is algebraic over $\mathbb{Q}$. We consider this case in Section 3.1.
2. $[\mathbb{Q}(s) : \mathbb{Q}] = \infty$, s is transcendental over $\mathbb{Q}$. We consider this case in Section 3.2.

3.1 Algebraic slopes

When s is algebraic over $\mathbb{Q}$, we will first argue that we can assume the coordinates of the points in V have no common divisors. We know that all $x_{ij}, y_{ij} \in \mathbb{Q}(s)$.

Observation 2 *W.l.o.g. we can assume that all $x_{ij}, y_{ij} \in O_{\mathbb{Q}(s)}$ and that they do not have a common divisor.*

Proof: Each element of $\mathbb{Q}(s)$ can be written as a fraction p/q with $p, q \in O_{\mathbb{Q}(s)}$. Therefore, we can multiply all coordinates by q for each x_{ij} and y_{ij} to ensure all coordinates are in $O_{\mathbb{Q}(s)}$. Furthermore, each element of $O_{\mathbb{Q}(s)}$ can be written (possibly in multiple ways) as a product of irreducible elements. A set of elements $S \subset O_{\mathbb{Q}(s)}$ has a common divisor d if each element $s_i \in S$ can be written as $d \cdot r_i$ with $r \in O_{\mathbb{Q}(s)}$. In this case, we can take the set $R = \{r_i | s_i \in S\}$ instead. Since the sequences of irreducible elements are finite, this eventually results in a set without a common divisor. ⊠

As a result, we have $x_i, y_i \in O_{\mathbb{Q}(s)}\langle E\rangle$. Consider any pair of points $v, v' \in V$, and the horizontal and vertical distances Δx and Δy between them. These points are connected by a path $v_1, v_2, \ldots, v_m$ in G, where $v_1 = v$ and $v_m = v'$. Denote by $(\Delta x_i, \Delta y_i)$ the horizontal and vertical distance between v_i and v_{i+1}. Note that Δx_i and Δy_i are also in $O_{\mathbb{Q}(s)}\langle E\rangle$.

Lemma 1 *Given a set of points in the plane with a rectilinear axis-aligned graph on them, there cannot be another rectilinear graph with algebraic slope (other than 0) on the same point set.*

Proof: Assume for contradiction that there is such a graph G' with slope s. Then we know that there exists a path in G' from v_i to v_{i+1}, see Figure 3. This path uses edges with slope s or $-\frac{1}{s}$, so when following this path we move over distances (a, sa) or $(sb, -b)$. Since all point coordinates are in $O_{\mathbb{Q}(s)}\langle E\rangle$, we know that $a, b \in O_{\mathbb{Q}(s)}\langle E\rangle$. In total we move from v_i to v_{i+1} over a distance $(a_i + sb_i, sa_i - b_i)$ where $a_i, b_i \in O_{\mathbb{Q}(s)}\langle E\rangle$.

Since G is axis-aligned, every edge between two points v_i and v_{i+1} is either horizontal or vertical. If it is horizontal, we have:

$$\Delta y_i = sa_i - b_i = 0 \qquad \Delta x_i = a_i + sb_i = a_i + s^2 a_i = (1+s^2)a_i$$

If it is vertical, we have:

$$\Delta x_i = a_i + sb_i = 0 \qquad \Delta y_i = sa_i - b_i = -s^2 b_i - b_i = -(1+s^2)b_i$$

In other words, all Δx_i and Δy_i are multiples of $1+s^2$. Now write $\Delta x_i = \sum_j \Delta x_{ij} e_j$ and $\Delta y_i = \sum_j \Delta y_{ij} e_j$, and also write $a_i = \sum_j a_{ij} e_j$ and $b_i = \sum_j b_{ij} e_j$. Clearly $\Delta x_{ij}, \Delta y_{ij}, a_{ij}, b_{ij} \in O_{\mathbb{Q}(s)}$. Since the elements of E are linearly independent over $\mathbb{Q}(s)$, it follows that $\Delta x_{ij} = (1+s^2)a_{ij}$ for horizontal segments and $\Delta y_{ij} = -(1+s^2)b_{ij}$ for vertical segments for all i, j.

