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Abstract. Geometric intersection graphs are graphs determined by the intersections of certain geometric objects. We study the complexity of visualizing an arrangement of objects that induces a given intersection graph. We give a general framework for describing classes of geometric intersection graphs, using arbitrary finite base sets of rationally given convex polygons and rationally-constrained affine transformations as similarity maps. We prove that for every class of intersection graphs that fits this framework, the graphs in this class have a representation in integers using only polynomially many bits. Consequently, the recognition problem of these classes is in NP (and thus usually NP-complete). We also give an exponential algorithm to find suitable plane representations ('drawings'), if a graph class fits the framework.

1 Introduction

A *geometric intersection graph* is the intersection graph of a finite set of geometric objects. That is, each vertex corresponds to one of the objects and there is an edge between two vertices if and only if their corresponding objects intersect. The set of objects is a *representation* of the graph. Class of geometric intersection graphs are obtained if one allows objects that are similar to certain base objects specific for that class only. To visualize a geometric intersection graph, drawing the objects of a representation can be more informative than drawing the graph itself. Therefore we study the complexity of representations of geometric intersection graphs. In particular, we consider the following problems: do representations in polynomial space exist, and how can drawings be effectively found?

Understanding Geometric Intersection Graphs Geometric intersection graphs arise naturally in many applications. They are used e.g. in modeling wireless communication networks, where the geometric objects model the transmission ranges of the different devices in the network. This has lead to the study of the well-known class of (unit) disk graphs and several other classes of intersection graphs, such as box graphs [15, 29, 32, 38, 39].

Geometric intersection graph models are normally based on the use of homothetic copies of a single base object, thus allowing translations and scalings of this object only. Certainly the notion of similarity is broader, and one may want to take a larger variety of base objects into account in defining a class. To obtain a thorough understanding of geometric intersection graphs, we aim at a more general conceptual model.

Definition 1. *A* signature *is any 2-tuple $\mathcal{P} = \langle S, T \rangle$ where:* (a) *S is a finite base set of geometric objects in the plane: $S = \{o_1, \cdots, o_m\}$, with each object in the base set containing the origin, and* (b) *T map every object $o \in S$ to a finite set of* similarity templates *that determine how objects similar to o can be obtained.*

* Extended version of [41], presented at the 18th Int. Symposium on Graph Drawing, Konstanz, 2010.

A similarity template is any family of similarity transforms of some kind, e.g. a rotation over some angle, followed by an arbitrary translation. More generally, a similarity template t is any parametric family of bi-continuous functions $t(w_1, \cdots, w_k) : \mathbb{R}^2 \to \mathbb{R}^2$ (for some fixed k) which are all shape-preserving in some sense, with the w_i's ranging over e.g. $\mathbb{R}_+$.

Definition 2. *Given a signature $\mathcal{P} = \langle S, T \rangle$, a graph G is called a* $\mathcal{P}$-intersection graph *if it is the intersection graph of a finite set of objects $\mathcal{O}_1, \cdots, \mathcal{O}_n$, where every $\mathcal{O}_i$ ($1 \leq i \leq n$) is similar to an object $o \in S$, i.e. obtained from o using a transformation that fits a similarity template from $T(o)$.*

Problem Definitions In order to visualize $\mathcal{P}$-intersection graphs, it is crucial to know the complexity of their representation. In particular, we want to know whether representations exist that require only polynomial space, i.e. polynomially many bits. Let us assume from now on that all objects we consider are fully specified, both for localizing and drawing them, by only finitely many parameters.

Definition 3. *(i) A $\mathcal{P}$-intersection graph with n vertices is said to be* polynomially represented *(using polynomial p), if it is the intersection graph of a finite set of objects $\mathcal{O}_1, \cdots, \mathcal{O}_n$, where every $\mathcal{O}_i$ ($1 \leq i \leq n$) is similar to an object in S according to an allowed template of T, and has all its specifying parameters equal to rationals $\frac{a}{b}$ with $|a|, |b| \leq 2^{p(n)}$.*

(ii) A class $\mathcal{C}$ of $\mathcal{P}$-intersection graphs is said to be polynomially represented *if there is a polynomial $p = p(n)$ such that every graph in $\mathcal{C}$ is polynomially represented using p.*

Given an arbitrary graph G, we like to determine whether it is a geometric intersection graph and be able to visualize it by a set of objects in the plane if it is and can be done in a feasible way. This leads to the following problems.

$\mathcal{P}$-Intersection Graph Recognition
Given a graph G, decide whether G is a $\mathcal{P}$-intersection graph.

$\mathcal{P}$-Intersection Graph Construction (or -Visualization)
Given a graph G that is known to be a $\mathcal{P}$-intersection graph, construct ('draw') a representation of G as the intersection graph of a set of objects in the plane according to signature $\mathcal{P}$.

We consider the complexity of both problems for $\mathcal{P}$-intersection graphs. In particular we consider whether or not such graphs have a polynomial representation, for any signature $\mathcal{P}$.

Previous Work The complexity of the recognition problem and the size of representations (in bits) have been studied for many classes of geometric intersection graphs. An overview of some of the most prominent results in this area is given in Table 1.

The recognition problem for geometric intersection graphs can be nontrivial. For example, for disk graphs, the problem is algorithmically decidable ([38], Sect. 4.3), known to be NP-hard [24] and in PSPACE [15, 25], but it is open whether the problem actually is in NP. This holds even for the more restricted class of unit disk graphs.

The complexity or size of representing an intersection graph on the integer grid is an equally challenging problem. A first question is whether various classes of geometric intersection graphs actually have a representation using only rational coordinates. This was shown e.g. for the very general class of intersection graphs of so-called scalable objects [39]. If one

Graph Class	Objects	Recognition	Repr.	Reference
interval	intervals	linear	poly	[1]
unit interval	unit intervals	linear	poly	[7, 10]
(unit) circular-arc	(unit) arcs	linear	poly	[16, 27, 30]
(unit) disk	(unit) disks in $\mathbb{R}^2$	NP-h, $\in$PSPACE	exp.	[2, 24, 15, 31]
string	simple curves in $\mathbb{R}^2$	NP-c	exp.	[19–21, 15, 36]
tolerance	intervals w/tolerances	$\in$ NP	poly	[14]
segment	line segments in $\mathbb{R}^2$	NP-h	exp.	[23]
planar	line segments in $\mathbb{R}^2$	linear	poly	[5, 9]
box (rectangle)	rectangles	NP-c	poly	[22, 28]
unit square	unit squares	NP-c	poly	[2, 8]
square	squares	NP-h, $\in NP$	*poly*	*
max-tolerance	semi-squares	NP-h, $\in NP$	*poly*	[17], *
polygon intersect.	homoth. conv. polygons	NP-h	-	[26, 34]
polygon intersect.	rat. repr. conv. polygons	$\in NP$	*poly*	*
convex intersect.	convex sets $\subset \mathbb{R}^2$	NP-h, $\in$PSPACE	exp.	[19, 23, 34]

Table 1. Some classes of geometric intersection graphs. The first column gives the graph class, the second column the objects in representations of the graphs in the class, the third gives the complexity of the recognition problem, the fourth the size of a representation of the intersection graph (polynomial or exponential), and the fifth gives references where the results can be found. The contributions of this paper are marked in italics and we use * to refer to the current paper.

however only allows objects to touch and not to overlap (so-called *contact graphs*), this is no longer guaranteed [4].