Now $s^2 \in \mathbb{Q}(s)$, so we can write $s^2 = p/q$ with $p, q \in O_{\mathbb{Q}(s)}$ and such that p and q do not contain a common irreducible factor in any factorisation. This means that, if $\overline{v_i v_{i+1}}$ is horizontal, we have:

$$\Delta x_{ij} = (1 + p/q)a_{ij} = (p+q)a_{ij}/q$$

And if it is vertical, we have:

$$\Delta y_{ij} = -(1 + p/q)b_{ij} = -(p+q)b_{ij}/q$$

Since q does not divide $p+q$ (unless it is 1), $p+q$ is in $O_{\mathbb{Q}(s)}$ and divides Δx_{ij} and Δy_{ij}. Since $\Delta x = \sum_{i,j} \Delta x_{ij} e_j$, it follows that $p+q$ divides Δx, and similarly that $p+q$ also divides Δy. So, any two points v and v' are a $O_{\mathbb{Q}(s)}\langle E\rangle$-multiple of $p+q$ away from each other in both horizontal and vertical direction, which contradicts the assumption that their coordinates have no common divisor. ⊠

3.2 Transcendental slopes

When s is transcendental, every element $w \in \mathbb{Q}(s)$ can be written in the form

$$w = \frac{\sum_{0 \le l \le h} w_l s^l}{\sum_{0 \le l' \le h'} w'_{l'} s^{l'}}$$

for some $h, h' \in \mathbb{N}$, and $w_l, w'_{l'} \in \mathbb{Q}$.

Observation 3 *W.l.o.g. we can assume that we can write:*

$$x_{ij} = \sum_{0 \le l \le h} x_{ijl} s^l \quad \text{and} \quad y_{ij} = \sum_{0 \le l \le h} y_{ijl} s^l$$

where $h \in \mathbb{N}$ and $x_{ijl}, y_{ijl} \in \mathbb{Z}$.

Proof: We can get rid of the expression below the division line of each element w, by multiplying it and all other elements by $\sum_{0 \le l' \le h'} w'_{l'} s^{l'}$. ⊠

Now $x_{ij}, y_{ij} \in \mathbb{Z}\langle s, \ldots, s^h\rangle$. Assume $h \ge 2$ (if it is smaller, just add some 0's to the descriptions of the coordinates). We now also know that $(1+s^2) \in \mathbb{Z}\langle s, \ldots, s^h\rangle$. We can assume w.l.o.g. that not all of x_{ij}, y_{ij} can be written as $(1+s^2)w$ for some $w \in \mathbb{Z}\langle s, \ldots, s^h\rangle$ (otherwise we could divide everything by $(1+s^2)$).

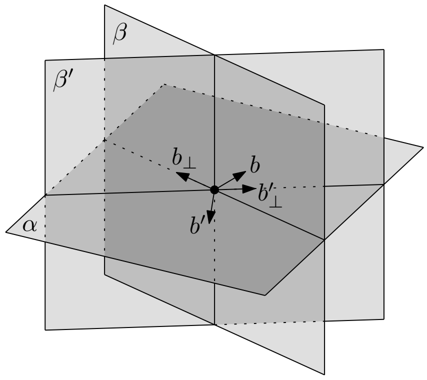

Figure 4: The plane α spanned by b and b', and the subspaces β and β' that project onto $b_\perp$ and $b'_\perp$. In the figure, the subspaces are planes since $d = 3$, but in general this is not the case.

Lemma 2 *Given a set of points in the plane with a rectilinear axis-aligned graph on them, there cannot be another rectilinear graph with transcendental slope on the same point set.*

Proof: Assume that such a graph G' with slope s exists. As in the proof of Lemma 1, we can again argue that for horizontal edges $\overline{v_i v_{i+1}}$ we have:

$$\Delta x_{ij} = (1 + s^2)a_{ij}$$

and for horizontal vertical segments we have:

$$\Delta y_{ij} = -(1 + s^2)b_{ij}$$

This time, however, $a_{ij}, b_{ij} \in \mathbb{Z}\langle s, \ldots, s^h\rangle$. This clearly contradicts our assumption. ⊠

We can summarise Lemmata 1 and 2 in the following result:

Theorem 1 *Given a set of points in the plane, there can be at most one orientation that allows a connected rectilinear graph that has these points as its vertices.*

3.3 Higher dimensions

We now show how to extend the result to any dimension. Let V now be a set of n points in $\mathbb{R}^d$. Assume that V is d-dimensional: it is not contained in any subspace of $\mathbb{R}^d$.