The second question is whether the rationals or integers involved in a representation can be stored using polynomially many bits. For (unit) disk graphs, this question was answered negatively only recently [31].

Our Results The given framework allows us to prove various structural properties of geometric intersection graphs. We apply the framework to define classes of generalized geometric intersection graphs that use arbitrary, finite non-empty sets of rationally given convex polygons as base sets (see Section 2). We will use arbitrary rationally-constrained affine similarity transformations in the templates.

The framework allows for several topological considerations. For example, in Section 3 we show that it is irrelevant whether the objects from which the intersection graphs are built, i.e. convex polygons in this case, are open or closed.

As a main result, we prove that any (generalized) convex polygon intersection graph in our framework has a polynomial-size representation on the integer grid. This contrasts the known fact that the intersection graphs of arbitrary convex sets in $\mathbb{R}^2$ may require exponentially-sized representations in the worst case [34]. The main result is presented in Section 4.

The polynomial representation result enables us to settle, in a very general way, the question left open by the recent NP-hardness proof of the recognition problem for intersection graphs of homothetic copies of a single convex polygon [26], namely whether this problem is in NP[3]. The results of this paper immediately imply that this problem is indeed in NP and thus NP-complete, even in the generalized case with any finite number of rationally given base polygons. Moreover, we give an exponential algorithm to determine whether a given graph is

[3] NP-completeness in this case was stated in Theorem 20 of [34] but only NP-hardness was proved there. However, a proof of membership in NP was reportedly sketched during the oral defense of [34], based on an extension of Theorem 1.1 (ii) sub (a) of [23], cf. [35].

an intersection graph within the above framework. The algorithm is constructive and returns a visualization of the arrangement of objects in a representation of the given graph, if such a representation exists. For the case of intersection graphs of homothetic copies of a single convex polygon the problem was proved to be in PSPACE in [23].

Finally, by applying the same techniques and ideas, one can prove for instance that max-tolerance graphs and contact graphs of homothetic convex polygons have polynomial representations, and that their recognition problems thus are in NP. Further applications are given in Section 5.

2 $\mathcal{P}$-intersection graphs

$\mathcal{P}$-intersection graphs give a very general framework for studying geometric intersection graphs. We use this framework to consider $\mathcal{P}$-intersection graphs for signatures $\mathcal{P} = \langle S, T \rangle$, where S is any finite non-empty set of (closed and) rationally given convex base polygons, i.e. of convex polygons specified by means of vertices with rational coordinates.

We also tune the choice of similarity templates. Similarity templates were described as being (infinite) parametric families of bi-continuous functions $t = t(\bar{w}) : \mathbb{R}^2 \to \mathbb{R}^2$ which preserve shapes according to some notion of similarity. We may even go one step further and say that similarity templates must be *smooth* in the sense that images of base objects under $t(\bar{w})$ and $t(\bar{z})$ are 'almost equal' if $\|\bar{w} - \bar{z}\|$ is 'small'. Let $D \subset \mathbb{R}^2$ be any (open) bounded set, in particular any such set that encloses the base objects.

Definition 4. *A similarity template $t = t(\bar{w})$ is called* smooth on D *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\bar{w}, \bar{z}, x, y$ with $\|\bar{w} - \bar{z}\| < \delta$ and $(x, y) \in D$ one has $\|t(\bar{w})(x, y) - t(\bar{z})(x, y)\| < \epsilon$.*

A similarity template will be called *smooth* if it is smooth on every D. Smoothness is a desirable property of sets of similarity transformations, as it preserves continuity over the base polygons.

For $\mathcal{P} = \langle S, T \rangle$-intersection graphs with S any finite non-empty set of convex base polygons as above, we restrict ourselves to similarity templates that are families of *linearly parameterized affine transformations* over the rationals as follows.

Definition 5. *A* linear similarity template $t = t_{\alpha,\beta,\gamma,\delta}$ *is a family of affine transformations of the form $x \to u + Q(v)x$, where: (i) $\alpha, \beta, \gamma, \delta$ are rationals such that $\alpha\delta - \beta\gamma \neq 0$, (ii) $u = (u_1, u_2)$ is any 2-dimensional vector, and (iii) $Q(v) = v\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$ is a 2×2 matrix with factor v satisfying $v > 0$.*

A linear similarity template has two parameters: the vector u (the translation vector) and the scalar v (the scaling factor of the distortion matrix). Neither of them needs to be rational. The constraint $v > 0$ keeps $Q(v)$ nonsingular and thus guarantees that template mappings are always topological isomorphisms.

Applying linear similarity transformations $x \to u + Q(v)x$ to objects o is the same as applying regular homothetic transformations to a base of new objects $Q(o)$. However, the framework gives us the conceptual generality we want. It allows us to vary the assignment of similarity templates while keeping the set of base polygons fixed, which is attractive in modeling applications . The essential generalization is that the number of distinct base polygons can be arbitrary (but finite).

Lemma 1. *Linear similarity templates are smooth.*

Proof. Consider any linear similarity template $u + Q(v)x$, with u and v varying and Q fixed. Let D be any bounded set, with $g_D \in \mathbb{R}$ such that $\|x\| \leq g_D$ for all $x \in D$, and $\epsilon > 0$. Note that $\|(u + vQx) - (u' + v'Qx)\| \leq \|u - u'\| + \|v - v'\|\|Q\|\|x\| \leq \|u - u'\| + \|v - v'\|g_D\|Q\|$. By choosing $\delta = \frac{\epsilon}{1+g_D\|Q\|}$, smoothness on D follows. □

Note: From now on all similarity templates we consider are assumed to be linear.

An affine transformation $u + Q(v)x \in t$ where t is any template assigned to a polygon in S by T and v a scalar, is called a $\mathcal{P}$*-transformation*. Many familiar types of transformations (combined with scaling) can occur as $\mathcal{P}$-transformations:

- *shift*: $\alpha = \delta = 1$, $\beta = \gamma = 0$, $v = 1$.
- *scaling*: $\alpha = \delta = 1$, $\beta = \gamma = 0$ (enlarging with $v \geq 1$, shrinking with $v < 1$).
- *skewed scaling*: $\beta = \gamma = 0$.
- *horizontal shear*: $\alpha = \delta = 1$, $\beta \neq 0$, $\gamma = 0$, $v = 1$.
- *vertical shear*: $\alpha = \delta = 1$, $\beta = 0$, $\gamma \neq 0$, $v = 1$.
- *rotation*: $\alpha = \delta \approx \cos\phi$, $\beta = -\gamma \approx \sin\phi$, $v = 1$.
- *reflexion*: $\alpha = -\delta \approx \cos\phi$, $\beta = \gamma \approx \sin\phi$, $v = 1$.
- *mirroring*: $\alpha = \delta = -1$, $\beta = \gamma = 0$, $v = 1$.

T may assign different sets of templates to different objects, without any dependency between them. Compositions of standard templates i.e. of the transformations they represent are allowed provided these are represented again in T.

Given a template t and $v < v'$, the transformation of a base polygon o using $u + Q(v)x \in t$ is a convex polygon strictly contained in its transformation obtained using $u + Q(v')x \in t$. In fact, the latter polygon is the enlargement of the former by a factor $\frac{v'}{v}$. We often write Q for $Q(v)$ from now on.

Definition 6. *Given a signature $\mathcal{P} = \langle S, T \rangle$, an object $\mathcal{O}$ is said to be* similar *to an object $o \in S$ if $\mathcal{O}$ can be obtained from o by applying a $\mathcal{P}$-transformation to it.*

Lemma 2. *The notion of similarity is well-founded, i.e. it is decidable, for any given signature $\mathcal{P} = \langle S, T \rangle$ and convex polygon $\mathcal{O}$, whether $\mathcal{O}$ is similar to a base polygon in S under $\mathcal{P}$-transformation.*

Proof. Consider any convex polygon $\mathcal{O}$ and suppose it is similar to base polygon $o \in S$. Let the $\mathcal{P}$-transformation that maps o to $\mathcal{O}$ be $(u_1, u_2) + Qx$ with $Q = v\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$. Now note that u_1, u_2, v are uniquely determined. To see this, let $(r_1, r_2), (s_1, s_2)$ be two vertices of $\mathcal{O}$ with $r_1 \neq s_1$, and let $(d_1, d_2), (e_1, e_2)$ be the vertices of o mapped onto them under $(u_1, u_2) + Qx$. Then:

$$u_1 + v(\alpha d_1 + \gamma d_2) = r_1$$
$$u_1 + v(\alpha e_1 + \gamma e_2) = s_1$$

and v and hence u_1 (and u_2) follow by straight elimination. Consequently, to decide whether $\mathcal{O}$ can be obtained from a base polygon under $\mathcal{P}$-transformation it suffices to consider every base polygon o and every similarity template t assigned to it under T, and for every choice of vertices determine the parameters of the template as above, and verify whether the resulting $\mathcal{P}$-transformation indeed maps o to $\mathcal{O}$ (by checking that all vertices are mapped correctly and in order). □

We will need to scale polygons in several of our later arguments. Consider the (polynomial) representation of any $\mathcal{P}$-intersection graph and an arbitrary object $\mathcal{O}$ that occurs in it.

Lemma 3. *Let $\mathcal{P} = \langle S, T \rangle$ be as above, and let $\mathcal{O}$ be similar to $o \in S$.*
(i) For every $\mu > 0$, the scaling of $\mathcal{O}$ by a factor μ is similar to o as well.
(ii) If $\mathcal{O}$ is polynomially represented and $\mu > 0$ a polynomially represented rational, then the scaling of $\mathcal{O}$ by factor μ is also polynomially represented.
(iii) If $\mathcal{O}$ is polynomially represented and $\rho > 0$ a polynomially represented rational, then there is an enlargement of $\mathcal{O}$ by an additive margin δ with $0 < \delta < \rho$ that is again polynomially represented. This holds similarly for reductions.

Proof. *(i)* Let $\mathcal{O}$ be obtained from o by applying $\mathcal{P}$-transformation $u + Qx$ from template t. Now $u + \mu Qx$ is a $\mathcal{P}$-transformation in t as well, for every $\mu > 0$. The result follows.

(ii) Let $\mathcal{O}$ be polynomially represented. Let the $\mathcal{P}$-transformation that maps $o \in S$ to $\mathcal{O}$ be $(u_1, u_2) + Qx$ with $Q = v\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$. In the proof of Lemma 2 it was shown that v, u_1 and u_2 are uniquely determined in terms of vertices of o and $\mathcal{O}$. As (d_1, d_2) and (e_1, e_2) are fixed rationals as part of the specification of o and $(r_1, r_2), (s_1, s_2)$ are polynomially represented, it easily follows that u_1, u_2, v are polynomially represented as well. If $\mu > 0$ is polynomially represented, then $u + \mu Qx$ applied to o makes the resulting polygon, the scaling of $\mathcal{O}$ by μ, polynomially represented again. The corresponding $\mathcal{P}$-transformation conforms to the original template.

(iii) Given $\mathcal{O}$, let $o \in S$ and $(u_1, u_2) + Qx$ be as above. Write $Q = vR$ with $R = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$. Now consider the enlargement of $\mathcal{O}$ to the convex polygon obtained from o by using $(u_1, u_2) + (v + \delta)Rx = (u_1, u_2) + vRx + \delta Rx$, for some $\delta > 0$ to be determined. If x traverses the boundary of o, its image $(u_1, u_2) + vRx$ traverses the boundary of $\mathcal{O}$ and δRx is the added margin to it. Note that, as x moves around the boundary of o, quasi-convexity of $||Rx||$ implies that its value is maximized in one the vertices of o, say (d_1, d_2). Thus

$$0 < ||\delta Rx|| = \delta ||Rx|| \leq \delta ||R(d_1, d_2)|| \leq \rho$$

provided $\delta \leq \frac{\rho}{||R(d_1, d_2)||}$. As the coefficients of R and d_1, d_2 are fixed rationals and ρ is a polynomially bounded rational, $\delta > 0$ can easily be chosen rational and polynomially bounded so this holds. Observe also that, if $(u_1, u_2) + vRx$ is an allowed similarity transformation, then so is $(u_1, u_2) + (v + \delta)Rx$, as $v + \delta > v > 0$.

By the same argument, one can obtain a polynomially represented polygon that is a reduction of $\mathcal{O}$ by a nonzero reducing margin bounded by ρ, by a similarity transformation of the same template, provided that δ is chosen so $v - \delta > 0$ as well. This can always be achieved by imposing the additional constraint that $\delta \leq \frac{1}{2}v$. $\square$

Given any $\mathcal{P}$-intersection graph, it is important to note that its representations allow scaling (i.e. of the entire configuration), by any scaling factor > 0.

We will later need the following observation. Because the matrices $Q(v)$ in similarity templates are nonsingular, $\mathcal{P}$-transformations map convex polygons 1-1 onto convex polygons. Thus vertices of the latter are images of vertices of the former and edges of the latter are images of edges of the former, and vice versa. Recall from convex analysis that a convex polygon can be given by its *defining inequalities*.