Let G be a connected rectilinear graph on V and let $B = b_1, \ldots, b_d$ be its orientation.

Lemma 3 *Any other rectilinear connected graph G' on V must have the same orientation as G.*

Proof: By our assumption $\mathbb{R}^d$ is the smallest space that contains V, hence any graph on V has dimension d.

Assume for a contradiction that there exists a rectilinear connected graph G' on V with a different orientation B'. This means that there exists an edge $e \in G$ and an edge $e' \in G'$ such that e is not parallel nor perpendicular to e'. Let $b \in B$ be the direction of e, and $b' \in B'$ be the direction of e'. Let α be the plane spanned by b and b'. Let β be the subspace perpendicular to b, and β' the subspace perpendicular to b'. Clearly $\alpha \perp \beta$ and $\alpha \perp \beta'$. Figure 4 depicts this situation.

Consider the projection π_α of the set of points V to α along a vector e_α perpendicular to α. Let G_α be the graph obtained by projecting G to α. Namely, the set of vertices of G_α is the set of points $\pi_\alpha(V)$ and the set of edges is $\{e' \mid e' = \pi_\alpha(e), e \in G, e' \text{ is not a point}\}$. Thus any connected pair of vertices of G is represented by either a connected pair of connected vertices or by a single vertex in G_α. Thus if G is connected, so is G_α.

Now consider an edge $e \in G$. If e has direction b, then its projection does not change it: $\pi_\alpha e = e$. On the other hand, if e has another direction, it must be perpendicular to b, because G is rectilinear. Therefore, $e \subset \beta$. since β is perpendicular to α, e will project to an edge with direction $b_\perp$. We conclude that G_α is a rectilinear graph on $\pi_\alpha(V)$.

Exactly the same reasoning can be done for G'_α and an edge $e' \in G'$ to show that G'_α is a rectilinear connected graph on $\pi_\alpha(V)$. Thus we have two rectilinear connected graphs G_α and G'_α such that G_α contains edges parallel (or perpendicular) to b and G'_{alpha} contains edges parallel (or perpendicular) to b'. Pince b and b' have different slopes on α, this is impossible by Theorem 1. ⊠

We can summarise:

Theorem 2 *Given a set of points in $\mathbb{R}^d$, there can be at most one orientation that allows a connected rectilinear graph that has these points as its vertices.*

4 Finding the right orientation

In this section, we discuss the algorithmic question: given a set of points in the plane, can we *find* a slope s such that the graph with edges of slope s and $-1/s$ is connected?

A trivial approach takes $O(n^2 \log n)$ time. Consider all pairs of points and the line segment connecting them. Sort those segments by slope. For each slope that has at least $n-1$ segments (together with its perpendicular slope), test whether they form a connected graph. The most expensive step here is sorting the directions. A long-standing open problem is whether this can be done any faster than in $O(n^2 \log n)$ time [1, 11, 20]. However, we don't really need to sort all directions, which allows a slightly faster algorithm.

Consider the problem in dual space: we map a point (a, b) in primal space to the line $y = ax + b$ in dual space. Our set of primal points becomes a set of dual lines, our primal slope a dual x-coordinate, and two primal points are connected by a line segment of slope s if the two corresponding dual lines intersect at x-coordinate s. Figure 5 shows an example point set with a connected rectilinear graph, and its dual analogue. This way of viewing the problem suggests that to solve it, we just have to sweep two vertical lines (at $x = -1/s$ and $x = s$) simultaniously over the dual plane, and keep track of the intersection points on those lines. We can compute the arrangement of the lines in $O(n^2)$ time, using a topological sweep algorithm [8]. However, we cannot just sweep the vertical lines and treat all encountered intersection points in order, since sorting them faster than $O(n^2 \log n)$ time is the same open problem as mentioned above.

Therefore, note that if we want to find a connected graph, each point should be connected to at least one other point. Let p be a random point (line in dual space). Since it must be connected to some other point, we only have to consider slopes that involve p, of which there are only a linear number. In dual space, this means we only need to stop at the events along line p. At each such event, we need to identify all arrangement vertices on the two vertical sweep lines, and check whether they form a connected (dual) graph.