Lemma 4. *(i) Let Q be a nonsingular 2×2 matrix, and o a plane convex polygon containing the origin. When Q transforms o and $\det Q > 0$, then defining inequalities are mapped to*

defining inequalities with preservation of the inequality sign. If $\det Q < 0$*, the inequality signs are reversed.*
(ii) A $\mathcal{P}$*-transformation* $(u_1, u_2) + Qx$ *with* $Q = v\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$ *maps the line* $ax + by + c = 0$ *onto the line* $(a\delta - b\beta)x + (b\alpha - a\gamma)y + (\alpha\delta - \beta\gamma)vc - (a\delta - b\beta)u_1 - (b\alpha - a\gamma)u_2 = 0$.

Proof. (i) Let $Q = \left(\begin{smallmatrix} e_1 & f_1 \\ e_2 & f_2 \end{smallmatrix}\right)$ be nonsingular. Qx maps points to points and lines to lines. In particular, given a line defined by $ax + by + c = 0$, this line is mapped 1-1 onto the line of points (x', y') with the property that $Q^{-1}\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = \frac{1}{\det Q}\left(\begin{smallmatrix} f_2x' - f_1y' \\ -e_2x' + e_1y' \end{smallmatrix}\right)$ satisfies $ax + by + c = 0$. Thus line $ax + by + c = 0$ is mapped to line $(af_2 - be_2)x + (be_1 - af_1)y + (\det Q)c = 0$.

Let $ax + by + c = 0$ be an edge of o. Because the origin is $\in o$, the edge leads to a defining inequality $ax + by + c \leq 0$ (if $c < 0$) or $ax + by + c \geq 0$ (if $c > 0$) of o, respectively. Now apply Q. As the origin is contained in Qo, the $\leq$- and $\geq$-signs in the defining inequalities of Qo remain as they were for o or are reversed, depending on the sign of $\det Q$.

(ii) This follows by applying the above calculation and extending it to $x \to u + Qx$, with u and Q as specified. □

Lemma 4 enables one to determine exactly how the defining inequalities of a base polygon are transformed under a $\mathcal{P}$-transformation.

3 Open versus closed objects

Let $\mathcal{P} = \langle S, T \rangle$ be a signature as above. What happens if we let S consist of *open* convex polygons instead of closed ones? In the case of disk graphs, it is known that taking open or closed disks does not change the class of intersection graphs [40]. In [39] this was shown for the intersection graphs of all 'scalable' geometric objects. Also, it was shown that in the case of disk and unit disk graphs, even polynomial representation is preserved.

We consider here the case of $\mathcal{P}$-intersection graphs, emphasizing polynomial representation. We show that for $\mathcal{P}$-intersection graphs the closed and open cases are again equivalent. We use the following facts.

Lemma 5. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be two disjoint convex polygons in the plane, both having nonempty interior. The (shortest) distance between* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *is realized as the distance between a vertex of one polygon and an edge of the other.*

Proof. This follows by a simple extension of the proof of Lemma 2.1 in [12]. □

Lemma 6. *Let* a, b, c, v_1, v_2 *be rationals with numerator and denominator bounded in absolute value by* q *for some* $q > 0$*. If the following fraction is* $\neq 0$*, then*

$$\frac{|av_1 + bv_2 + c|}{\sqrt{a^2 + b^2}} \geq \frac{1}{2q^5}.$$

Proof. Write each a, b, c, v_1, v_2 in rational form $\frac{\alpha}{\beta}$ with $|\alpha|, |\beta| \leq q$ and substitute this into the given fraction. This leads to a formula of the form

$$\frac{|w_1|}{|z_1z_2z_3z_4z_5|} \cdot \frac{|z_1z_2|}{\sqrt{z_6^2z_7^2 + z_8^2z_9^2}}$$

with w_1 and z_i $(1 \leq i \leq 9)$ integers, and the z_i all numerators or denominators of the given numbers (or 1). If the fraction is not 0, then $|w_1| \geq 1$. By using that $|z_i| \leq q$ for every i $(1 \leq i \leq 9)$, the lower bound follows. □

We prove the equivalence of the closed and open cases for $\mathcal{P}$-intersection graphs in two steps.

Lemma 7. *Every polynomially represented $\mathcal{P}$-intersection graph using a base set of closed convex polygons can be obtained as a polynomially represented $\mathcal{P}$-intersection graph using a base set of open convex polygons.*

Proof. Let $\mathcal{P} = \langle S, T \rangle$ be a signature, $S = \{o_1, \cdots, o_m\}$ its base set of closed convex polygons. Let G be the intersection graph defined by the objects $\mathcal{O}_1, \cdots, \mathcal{O}_n$, where $\mathcal{O}_i$ $(1 \leq i \leq n)$ is similar to object $o_{s_i} \in S$ for some $s_i \in \{1, \cdots, m\}$ and only has vertices with rational coordinates $\frac{a_i}{b_i}$ with $|a_i|, |b_i| \leq 2^{p(n)}$, for some fixed polynomial $p(n)$.

Let $\mathcal{P}' = \langle S', T \rangle$ be the signature obtained from $\mathcal{P}$ in which every (closed) base polygon o is replaced by its interior o°. G can now be viewed as the intersection graph of the objects $\mathcal{O}_1^\circ, \cdots, \mathcal{O}_n^\circ$, provided that no intersections of the polygons are lost by restricting to the interiors. Intersections are lost precisely when there are (closed) polygons $\mathcal{O}_i$ and $\mathcal{O}_j$ that touch. We show that we can slightly alter the polygons $\mathcal{O}_i$ so this does not occur, while preserving G as the intersection graph.

Suppose any of the closed polygons, say $\mathcal{O}_i$, touches several of the other polygons $\mathcal{O}_j$. By enlarging $\mathcal{O}_i$ by a small but nonzero margin μ, we can eliminate the touchings and make $\mathcal{O}_i$ have a nontrivial overlap with each $\mathcal{O}_j$. However, in enlarging it (and enlarging all other polygons for which a similar step is carried out) we must make sure that no spurious intersections with any objects $\mathcal{O}_r$ disjoint from $\mathcal{O}_i$ are created. Suppose $\mathcal{O}_i$ and $\mathcal{O}_r$ do not intersect (and thus are fully disjoint). By Lemma 5, the distance between the two polygons is realized by the distance between a vertex, say $v = (v_1, v_2)$ of one of them and the line of an edge, say $ax + by + c = 0$ of the other. By common analytic geometry, this distance is given by:

$$\frac{|av_1 + bv_2 + c|}{\sqrt{a^2 + b^2}}.$$

Now the numerators and denominators of v_1, v_2 are $\leq q = 2^{p(n)}$. Also, $ax + by + c = 0$ connects two vertices of a polygon in the set, thus two points with rational coordinates with numerators and denominators $\leq q$. It follows that a, b, c are all rational, with numerators and denominators $\leq 4q^4$. By Lemma 6 the distance is thus at least $\frac{1}{dq^{20}}$ for some constant $d > 0$. Hence, if we enlarge $\mathcal{O}_i$ by a nonzero margin of $\mu \leq \frac{1}{3dq^{20}}$, then disjointness with every disjoint $\mathcal{O}_r$ is maintained (taking into account that the latter may also be enlarged by the same factor).