In order to find the crossings quickly, we need to maintain the sorted order in which the lines cross the sweeplines. If we know the sorted order at candidate slope (x-coordinate) s_i, and the next candidate is at s_{i+1}, let v_i be the number of vertices of the arrangement between lines $x = s_i$ and $x = s_{i+1}$. We will traverse the arrangement from s_i to s_{i+1} and update the sorted order, which takes $O(n + v_i)$ time. Since the total number of vertices is $O(n^2)$ and the total number of candidate slopes is $O(n)$, this takes $O(n^2)$ time in total.

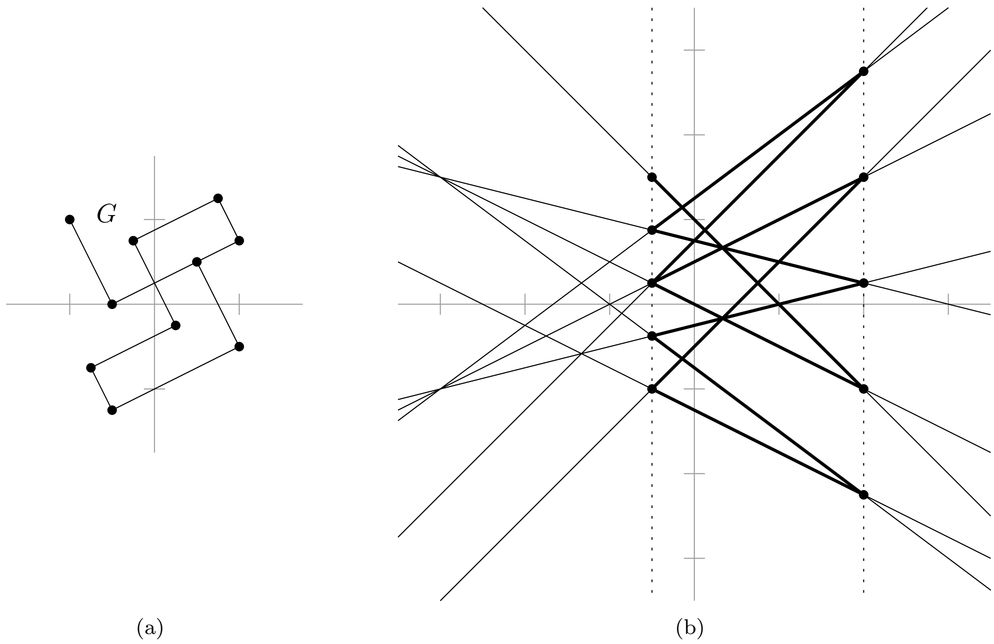

Figure 5: (a) A rectilinear graph in the plane with edges of slope 2 and $-\frac{1}{2}$. (b) The dual, with intersections at the vertical lines $x = 2$ and $x = -\frac{1}{2}$.

Once we have all vertices on a candidate pair of vertical lines, we have to check whether they form a connected graph. This can be done in linear time using a simple depth-first search, if we store the appropriate pointers in the arrangement. Since there are a linear number of candidate slopes, this also takes $O(n^2)$ time in total.

Theorem 3 *Given a set of points V, we can determine if there exists a slope s (and return it if it exists) such that there is a rectilinear graph with V as vertex set and slope s in $O(n^2)$ time.*

4.1 Non-crossing graphs

If the goal is to find a slope that allows a *planar* connected rectilinear graph (a simple polygon, for example), this is NP-hard. Since we know that there is at most one such slope, we can use the algorithm in the previous section to find it. Then we can compute the maximal rectilinear graph with this slope. However, now we need to decide whether this graph has a planar connected subgraph. This was proven NP-complete by Jansen and Woeginger [14].

5 Conclusion

We have proven that, given a point set in $\mathbb{R}^d$, there can exist at most one orientation (up to trivial rotations) such that the maximal rectilinear graph on the points in that orientation is connected. However, to find this orientation remains an interesting challange. We have shown that this can be done in $O(n^2)$ time for a 2-dimensional point set, but we see no reason why this should be the right bound. As soon as we require a solution to be planar though, the problem becomes NP-hard. Also, the algorithmic question in higher dimensions is still open.

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