As the margin μ is independent of the specific $\mathcal{O}_i$ chosen, the enlargement can be carried out simultaneously for all polygons. For every $\mathcal{O}_i$, it preserves all the intersections with other polygons, introduces no new ones, and has the effect that every polygon $\mathcal{O}_i$ that it touched now has a nontrivial overlap with it as well (and thus their interiors overlap). By Lemma 2, the enlargement of every $\mathcal{O}_i$ by a nonzero margin at most μ can be achieved while preserving similarity and polynomial representation. Thus, G is a $\mathcal{P}'$-intersection graph and polynomially represented. □

Lemma 8. *Every polynomially represented $\mathcal{P}$-intersection graph with a base set of open convex polygons can be obtained as a polynomially represented $\mathcal{P}$-intersection graph with a base set of closed convex polygons.*

Proof. Let $S = \{o_1, \cdots, o_m\}$ be the base set of open convex polygons. Let G be the intersection graph defined by objects $\mathcal{O}_1, \cdots, \mathcal{O}_n$, where $\mathcal{O}_i$ $(1 \le i \le n)$ is similar to object $o_{s_i} \in S$ for some $s_i \in \{1, \cdots, m\}$ and only has vertices with rational coordinates $\frac{a_i}{b_i}$ with $|a_i|, |b_i| \le p(n)$, for some fixed polynomial $p(n)$. Let $\mathcal{O}_i$ be obtained by applying an allowed affine transformation A_i to o_{s_i}.

Let $\mathcal{P}' = \langle S', T\rangle$ be the signature obtained from $\mathcal{P}$ in which every (open) base polygon o is replaced by its closure $\overline{o}$. $\mathcal{P}'$ now has a base set of closed convex polygons. Clearly $A_i(x)$ maps $\overline{o_{s_i}}$ to $\overline{\mathcal{O}_i}$ and we can view G as the intersection graph of the objects $\overline{\mathcal{O}_1}, \cdots, \overline{\mathcal{O}_n}$, provided we can avoid the 'unwanted' intersections we get from pairs of open polygons $\mathcal{O}_i$ and $\mathcal{O}_j$ that touch.

Suppose any of the open polygons, $\mathcal{O}_i$, touches several other polygons $\mathcal{O}_j$. By reducing $\mathcal{O}_i$ by a small but nonzero margin μ, we can eliminate the touchings. However, we must make sure that in doing so existing intersections with objects $\mathcal{O}_r$ are preserved. Thus, suppose $\mathcal{O}_i$ and $\mathcal{O}_r$ intersect. Their intersection is an open, thus nontrivial convex polygon, say $\mathcal{C}_{i,r}$. We note that the vertices of $\mathcal{C}_{i,r}$ are easily computed and are either a vertex of $\mathcal{O}_i$, a vertex of $\mathcal{O}_r$, or the intersection of two edges of these two polygon. In all cases the vertices are (rational and) polynomially represented again.

Let $G_{i,r}$ be the geometric centroid of $\mathcal{C}_{i,r}$, i.e. of its vertices. $G_{i,r}$ lies inside the (convex) polygon, and its coordinates are the average of the coordinates of $\mathcal{C}_{i,r}$, thus polynomially represented again. The shortest distance of $G_{i,r}$ to any of the edges $ax + by + c = 0$ of $\mathcal{C}_{i,r}$ (which necessarily is an edge of $\mathcal{O}_i$ or $\mathcal{O}_r$) is at least $\mu = \frac{1}{dq^{20}}$ as in the previous proof, for some constant $d > 0$ and $q = q(n)$ polynomial in $2^{p(n)}$. The (open) ball around $G_{i,r}$ with radius μ will be entirely contained in $\mathcal{C}_{i,r}$. If we reduce ('shrink') both $\mathcal{O}_i$ and $\mathcal{O}_r$ by a nonzero margin of, say, at most $\frac{1}{3}\mu$, then their intersection remains nonempty, as it continues to contain $G_{i,r}$ and the ball of radius $\frac{2}{3}\mu$ around it.

As μ is independent of the specific $\mathcal{O}_i$ and $\mathcal{O}_r$, this can be done simultaneously for all polygons. By Lemma 2, reducing every $\mathcal{O}_i$ by the nonzero margin of at most $\frac{1}{3}\mu$ can be done in a way that preserves similarity and polynomial representation. Thus, G is a $\mathcal{P}'$-intersection graph and polynomially represented. $\square$

From the two lemmas we conclude:

Theorem 1. *Every polynomially represented $\mathcal{P}$-intersection graph with a base set of closed convex polygons can be obtained as a polynomially represented $\mathcal{P}$-intersection graph with a base set of open convex polygons, and vice versa.*

In the sequel we will only work with base sets of closed convex polygons.

4 Representing $\mathcal{P}$-intersection graphs

Let G be a $\mathcal{P}$-intersection graph as above and let some geometric representation of G as $\mathcal{P}$-intersection graph be given. The representation of G can be viewed as a feasible solution of a *model*, namely of the model that defines the exact pattern of intersections and non-intersections between the polygons $\mathcal{O}$ in the representation. In this section, we will design a suitable LP model for this. Every feasible solution of the model will imply a geometric representation of G 9and vice versa). We will show that the model, if it has a feasible solution, also has a feasible solution that is polynomially represented, thus implying a geometric representation of G with this property. A similar approach was used in [14, 23, 40].

We heavily rely on the following fact. We include a proof for the sake of completeness, although the result appears to be folklore (cf. [17, 25])[4].

Lemma 9. *Two closed convex polygons $\mathcal{O}_i$ and $\mathcal{O}_j$ in the plane are disjoint if and only if they can be separated by a line that precisely coincides with an edge of one of them.*

Proof. The only if-part is the only nontrivial part. Let $\mathcal{O}_i$ and $\mathcal{O}_j$ be disjoint polygons as given. As $\mathcal{O}_i$ and $\mathcal{O}_j$ are compact, the Separating Hyperplane Theorem tells us that they can be strictly separated by a line l. Orient the plane so l is vertical, and assume w.l.o.g. that $\mathcal{O}_i$ is strictly to the left and $\mathcal{O}_j$ strictly to the right of l. Let p be a vertex of $\mathcal{O}_i$ or $\mathcal{O}_j$ closest to l (say $p \in \mathcal{O}_i$). Let m and m' be the two edges of $\mathcal{O}_i$ incident to p. If one of these edges runs parallel to l, we are done. Thus we may assume that both m and m', when extended to full lines of the same name, intersect l. Assume w.l.o.g. that to the right of p, m' runs above m.

If $\mathcal{O}_j$ lies strictly above line m or strictly below line m', then m respectively m' is the desired separating line and we are done. Thus assume that $\mathcal{O}_j$ is incident to both m and m'.

Consider any half-space h that defines (an edge of) $\mathcal{O}_j$, but does not contain p. (Such a halfspace exists because $p \notin \mathcal{O}_j$.) Let q (resp. q') be an intersection point of m (resp. m') with $\mathcal{O}_j$. As h necessarily contains q and q', but not p, the entire wedge enclosed by m and m' to the left of p is not contained in h. Since $\mathcal{O}_i$ is contained in this wedge, the line defining h satisfies the lemma. $\square$

Let G be the $\mathcal{P}$-intersection graph of the convex polygons $\mathcal{O}_1, \cdots, \mathcal{O}_n$, where $\mathcal{O}_i$ ($1 \leq i \leq n$) is similar to polygon $o_{s_i} \in S$ for some $s_i \in \{1, \cdots, m\}$. Let $\mathcal{O}_i$ be the result of applying the transformation $u_i + Q_i x = \binom{u_{i,1}}{u_{i,2}} + v_i \begin{pmatrix} \alpha_i & \gamma_i \\ \beta_i & \delta_i \end{pmatrix} \binom{x}{y}$ to o_{s_i}, with suitable $u_{i,1}, u_{i,2}$ and v_i ($1 \leq i \leq n$), all conforming to some template $t = t_i$ applicable to o_{s_i}. Let o_{s_i} ($1 \leq i \leq n$) have k_i vertices and (thus) k_i edges. All data related to o_{s_i} (vertices, edges, defining inequalities) will be super-indexed by (i).

4.1 Helpful inequalities

Consider any two polygons $\mathcal{O}_i, \mathcal{O}_j$ and suppose we want to express that they are *disjoint*. By Lemma 9 there must be a defining inequality of (say) $\mathcal{O}_i$ such that all of $\mathcal{O}_j$ does not satisfy it, precisely expressing that the polygons do not intersect. *Which* of $\mathcal{O}_i, \mathcal{O}_j$ to take and which of their defining inequalities, follows from the given geometric representation of G. Say the polygon to take is $\mathcal{O}_i$ and that the relevant defining inequality to take is the one obtained by applying t_i to the defining inequality $a^{(i)}x + b^{(i)}y + c^{(i)} \leq / \geq 0$ of o_{s_i}. Lemma 4 implies that this defining inequality of $\mathcal{O}_i$ can then be written as

$$(a^{(i)}\delta_i - b^{(i)}\beta_i)x + (b^{(i)}\alpha_i - a^{(i)}\gamma_i)y + (\alpha_i\delta_i - \beta_i\gamma_i)v_i c^{(i)} - (a^{(i)}\delta_i - b^{(i)}\beta_i)u_{i,1} - (b^{(i)}\alpha_i - a^{(i)}\gamma_i)u_{i,2} \leq / \geq 0.$$

Each vertex of $\mathcal{O}_j$ is obtained from a vertex $\binom{d^{(j)}}{e^{(j)}}$ of o_{s_j} under the mapping $u_j + Q_j x$, and can thus be written as $\binom{u_{j,1} + \alpha_j d^{(j)} v_j + \gamma_j e^{(j)} v_j}{u_{j,2} + \beta_j d^{(j)} v_j + \delta_j e^{(j)} v_j}$. To express that $\mathcal{O}_j$ is disjoint of $\mathcal{O}_i$ it now suffices to express that *none* of the k_j vertices of $\mathcal{O}_j$ satisfy this particular, defining inequality. It leads to k_j constraints of the form

$$\begin{aligned} &DISJ_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \quad := \\ &\quad (a^{(i)}\delta_i - b^{(i)}\beta_i)(u_{j,1} + \alpha_i d^{(j)} v_j + \gamma_j e^{(j)} v_j) + (b^{(i)}\alpha_i - a^{(i)}\gamma_i)(u_{j,2} + \beta_j d^{(j)} v_j + \delta_j e^{(j)} v_j) \\ &\quad + (\alpha_i\delta_i - \beta_i\gamma_i)v_i c^{(i)} - (a^{(i)}\delta_i - b^{(i)}\beta_i)u_{i,1} - (b^{(i)}\alpha_i - a^{(i)}\gamma_i)u_{i,2} \qquad > / < \quad 0, \end{aligned}$$

[4] A more general result for all dimensions was recently given in [42].

one for every vertex of $\mathcal{O}_j$. In fact we can strengthen each inequality to "$\geq$ some positive margin or "$\leq$ some negative margin" respectively (for some real nonzero margin), by evaluating the inequalities in the given geometric representation of G. Note that the inequalities are homogeneous in $u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j$. Thus, by multiplying all $u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j$ $(1 \leq i, j \leq n)$ by a same factor $\mu \geq 1$ large enough and rescaling the variables, the constraints continue to express that the embedding realizes G (by homogeneity of the inequalities), but now we can assume w.l.o.g. that the inequalities can be written as

$$DISJ_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \leq -1/\geq 1 \text{ for } k_j \text{ nodes and relevant } 1 \leq i, j \leq n.$$

(We will in fact choose μ large enough so a number of further goals w.r.t. the other constraints are all achieved as well, as explained below.)

Next suppose we want to express that $\mathcal{O}_i, \mathcal{O}_j$ *overlap.* Now we must express that there is a point $(x_{i,j}, y_{i,j})$ of (say) $\mathcal{O}_i$, satisfying the defining inequalities of both $\mathcal{O}_i$ and $\mathcal{O}_j$. This leads to a set of $k_i + k_j$ linear constraints

$$IN_i(x_{i,j}, y_{i,j}, u_{i,1}, u_{i,2}, v_i) \leq / \geq 0,$$
$$IN_j(x_{i,j}, y_{i,j}, u_{j,1}, u_{j,2}, v_j) \leq / \geq 0,$$

one for each defining inequality of $\mathcal{O}_i$ and $\mathcal{O}_j$. The inequalities are homogeneous in $x_{i,j}, y_{i,j}, u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j$ and thus they scale along with the scaling of the DISJ-inequalities.

Observe that the constraints "$v_i > 0$" may be replaced by "$v_i \geq$ some positive margin" in all cases as before, by using the data from the given embedding. If we multiply all variables by a factor $\mu \geq 1$ large enough and rescale the variables accordingly, we can achieve that the constraints continue to express that the model realizes G, but now we can also assume w.l.o.g. that $v_i \geq 1$ for $1 \leq i \leq n$.

Finally, note that the entire arrangement of polygons in the plane realizing G can be shifted over any fixed vector we want. Thus, in particular and without loss of generality we may assume that $u_{i,1}, u_{i,2} \geq 0$ for every $1 \leq i \leq n$.

The resulting LP model so far describes a possible geometric representation of G and has at least one feasible solution (which was used to define the model).

4.2 Assembling the model

The model is completely defined when we define the situation (intersection or non-intersection) for every pair $\mathcal{O}_i, \mathcal{O}_j$. First include all constraints of the defining affine transformations $u_i + Q_i x = \binom{u_{i,1}}{u_{i,2}} + v_i \begin{pmatrix} \alpha_i & \gamma_i \\ \beta_i & \delta_i \end{pmatrix} \binom{x}{y}$, following the templates t_i that are used and taking the scalings into account the conditions $\alpha_i \delta_i - \beta_i \gamma_i \neq 0$ hold by definition):

$u_{i,1}, u_{i,2} \geq 0$
$v_i \geq 1$

Next consider all $\frac{1}{2} n(n-1)$ pairs $\mathcal{O}_i, \mathcal{O}_j$ and express the model inequalities for each pair. For any given pair $\mathcal{O}_i, \mathcal{O}_j$ we have a set of k_{ij} or $l_{i,j}$ inequalities respectively of the following form:

if $\mathcal{O}_i, \mathcal{O}_j$ must be disjoint:
$k_{ij} \leq \max\{k_i, k_j\}$ inequalities of type
$DISJ_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \leq -1/\geq 1.$

if $\mathcal{O}_i, \mathcal{O}_j$ must intersect (m cases):
$l_{ij} = k_i + k_j$ inequalities of type
$IN_i(x_{i,j}, y_{i,j}, u_{i,1}, u_{i,2}, v_i) \leq / \geq 0$, respectively
$IN_j(x_{i,j}, y_{i,j}, u_{j,1}, u_{j,2}, v_j) \leq / \geq 0.$

We now bring the linear system into *standard form* by introducing nonnegative slack variables $z_i, w_{ij1}, \cdots, w_{ijk_{ij}}, z_{ij1}, \cdots, z_{ijl_{ij}}$ that turn the inequality constraints into equality constraints:

$v_i - z_i = 1$,

if $\mathcal{O}_i, \mathcal{O}_j$ must be disjoint:
k_{ij} inequalities $DISJ_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \pm w_{ijr} = \mp 1$ (where w_{ijr} is used in the r-th inequality),

if $\mathcal{O}_i, \mathcal{O}_j$ must intersect:
l_{ij} inequalities $IN_i(x_{i,j}, y_{i,j}, u_{i,1}, u_{i,2}, v_i) \pm z_{ijr} = 0$ and $IN_j(x_{i,j}, y_{i,j}, u_{j,1}, u_{j,2}, v_j) \pm z_{ijr} = 0$ (where z_{ijr} is used in the r-th inequality),

now with the standard constraints

$u_{i,1}, u_{i,2}, v_i, x_{i,j}, y_{i,j}, z_i \geq 0$ for all $1 \leq i \leq n$,
$w_{ij1}, \cdots, w_{ijk_{ij}} \geq 0$ for all $1 \leq i < j \leq n$,
$z_{ij1}, \cdots, z_{ijl_{ij}} \geq 0$ for all $1 \leq i < j \leq n$.

The linear equations of the model all have *rational* coefficients.

4.3 Solving the model

Because S is finite, there is a constant k such that $k_i \leq k$ for every $1 \leq i \leq n$. As T only has finite many different templates, there is a constant $q \geq 1$ such that for every $1 \leq i \leq n$, the numerators and denominators of the (rational) coefficients of the defining inequalities of every o_{s_i} and of the rationals $\alpha_i, \beta_i, \gamma_i, \delta_i$ are all $\leq 2^q$. This implies that the coefficients in the linear inequalities $DISJ_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \leq -1/\geq 1$ and $IN_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \leq 0/\geq 0$ are all rationals with numerators and denominators $\leq 2^{dq}$ for a small integer constant $d \geq 1$. The same bound holds for the coefficients in the standard form.

Let $N = n + \sum_{ij} k_{ij} + \sum_{ij} l_{ij} \leq n + \frac{1}{2}kn(n-1)$. The system of linear equalities can be written in matrix-vector form: $\mathbf{Ax} = \mathbf{b}$ with the constraint $\mathbf{x} \geq 0$, where

$\mathbf{A}$ is a N by $N + 3n + 2m$ all-rational matrix

all entries a of $\mathbf{A}$ have numerator and denominator $\leq 2^{dq}$

N columns of $\mathbf{A}$ are unit vectors, namely the columns corresponding to the variables $z_i, w_{ij1}, \cdots, w_{ijk_{ij}}, z_{ij1}, \cdots, z_{ijl_{ij}}$ ($1 \leq i \leq n$ and $1 \leq i \leq j \leq n$ resp.)

$\mathbf{x} = (\cdots, u_{i,1}, u_{i,2}, \cdots, v_i, \cdots, x_{i,j}, y_{i,j}, \cdots, z_i, \cdots, w_{ij1}, \cdots, w_{ijk_{ij}}, \cdots, z_{ij1}, \cdots, z_{ijl_{ij}}, \cdots)^T$

$\mathbf{b} = (\cdots, 1, \cdots, \mp 1, \cdots, 0, \cdots)^T$, with all entries rational, in fact ± 1 or 0.

The term 'unit vector' is used to denote any column that has only one nonzero entry, with this entry being ± 1. It follows that $\text{rank}(\mathbf{A}) = N = O(n^2)$.

Theorem 2. *The LP model has an all-rational solution for $u_{i,1}, u_{i,2}, v_i$ with numerators and denominators bounded in absolute value by $2^{\mathcal{O}(n^4)}$.*

Proof. Because $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{x} \geq 0$ has a feasible solution (by the given geometric representation of G), the Fundamental Theorem of Linear Programming ([6]) implies that it also has a *basic* feasible solution. As $\text{rank}(\mathbf{A}) = N$ (= the number of rows), this basic feasible solution has (at least) $3n + 2m$ of the coordinates of $\mathbf{x}$ equal to 0, whereas the N-by-N submatrix $\mathbf{A}'$ consisting of the columns corresponding to the other coordinates is invertible and satisfies $\mathbf{A}'\mathbf{x}' = \mathbf{b}$ (with $\mathbf{x}' \geq 0$), where $\mathbf{x}'$ is the subvector of $\mathbf{x}$ consisting of these other coordinates. Hence, by Cramer's rule [37], it follows that $(\mathbf{x}')_i = \frac{\det \mathbf{A}'_{\mathbf{i}}}{\det \mathbf{A}'}$, where $\mathbf{A}'_{\mathbf{i}}$ is the matrix formed by replacing the i-th column of $\mathbf{A}'$ by $\mathbf{b}$. This leads to a solution $u_{i,1}, u_{i,2}, v_i$ $(1 \leq i \leq n)$ of the model that satisfies:

$$\begin{aligned} u_{i,1} &= 0 \text{ or } u_{i,1} = \tfrac{\det \mathbf{A}'_{\mathbf{i_1}}}{\det \mathbf{A}'} \\ u_{i,2} &= 0 \text{ or } u_{i,2} = \tfrac{\det \mathbf{A}'_{\mathbf{i_2}}}{\det \mathbf{A}'} \\ v_i &= 0 \text{ or } v_i = \tfrac{\det \mathbf{A}'_{\mathbf{i_3}}}{\det \mathbf{A}'} \end{aligned}$$

for suitable indices i_1, i_2, i_3 for every i. (We omit the similar statements for the other variables in the model because they are not needed here. By the constraints $v_i - z_i = 1$, none of the v_i can be zero. Thus, in particular the n columns corresponding to these variables must all occur in the basis.) Observe that, because $\mathbf{A}'$ and $\mathbf{A}'_{\mathbf{i}}$ are rational matrices, their determinants are rational as well.

Recall that the nonzero entries of $\mathbf{A}'$ are all of the form $\frac{f}{h}$ with f, h integer and $|f|, |h| \leq 2^{dq}$. Let H_i be the product of all $|h|$-values that occur as denominators in the i-th column and let $H = \prod_1^N H_i$, thus $H \leq 2^{dq \cdot N \cdot N}$. Then $\det \mathbf{A}' = \frac{1}{H} \det \mathbf{A}''$, where $\mathbf{A}''$ is obtained from $\mathbf{A}'$ by multiplying the elements in the first column by H_1, the elements in the second column by H_2, etc. Now $\mathbf{A}''$ is an all-integer matrix with elements bounded by $2^{dq \cdot N}$ (noting that in every entry one h-factor of H_i cancels against the denominator of that entry).

By Hadamard's inequality for matrices $\mathbf{U} = (\mathbf{u_1} \cdots \mathbf{u_N})$, $|\det \mathbf{U}| \leq \|\mathbf{u_1}\| \cdots \|\mathbf{u_N}\|$. Apply this to $\det \mathbf{A}''$ or the similar determinant in the case of each $\mathbf{A}'_{\mathbf{i}}$. The matrices have N columns with integer entries bounded by $2^{dq \cdot N}$. Thus:

$$F = |\det \mathbf{A}''| \leq \left(\sqrt{N 2^{2dqN}}\right)^N \leq N^{\frac{1}{2}N} 2^{dqN^2} \leq 2^{2dqN^2}.$$

This shows that $\det \mathbf{A}' = \frac{F}{H}$ with F, H integers with $|F|, |H| \leq 2^{gn^4}$ for some constant g. The same bounds hold for the other determinants. The theorem now follows. □

The bound in Theorem 2 can be improved (cf. [33]) but is sufficient for obtaining the following, main result.

Theorem 3. *Let G be a $\mathcal{P}$-intersection graph. Then G has a polynomial representation, even fully in integers.*

Proof. By Theorem 2, G has a representation as $\mathcal{P}$-intersection graph using similarity transformations with $u_{i,1}, u_{i,2}, v_i$ rational numbers and numerators and denominators bounded in absolute value by $2^{\mathcal{O}(n^4)}$. By scaling the space by a factor equal to the product of the $3n$ denominators of these rational numbers, we obtain a representation in which all $u_{i,1}, u_{i,2}, v_i$ are integer, with absolute values bounded by $2^{\mathcal{O}(n^4) + 3n\mathcal{O}(n^4)} \leq 2^{\mathcal{O}(n^5)}$.

To obtain an all integer representation, apply a final scaling by a factor equal to the product of the denominators of all nonzero coefficients $\alpha, \beta, \gamma, \delta$ that occur in the templates and all denominators of the nonzero coordinates of the vertices of the polygons in S. This factor is bounded by a constant, given $\mathcal{P}$. □

Corollary 1. *The recognition problem for every class of $\mathcal{P}$-intersection graphs is in NP.*

Although the above arguments seem to rely on being given some representation of the $\mathcal{P}$-intersection graph, this is in fact not necessary. It suffices to know for each vertex of G which base polygon of S and which transformations of T to use, and for any nonadjacent pair of vertices which defining inequality of the objects representing these vertices to use to express disjointness. Given G and a signature $\mathcal{P} = \langle S, T \rangle$, this information is easily quantified over to find a representation of $\mathcal{P}$.

Corollary 2. *The construction ('drawing') problem of any class of $\mathcal{P}$-intersection graphs can be solved algorithmically, in exponential time.*

5 Applications

The notions of signatures and $\mathcal{P}$-intersection graphs are very useful in modeling classes of intersection graphs, particularly when combined with the generic theorems presented above. We list a few applications.

Square intersection graphs It is known that unit square graphs have polynomial size representations [8]. We can easily extend this now to *square intersection graphs*. Recall that NP-hardness of the recognition problem of unit square intersection graphs was proved in [2]. For general square graphs, NP-hardness follows from the recent results in [26].

Theorem 4. *Square intersection graphs have polynomial-size integer representations. Their recognition problem is in NP (and thus NP-complete).*

Proof. Define signature $\mathcal{P} = \langle S, T \rangle$ with S consisting of a unit square around the origin, and T consisting of the template $t : u + v\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Square intersection graphs are precisely the $\mathcal{P}$-intersection graphs for this signature $\mathcal{P}$. Now apply Theorem 3 and Corollary 1. □

Polynomial representation for unit square graphs follows more directly but may also be shown by the LP-based argument, provided we add the equations $v_i = v_j$ to the LP model.

Max-tolerance graphs Kaufmann et al. [17] showed that max-tolerance graphs are precisely the intersection graphs of so-called semi-squares. (A semi-square is 'a square with one half cut off along the bottom-right to top-left diagonal'.) In the same way as in the previous example, one can show that semi-square intersection graphs, thus max-tolerance graphs, have polynomial-size integer representations and an NP-recognition problem.

As Kaufmann et al. [17] proved the recognition problem for max-tolerance graphs to be NP-hard, it follows that this problem is in fact NP-complete.

Intersection graphs of homothetic polygons Kratochvíl and Pergel [26] initiated a general study of the intersection graphs that can be formed using homothetic copies of a single convex polygon P, or *P_{hom}-intersection graphs*. (We assume that P is always finitely given, in rational coordinates.) They show that the recognition problem for P_{hom}-intersection graphs is NP-hard. One can strengthen this as follows (see also the footnote on page 3):

Theorem 5. *P_{hom}-intersection graphs have polynomial-size integer representations. Their recognition problem is in NP (and thus NP-complete).*

Proof. Define a signature $\mathcal{P} = \langle S, T \rangle$ with $S = \{P\}$ and T assigning the homothetic transformations to P. P_{hom}-intersection graphs are precisely the $\mathcal{P}$-intersection graphs for this signature. The result follows as before. □

In [26], Kratochvíl and Pergel also define P_{hom}*-contact graphs*, where intersections are restricted to being *contacts* only. They pose as an open problem to determine the complexity of recognizing P_{hom}-contact graphs. By modifying the LP model, one can show by the same technique as developed in Section 4 that P_{hom}-contact graphs have polynomial-size integer representations. Thus the recognition problem for P_{hom}-contact graphs is *in NP*. It remains open whether this problem is NP-complete.

Note added in proof. In recent work jointly with Tobias Müller (CWI, Amsterdam), tight upper- and lower bounds have been obtained on the number of bits needed for representing convex polygon intersection graphs [33].

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